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A QUANTITATIVE VERSION OF KRASNOSEL'SKII'S THEOREM IN R^2

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This work concerns a quantitative version of Krasnosel'skii's theorem in R^2, and the following result is obtained: Let S be a nonempty compact subset of R^2 having n points of local nonconvexity. Then the kernel of S contains an interval of radius $\varepsilon > 0$ if and only if every $f(n) = \max\{4, 2n\}$ points of S see via S a common interval of radius ε. The number $f(n)$ in the theorem is best possible for every $n \geqq 1$.

We begin with some preliminary definitions: Let S be a subset of R^d. A point x in S is said to be a *point of local convexity* of S if and only if there is some neighborhood N of x such that $S \cap N$ is convex. In case S fails to be locally convex at point q in S, then q is called a *point of local nonconvexity* (lnc point) of S. For points y and z in S, we say *y sees v zia S* if and only if the corresponding segment $[y, z]$ lies in S. Set S is called *starshaped* if and only if there is some point p in S such that, for every x in S, p sees x via S, and the set of all such points p is called the (convex) kernel of S, denoted $\ker S$.

A well-known theorem of Krasnosel'skii [3] states that if $S \neq \emptyset$ is a compact set in R^d, then $\ker S$ is nonempty if and only if every $d+1$ points of S see a common point via S. Furthermore, quantitative analogues in [6, Theorem 6.19] and in [2] reveal that for S compact in R^d, $\ker S$ contains a d-dimensional neighborhood of radius $\varepsilon > 0$ if and only if every $d+1$ points of S see via S such a d-dimensional ε-neighborhood. While other versions of the Krasnosel'skii theorem have been obtained for sets whose kernel is k-dimensional, $1 \leqq k \leqq d-1$ [2], these provide little information concerning the size of the kernel, and quantitative theorems for such sets promise to be difficult to formulate.

Here the problem is studied for starshaped sets in R^2, and a quantitative version of the Krasnosel'skii theorem involving the number n of lnc points of S is obtained. Finally, two examples reveal that the result in the theorem is best possible for every $n \geqq 1$.

The following familiar terminology will be used throughout the paper: Conv S, cl S, and rel int S will denote the convex hull, closure, and relative interior of set S, respectively. For $x \neq y$, $R(x, y)$ will represent the ray emanating from x through y, while $L(x, y)$ will be the line determined by x and y.

2. The results. We start with two lemmas which will be helpful in proving the main result of the paper. The first is a variation of a result by Valentine [7, Corollary 2].

LEMMA 1 (*Valentine's lemma*). *Let S be a closed set in R^d. If $[x, y] \cup [y, z] \subseteq S$ and no* lnc *point of S lies in* $\operatorname{conv}\{x, y, z\} \sim [x, z]$, *then* $\operatorname{conv}\{x, y, z\} \subseteq S$.

LEMMA 2. *Let S be a nonempty compact set in R^d. For $1 \leqq k \leqq d$,* $\ker S$ *contains a k-dimensional neighborhood of radius $\varepsilon > 0$ if and only if every finite subset of S sees via S a k-dimensional neighborhood of radius ε.*

Proof. The necessity of the condition is obvious. Proof of its sufficiency will require the Hausdorff metric defined on the collection of compact subsets of S. Let $\mathscr{F} = \{A$: A compact and $A \subseteq S\}$, and let d denote the Hausdorff metric for $\mathscr{F}$. By arguments given in Valentine [6, pp. 37-38] and in Nadler [4], $\mathscr{F}$ is bounded with respect to d, and $(\mathscr{F}, d)$ has the Bolzano-Weierstrass property. That is, every infinite subset of $\mathscr{F}$ has a limit point F in $\mathscr{F}$, and $F \neq \emptyset$.

For each x in S, define $\mathscr{C}_x = \{A$: A is the closure of a k-dimensional neighborhood of radius ε and x sees A via $S\}$. Using our hypothesis for the compact set S, it is easy to see that the collection of sets $\mathscr{C}_x$ has the finite intersection property. Furthermore, we assert that each $\mathscr{C}_x$ set is compact, and clearly it suffices to show that every infinite subset $\{A_n\}$ in $\mathscr{C}_x$ has a limit point in $\mathscr{C}_x$. By our earlier remarks, $\{A_n\}$ has a limit point $A \neq \emptyset$ in $\mathscr{F}$, and by an argument in [2, Lemma], $A \in \mathscr{C}_x$.

Hence $\{\mathscr{C}_x$: x in $S\}$ is a collection of compact sets having the finite intersection property, so $\cap \{\mathscr{C}_x$: x in $S\} \neq \emptyset$. For B in this intersection, B is the closure of a k-dimensional ε-neighborhood and $B \subseteq \ker S$, finishing the proof of Lemma 2.

THEOREM 1. *Let S be a nonempty compact set in R^2 having n* lnc *points. The kernel of S contains an interval of radius $\varepsilon > 0$ if and only if every $f(n) = \max\{4, 2n\}$ (or fewer) points of S see via S a common interval of radius ε. The number $f(n)$ is best possible for every $n \geqq 1$.*

Proof. By Lemma 2, it suffices to prove that for F any finite subset of S, F sees via S an interval of radius ε.

Clearly we may assume that the set Q of lnc points of S is nonempty, for otherwise the closed connected set S will be convex by Tietze's theorem [5], and the result will be trivial. Hence let $Q = \{q_1, \cdots, q_n\}$.

We begin with the case in which $S \neq \mathrm{cl\,int}\, S$. By [1, Theorem 1], $\ker S$ is at least 1-dimensional, so we let L be a line which contains a 1-dimensional subset of $\ker S$. The set $S \sim \mathrm{cl\,int}\, S$ must be a nonempty subset of L. By hypothesis, every 4 points of S see via S a common interval of radius ε. For a in $S \sim \mathrm{cl\,int}\, S$, a sees via S only points on L, so every 3 (and hence every 2) points of S see via S a common interval of radius ε on L. By results in [6, Theorem 6.19] and in [2], $\ker S$ must contain an interval of radius ε on L, and the proof of the theorem is complete.

Throughout the remainder of the proof, we assume that $S = \mathrm{cl\,int}\, S$. For convenience of notation, let $q = q_1$. We show that for some closed neighborhood N of q, $N \cap S$ is a union of two convex sets, each containing q. Since $\ker S$ is at least 1-dimensional, we may select a point b in $\ker S \sim Q \neq \emptyset$. Let H be the line determined by q and b, with H_1 and H_2 distinct open halfplanes determined by H. Since Q is finite, we may select a convex neighborhood U of $H \cap S$ so that $Q \cap U \subseteq H$. Again using the fact that Q is finite, choose a closed convex neighborhood N of q such that $N \cap Q = \{q\}$ and $N \subseteq U$. We assert that N has the required property: Since $N \subseteq U$, for points c and d in $N \cap S \cap H_1$, $\mathrm{conv}\,\{c, b, d\}$ can contain no member of Q. Therefore, by Valentine's Lemma, $[c, d] \subseteq S$. Thus $N \cap S \cap H_1$ is convex, as is $N \cap S \cap H_2$. Since $S = \mathrm{cl\,int}\, S$, $N \cap S = \mathrm{cl}\,(N \cap S \cap H_1) \cup \mathrm{cl}\,(N \cap S \cap H_2)$, and $N \cap S$ is indeed a union of two convex sets, each containing q. (In fact, the sets meet in an interval at q.)

Select points x and y in $N \cap S$ such that $(x, y) \cap S = \emptyset$ and such that no point of $[x, q)$ sees any point of $[y, q)$ via S. Since F is finite, an inductive argument reveals that x and y may be chosen so that no point of F which sees q lies in the open convex region bounded by $R(q, x)$ and $R(q, y)$, and we assume that x and y satisfy this requirement as well.

Let $x_1 = x$ and $y_1 = y$, and for $2 \leqq i \leqq n$, select points x_i and y_i corresponding to q_i in the manner described above. It is clear that for each i, any point of S seen by both x_i and y_i necessarily lies in the closed convex region R_i bounded by the rays $R(x_i, q_i) \sim [x_i, q_i)$ and $R(y_i, q_i) \sim [y_i, q_i)$. Hence $\ker S \subseteq \cap \{R_i\colon 1 \leqq i \leqq n\} \equiv R$. Furthermore, by hypothesis, points $x_1, \cdots, x_n, y_1, \cdots, y_n$ see via S a common interval I of radius ε, and $I \subseteq R$. To complete the proof, it suffices to show that for s in F and t in I, $[s, t] \subseteq S$.

We will assume that this does not occur to reach a contradiction. Now $\ker S$ has dimension at least 1, and we select point p in rel int $\ker S$. For future reference, notice that p cannot be an lnc point for S. Also, observe that since $p \in \ker S$, each x_i and y_i point necessarily sees the entire segment $[p, t]$ via S, $1 \leqq i \leqq n$. Because we are assuming that $[s, t] \nsubseteq S$, points s, t, p cannot be collinear, and we let L_1 denote the open halfplane determined by the line $L(s, p)$ and containing t, L_2 the opposite open halfplane. Since $[s, p] \cup [p, t] \subseteq S$ but $[s, t] \nsubseteq S$, there is some point r of $(t, p]$ closest to t for which $[s, r] \subseteq S$. Moreover, by Valentine's Lemma, $[s, r]$ must contain some lnc point of $\mathrm{cl}\, L_1 \cap S$, and clearly this lnc point is a member of Q, say q_1. Further, we assert that q_1 may be chosen distinct from s: If s were the only such lnc point on $[s, r]$, then since the set Q' of lnc points of $\mathrm{cl}\, L_1 \cap S$ is finite, we could select a neighborhood of $[s, r]$ disjoint from $Q' \sim \{s\}$, and by Valentine's Lemma, r would not be the closest point to t having the required property. Thus we may assume that $q_1 \neq s$.

For convenience of notation, let $q = q_1$, with $x = x_1$ and $y = y_1$ the corresponding points selected previously. Recall that no point of $[x, q)$ sees any point of $[y, q)$ via S. There are two cases to consider.

Case 1. Assume that $r = p$. Points x and y cannot both lie in the closed halfplane $\mathrm{cl}\, L_1$: Otherwise, since $p \in \ker S \subseteq R$, one of the two points, say y, would satisfy $q \in [p, y]$, and x could not see $[p, t]$ via S. Also, x and y cannot both lie in the opposite closed halfplane $\mathrm{cl}\, L_2$, for then q could not be an lnc point for $\mathrm{cl}\, L_1 \cap S$. Thus we may assume that $x \in L_1$ and $y \in L_2$. By a previous observation, p is not an lnc point for S so $q \neq p$. Using this together with the fact that $p \in \ker S$, it is not hard to see that y and s necessarily lie in the same open halfplane determined by $L(x, q)$. However, this forces s to lie in the open convex region bounded by rays $R(q, x)$ and $R(q, y)$. Since $s \in F$ and s sees q via S, this is impossible by our original selection of x and y. We have a contradiction, and we conclude that Case 1 cannot occur.

Case 2. Assume that $r \neq p$, and let M_1 denote the open halfplane determined by $L(s, r)$ and containing t, M_2 the opposite open halfplane. As in Case 1, points x and y cannot both lie in $\mathrm{cl}\, M_1$ for then they could not both see $[p, t]$ via S. A similar argument reveals that x and y cannot both lie in $\mathrm{cl}\, M_2$, so we may suppose that $x \in M_1$ and $y \in M_2$. Using the fact that x and y both see $[p, t]$ via S, it is easy to show that y and s lie in the same open halfplane determined by

$L(x, q)$, just as in Case 1. However, by our earlier argument, this contradicts the original choice of x and y, and Case 2 cannot occur either.

Since neither Case 1 nor Case 2 can occur, the assumption that $[s, t] \not\subseteq S$ must be false. We conclude that $[s, t]$ does indeed lie in S, and each point of F sees the interval I via S. By previous remarks, this completes the proof of Theorem 1.

The following examples reveal that the number $f(n)$ in the theorem is best possible for every $n \geqq 1$. In this setting, notice also that no finite Krasnosel'skii number exists independent of the number of lnc points of set S.

EXAMPLE 1. Set S be the set in Figure 1, with q the corresponding lnc point of S. Clearly for an appropriate $\varepsilon > 0$, every 3 points of S see via S some segment with endpoint q having radius ε, yet $\ker S = \{q\}$. Thus the number $f(1) = 4$ in Theorem 1 is best possible.

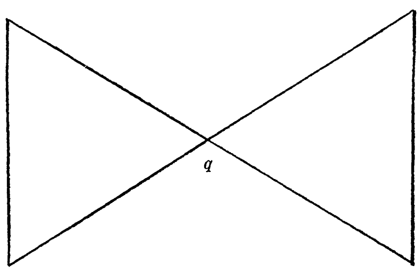

FIGURE 1

EXAMPLE 2. For $n \geqq 2$, consider the vertices of a regular $4n-$gon P, each vertex on the circumference of the unit circle U. Label the vertices of P in a clockwise direction by $p_1, p_2, \cdots, p_{4n} = p_0$ so that $p_1, p_2, \cdots, p_{2n}$ lie in the open upper halfplane. Assume that segments $T_0, T_1, \cdots, T_{2n+1}$ satisfy the following properties: T_0 contains p_0 and p_1, T_i contains p_{i-1} and p_{i+1} for $1 \leqq i \leqq 2n$, and T_{2n+1} contains p_{2n} and p_{2n+1}. Furthermore, for $0 \leqq i \leqq n$, segments T_i and T_{i+n+1} share one endpoint, and each segment has its remaining endpoint on U. Thus T_i and T_{i+n+1} determine a triangular region $S_i \equiv \operatorname{conv}(T_i \cup T_{i+n+1})$.

Define $S' = \cup \{S_i\colon 0 \leqq i \leqq n\} \cup P$. Notice that S' has exactly n lnc points $q_1, \cdots, q_n$, where $q_i \in T_{i+n} \cap T_i$, $1 \leqq i \leqq n$. Finally, intersect S' with an appropriate closed polygon to obtain a set S such that the lnc points of S are $q_1, \cdots, q_n$, each point of $S \cap S_i$ fails to see via S at most one of p_i, p_{i+n+1}, $1 \leqq i \leqq n-1$, and $p_{2n+1}, \cdots, p_{4n} \in \ker S$. Thus each point of S fails to see via S at most one of the points $p_1, \cdots, p_{2n}$. (In particular, S may be achieved by intersecting each set S_i with a suitable closed halfplane whose boundary contains $L(p_i, p_{i+1}) \cap L(p_{i+n+1}, p_{i+n})$.)

It is easy to see that every $2n-1$ points of S see via S a common diameter of U containing one of the points $p_1, \cdots, p_{2n}$. Thus every $2n-1$ points of S see via S a common interval of radius $\varepsilon = 1$. However, an appropriately chosen $2n$ points of S, one selected from each segment $T_1, \cdots, T_{2n}$, see no such interval. We conclude that for $n \geqq 2$, the number $f(n) = 2n$ in Theorem 1 is best possible.

Figure 2 illustrates the construction when $n = 3$.

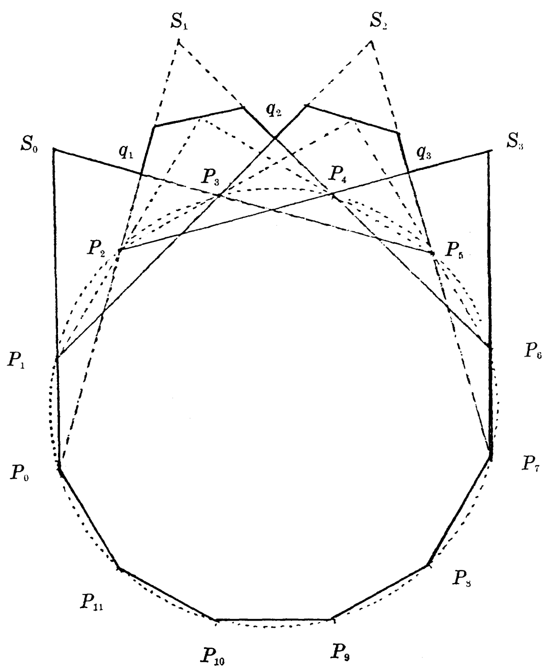

FIGURE 2

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