PACIFIC JOURNAL OF MATHEMATICS
Vol. 91, No. 1, 1980

EXTREMAL PROBLEMS ON NON-AVERAGING AND NON-DIVIDING SETS

H. L. ABBOTT

A set A of integers is said to be non-averaging if the arithmetic mean of two or more members of A is not in A. A is said to be non-dividing if no member divides the sum of two or more others. In this paper we investigate some of the many extremal problems which arise in connection with non-averaging and non-dividing sets.

1. Introduction. In [1] the author showed that a modification of an old argument of F. A. Behrend [3] could be used to disprove a conjecture of Erdös and Straus ([4] and [11]) on non-averaging sets. In the present paper the method of Behrend is put in a more general setting and we use it, together with a number of other devices, to derive several new results on non-averaging and non-dividing sets. In all of the questions we consider, however, the results obtained are far from being definitive.

2. The main theorem. The following theorem is a generalization of a result of Behrend on arithmetic progressions. In fact, Behrend's theorem is given as Corollary 3 below.

THEOREM 1. *Let l, B and t be positive integers exceeding* 1, *and suppose* $(l, B) = 1$. *Let*

$$s = tl^t(B-1)^2 \tag{1}$$

and let

$$n = B^t - 1 \, . \tag{2}$$

Then there exists a partition of $\{1, 2, \cdots, n\}$ *into s sets* $A_1, A_2, \cdots, A_s$ *such that for each m, $2 \leqq m \leqq l$, and each i, $1 \leqq i \leqq s$, no m members of A_i have arithmetic mean in A_i*

Proof. Write the numbers $1, 2, \cdots, n$ in base B so that if $1 \leqq a \leqq n$, we have

$$a = \sum_{i=0}^{t-1} d_i(a)B^i \, , \quad 0 \leqq d_i(a) \leqq B-1 \, .$$

Let $r = t(B-1)^2$ and partition $\{1, 2, \cdots, n\}$ into r sets $S_1, S_2, \cdots, S_r$ where

$$S_j = \left\{a\colon \sum_{i=0}^{t-1} d_i(a)^2 = j\right\} .$$

It will be useful to associate with a the lattice point $(d_0(a), d_1(a), \cdots, d_{t-1}(a))$ in E^t. Note that the lattice points corresponding to numbers in S_j lie on a sphere of radius $\sqrt{j}$.

Next partition S_j into $k = l^t$ sets, two numbers a and b in S_j being placed in the same set if $d_i(a) \equiv d_i(b) \pmod{l}$ for $i = 0, 1, \cdots, t-1$. Thus $\{1, 2, \cdots, n\}$ has been partitioned into $kr = tl^t(B-1)^2 = s$ sets $A_1, A_2, \cdots, A_s$.

Suppose that for some m, $2 \leqq m \leqq l$, and some i, $1 \leqq i \leqq s$, there are distinct numbers $y_0, y_1, \cdots, y_m$ in A_i such that

$$y_0 + y_1 + \cdots + y_{m-1} = my_m . \tag{3}$$

Define x_j for $j = 0, 1, \cdots, l$ by

$$x_j = \begin{cases} y_j & \text{if} \quad 0 \leqq j \leqq m \\ y_m & \text{if} \quad m \leqq j \leqq l . \end{cases} \tag{4}$$

It follows from (3) and (4) that

$$x_0 + x_1 + \cdots + x_{l-1} = lx_l . \tag{5}$$

From (5) it follows that

$$\sum_{j=0}^{l-1} d_0(x_j) = h + \mu B$$

and

$$ld_0(x_l) = h + \nu B$$

where $0 \leqq h \leqq B - 1$ and $0 \leqq \mu,\ \nu \leqq l - 1$. Thus

$$\sum_{j=0}^{l-1} d_0(x_j) = (\mu - \nu)B + ld_0(x_l) . \tag{6}$$

Now $d_0(x_0), d_0(x_1), \cdots, d_0(x_{l-1})$ belong to the same residue class modulo l and consequently l divides the left side of (6). Since $(l, B) = 1$, we must have $l \mid \mu - \nu$. However, since $|\mu - \nu| < l$, this gives $\mu = \nu$ and hence

$$\sum_{j=0}^{l-1} d_0(x_j) = ld_0(x_l) .$$

This argument may now be repeated to show that

$$\sum_{j=0}^{l-1} d_i(x_j) = ld_i(x_l) \quad \text{for} \quad i = 0, 1, \cdots, t-1 . \tag{7}$$

If $P_0, P_1, \cdots, P_l$ are the points of E^t corresponding to $x_0, x_1, \cdots, x_l$

then (7) is just the statement that P_l is the centroid of $P_0, P_1, \cdots, P_{l-1}$. Since the points lie on a sphere, we must have $P_0 = P_1 = \cdots = P_l$ and hence $x_0 = x_1 = \cdots = x_l$. It follows that $y_0 = y_1 = \cdots = y_m$ contrary to hypothesis. This completes the proof of the theorem.

3. Some consequences of the main theorem.

COROLLARY 1. *Denote by $f(n)$ the size of a maximal non-averaging subset of $\{1, 2, \cdots, n\}$. Then $f(n) > cn^{1/10}$.*

Proof. In Theorem 1 take $t = 5$, $B = l^2 + 1$, so that, by (1) and (2), $s = 5l^9$ and $n = B^5 - 1 \sim l^{10}$. One of the sets, say A_1, contains at least $[n/s] \sim l/5 \sim (1/5)n^{1/10}$ numbers. If $|A_1| \geqq l$, let A be any l-subset of A_1 and if $|A_1| < l$, let $A = A_1$. In both cases A is non-averaging and $|A| > cn^{1/10}$, as required.

REMARK 1. Corollary 1 appears in [1]. We point out that Straus [11] proved $f(n) > \exp(c\sqrt{\log n})$ and Erdös and Straus [4] proved $f(n) < cn^{2/3}$. It had been conjectured by Erdös and Straus that $f(n) < \exp(c\sqrt{\log n})$. Corollary 1, of course, shows that this conjecture is false. However, the following interesting question now arises: Does there exist a number α such that $f(n) = n^{\alpha + o(1)}$? It seems certain that such an α exists, but we have not been able to make any progress towards proving it.

COROLLARY 2. *Denote by $f_m(n)$ the size of a maximal subset A of $\{1, 2, \cdots, n\}$ with the property that no m members of A have arithmetic mean in A. Then, for each fixed $m \geqq 2$,*

$$f_m(n) > n \exp(-(2 + o(1))(2 \log m \log n)^{1/2}) .$$

Proof. In Theorem 1 take $l = m$ and put $B = m^{t/2} + 1$. (We suppose, without loss of generality, that t is even.) Then, by (1) and (2), $s = tm^{2t}$ and $n \sim m^{t^2/2}$. One of the sets contains at least $[n/s] \sim (1/t)m^{(1/2)t^2 - 2t}$ numbers and a simple calculation shows that

$$\frac{1}{t} m^{(1/2)t^2 - 2t} > n \exp(-(2 + o(1))(2 \log m \log n)^{1/2}) .$$

COROLLARY 3. (Behrend). *Denote by $r_3(n)$ the size of a maximal subset of $\{1, 2, \cdots, n\}$ not containing a three term arithmetic progression. Then*

$$r_3(n) > n \exp(-(2 + o(1))(2 \log 2 \log n)^{1/2}) .$$

Proof. Since $r_3(n) = f_2(n)$, the result follows from Corollary 2.

COROLLARY 4. (Moser [6]). *For positive integral k, let $W(k)$ denote the least integer such that if $\{1, 2, \cdots, W(k)+1\}$ is partitioned arbitrarily into k sets, one of the sets contains an arithmetic progression of length* 3. *Then*

$$W(k) > k^{c \log k}\,.$$

Proof. In Theorem 1 put $l = m = 2$ and determine t by

$$t \cdot 2^{3t} \leqq k < (t+1)2^{3t+3}\,. \tag{8}$$

By (1), $s = t \cdot 2^{3t}$ and if we put $B = 2^t + 1$ we get, by (2), $n \sim 2^{t^2}$. Then, by a simple calculation using (8), we get $W(k) \geqq W(s) \geqq n \sim 2^{t^2} > k^{c \log k}$.

Theorem 1 may also be used to show that various sets of integers, which arise in a natural way, contain large non-averaging subsets. We mention two examples.

COROLLARY 5. *Let $P = \{p : p \leqq n,\ p \text{ prime}\}$. Then P contains a non-averaging subset of size at least $cn^{1/10}/\log n$.*

Proof. In Theorem 1 take $t = 5$ and $B = l^2 + 1$, as in Corollary 1. One of the s sets contains at least $[\pi(n)/s] \sim n^{1/10}/5 \log n$ primes and the result follows.

COROLLARY 6. *Let Q_k denote the set of the kth powers not exceeding n. Then Q_k contains a non-averaging subset of size at least $c_k n^{1/8k^2+2k}$, where c_k is a constant depending only on k.*

Proof. In Theorem 1 take $t = 4k+1$, $B = l^{2k} + 1$ and note that one of the s sets contains at least $[n^{1/k}/s] \sim l/(4k+1) \sim (1/4k+1)n^{1/8k^2+2k}$ kth powers. The result follows.

REMARK 2. Corollary 6 includes Corollary 1 as the special case $k = 1$.

4. Additional results on finite non-averaging sets. It would be of interest to know whether there exists a number $\beta > 0$ such that every set of n integers contains a non-averaging subset of size at least n^β. We cannot answer this question, but we obtain a partial result in this direction as follows:

THEOREM 2. *Let* $m \geqq n$. *Then almost all* n*-subsets of* $\{1, 2, \cdots, m\}$ *contain a non-averaging subset of size at least* $c(f(n) \log\log n)^{1/2}/\log n$, *where* f *has the same meaning as in Corollary* 1 *and where almost all means all but* $o\left(\binom{m}{n}\right)$.

In order to prove the theorem we shall need the following lemma:

LEMMA 1. *There exists a partition of* $\{1, 2, \cdots, n\}$ *into* $k < 2n \log n/f(n)$ *non-averaging sets.*

Proof. Let A be a maximal non-averaging subset of $\{1, 2, \cdots, n\} = N$, so that $|A| = f(n)$. For integral λ let $A + \lambda = \{a + \lambda : a \in A\}$ and let $A_\lambda = (A + \lambda) \cap N$. It is clear that A_λ is non-averaging. Let $\lambda_0 = 0$ and suppose we have defined numbers $\lambda_0, \lambda_1, \cdots, \lambda_j$. Let $D_j = \{d : d \in N,\ d \notin A_{\lambda_i} \text{ for } i = 0, 1, 2, \cdots, j\}$. If $D_j \neq \varnothing$, then for every $d \in D_j$ and every $a \in A$, there exists an integer λ such that $\lambda + a = d$ and $0 < |\lambda| \leqq n$. Thus for some λ^*, $0 < |\lambda^*| \leqq n$, the equation $\lambda^* + a = d$ has at least $|D_j| f(n)/2n$ solutions $a \in A$, $d \in D_j$. Let $\lambda_{j+1} = \lambda^*$ and let $D_{j+1} = \{d : d \in N,\ d \notin A_{\lambda_i} \text{ for } i = 0, 1, \cdots, j + 1\}$. We have

$$|D_{j+1}| \leqq |D_j| - \frac{|D_j| f(n)}{2n} = |D_j|\left(1 - \frac{f(n)}{2n}\right).$$

Since $|D_0| = n - f(n) < n(1 - f(n)/2n)$ we get

$$|D_j| \leqq n\left(1 - \frac{f(n)}{2n}\right)^{j+1}.$$

Now choose $k = [(2n \log n)/f(n)]$. Then

$$|D_k| \leqq n\left(1 - \frac{f(n)}{2n}\right)^{k+1} < 1.$$

Thus $|D_k| = 0$ and the sets $A_{\lambda_0}, A_{\lambda_1}, \cdots, A_{\lambda_k}$ are non-averaging sets whose union is N. This implies the lemma.

REMARK 3. The idea used in the above proof seems to have been first used by G. G. Lorentz [6]. Subsequently it has been used by a number of other authors in many different situations. See, for example, [9] or [10] for a general discussion of the method and further references to the literature. We point out also that, with careful attention to detail the bound $k \leqq (n/f(n))(1 + \log f(n))$ can be obtained.

Proof of Theorem 2. The argument is similar to that used in

[8] and [2], but is somewhat more complicated. Let $w = m/n$ and partition $\{1, 2, \cdots, m\}$ into intervals $I_1, I_2, \cdots, I_n$ where

$$I_\alpha = \{a\colon (\alpha - 1)w < a \leqq \alpha w\} .$$

The first part of the argument involves showing that the elements of almost all n-subsets of $\{1, 2, \cdots, m\}$ are fairly well distributed among the intervals I_α. More precisely, we shall prove that if

$$\mu = \left[\frac{n \log\log n}{2 \log n}\right] \tag{9}$$

and if T denotes the number of n-subsets of $\{1, 2, \cdots, m\}$ which have elements in fewer than μ of the intervals I_α then

$$T = o\left(\binom{m}{n}\right) .$$

We may clearly suppose $m \geqq 2n$, since otherwise $T = 0$. We have

$$T \leqq \sum_{j=1}^{\mu-1} \binom{n}{j} \sum_{b_1+b_2+\cdots+b_j=n} \prod_{i=1}^{j} \binom{[w+1]}{b_i} \tag{10}$$

where, in the inner sum, the summation is over all compositions of n into j parts. In fact, (10) can be established as follows: $\binom{n}{j}$ is the number of ways of selecting j of the intervals I_α, say $I_{\alpha_1}, I_{\alpha_2}, \cdots, I_{\alpha_j}$ and $\prod_{i=1}^{j}\binom{[w+1]}{b_i}$ is the number of ways of selecting n integers, b_i of which are in I_{α_i}. From (10) we get

$$\begin{aligned}
T &\leqq \sum_{j=1}^{\mu-1} n^j \sum_{b_1+b_2+\cdots+b_j=n} \prod_{i=1}^{j} \frac{(w+1)^{b_i}}{b_i!} \\
&= \sum_{j=1}^{\mu-1} \frac{n^j (w+1)^n}{n!} \sum_{b_1+b_2+\cdots+b_j=n} \frac{n!}{b_1!\, b_2! \cdots b_j!} \\
&= \frac{(w+1)^n}{n!} \sum_{j=1}^{\mu-1} n^j j^n , \quad \text{by the multinomial theorem} \\
&\leqq \frac{(w+1)^n}{n!} n^{\mu-1} (\mu - 1)^{n+1} \\
&\leqq \frac{(2w)^n}{n!} n^{\mu} \mu^n \\
&\leqq \frac{1}{n!}\left(\frac{2m}{n}\right)^n n^{(n \log\log n)/(2\log n)} \left(\frac{n \log\log n}{2 \log n}\right)^n , \quad \text{by (9)} \\
&= \frac{m^n}{n!}\left(\frac{\log\log n}{\sqrt{\log n}}\right)^n = o\left(\frac{m^n}{n!\, 2^n}\right)
\end{aligned}$$

$$= o\left(\frac{m^n}{n!}\left(1 - \frac{n}{m}\right)^n\right), \quad \text{as} \quad m \geqq 2n$$

$$= o\left(\binom{m}{n}\right), \quad \text{as required.}$$

Let N be an n-subset of $\{1, 2, \cdots, m\}$ which has elements in at least μ of the intervals I_α and let $A = \{\alpha: I_\alpha \cap N \neq \varnothing\}$. For each $\alpha \in A$ choose $a_\alpha \in I_\alpha \cap N$ and let $A' = \{a_\alpha: \alpha \in A\}$. We now show that A' contains a non-averaging subset of size at least $c(f(n)\log\log n)^{1/2}/\log n$. Since $A' \subseteq N$, the theorem will then follow.

Partition $\{1, 2, \cdots, n\}$ into $k < 2n \log n/f(n)$ non-averaging sets via Lemma 1. One of these sets, say C, must be such that

$$q = |C \cap A| \geqq \left[\frac{\mu}{k}\right] > \frac{f(n)\log\log n}{(\log n)^2}\,. \tag{11}$$

Let $h = [\sqrt{q}]$ and for $\alpha \in C \cap A$ let

$$I_\alpha = I_\alpha^{(1)} \cup I_\alpha^{(2)} \cup \cdots \cup I_\alpha^{(h)}$$

where

$$I_\alpha^{(\nu)} = \left\{a: \left(\alpha - \frac{\nu}{h}\right)w < a \leqq \left(\alpha - \frac{\nu - 1}{h}\right)w\right\}\,.$$

Then, by the pigeon hole principle, there exists an integer ν_0 and a set $A^* \subset C \cap A$, $|A^*| = h$, such that $a_\alpha \in I_\alpha^{(\nu_0)}$ for each $\alpha \in A^*$. Let $A_1 = \{a_\alpha: \alpha \in A^*\}$. We claim that A_1 is non-averaging.

Suppose that $a_{\alpha_0}, a_{\alpha_1}, \cdots, a_{\alpha_p}$ $(p \leqq h - 1)$ are distinct members of A_1 satisfying

$$a_{\alpha_0} + a_{\alpha_1} + \cdots + a_{\alpha_{p-1}} = pa_{\alpha_p}\,. \tag{12}$$

We have

$$a_{\alpha_i} = \left(\alpha_i - \frac{\nu_0}{h}\right)w + b_i\,, \quad 0 < b_i \leqq \frac{w}{h}\,.$$

Thus (12) can be written as

$$w\left(p\alpha_p - \sum_{i=0}^{p-1}\alpha_i\right) = -pb_p + \sum_{i=0}^{p-1} b_i\,. \tag{13}$$

The conditions $0 < b_i \leqq w/h$ and $2 \leqq p \leqq h - 1$ imply that the right side of (13) lies strictly between $-w$ and w and must thererfore be 0. It follows that

$$\sum_{i=0}^{p-1}\alpha_i = p\alpha_p\,.$$

However, the numbers $\alpha_0, \alpha_1, \cdots, \alpha_p$ are in C and C is non-averaging. This is a contradiction. It follows that A_1 is non-averaging. Moreover, by (11),

$$|A_1| = h = [\sqrt{q}] > c(f(n) \log\log n)^{1/2}/\log n .$$

This completes the proof.

We conclude this section with an additional application of Lemma 1, which complements Corollary 5.

THEOREM 3. *Let $P = \{p: p \leqq n,\ p \text{ prime}\}$. Then p contains a non-averaging subset of size at least $cf(n)/(\log n)^2$.*

Proof. By Lemma 1, $\{1, 2, \cdots, n\}$ can be partitioned into $k < 2n \log n/f(n)$ non-averaging sets. One of these must contain at least $[\pi(n)/k] > cf(n)/(\log n)^2$ primes and the result follows.

5. Infinite non-averaging sets. In all of what follows α and β are numbers such that $n^\alpha \ll f(n) \ll n^\beta$. We prove first the following result, a weaker version of which was announced in [1].

THEOREM 4. *There exists an infinite non-averaging set A of positive integers whose counting function satisfies*

$$A(x) \gg x^{\alpha/(1+\beta)^2} .$$

Proof. Let $m > 1$ be a positive integer. Let $n_1 = m$ and let $n_k = [mn_{k-1}^{1+\beta} + 1]$ for $k = 2, 3, \cdots$. Let A_1 be a maximal non-averaging subset of $\{1, 2, \cdots, n_1\}$ and, for $k \geqq 2$, let A_k be a maximal non-averaging subset of $\{n_k + 1, n_k + 2, \cdots, n_k + n_{k-1}\}$. Let $A = \bigcup_{k=1}^{\infty} A_k$. Suppose now that m is chosen so that $|A_k| < (m/2)n_{k-1}^{\beta}$.

We now show that A is a non-averaging set. Suppose there are distinct numbers $a_0, a_1, \cdots, a_t \in A$ such that

$$a_0 + a_1 + \cdots + a_{t-1} = ta_t . \tag{14}$$

We may assume $a_0 < a_1 < \cdots < a_{t-1}$. Let $a_{t-1} \in A_k$. Suppose first that $k \geqq 3$. It is clear that not all of $a_0, a_1, \cdots, a_{t-1}$ are in A_k. Thus we may determine r, $1 \leqq r \leqq t-1$, such that $a_0 < a_1 < \cdots < a_{r-1} \leqq n_{k-1} + n_{k-2} < n_k + 1 \leqq a_r < \cdots < a_{t-1} \leqq n_k + n_{k-1}$. Then

$$\begin{aligned} (t-r)n_k &< a_0 + a_1 + \cdots + a_{t-1} \\ &< ra_{r-1} + (t-r)a_{t-1} \\ &< 2rn_{k-1} + (t-r)(n_k + n_{k-1}) \\ &= (t-r)n_k + (t+r)n_{k-1} \end{aligned} \tag{14}$$

$$
\begin{aligned}
&< (t - r)n_k + 2tn_{k-1} \\
&< (t - r)n_k + mn_{k-1}^{1+\beta}\,, \quad \text{as} \quad t \leqq |A_{k-1}| < \frac{m}{2} n_{k-1}^{\beta} \\
&< (t - r + 1)n_k \\
&\leqq tn_k\,.
\end{aligned}
$$

If $a_t \in A_l$ and $l \geqq k$ then $ta_t > tn_k > a_0 + a_1 + \cdots + a_{t-1}$, by (14) while if $l \leqq k - 1$ we have $ta_t \leqq t(n_{k-1} + n_{k-2}) \leqq 2tn_{k-1} < mn_{k-1}^{1+\beta} < n_k \leqq (t - r)n_k < a_0 + a_1 + \cdots + a_{t-1}$, by (14). This is a contradiction. The above argument does not apply verbatim to the case $k \leqq 2$, but the same method works. Thus A is non-averaging.

Let x be given and let k be determined by $n_k < x \leqq n_{k+1}$. We may suppose that x is so large that $k \geqq 3$. Then, if $n_k < x \leqq n_k + n_{k-1}$ we get $A(x) \geqq A(n_k) \geqq |A_{k-1}| \gg n_{k-2}^{\alpha} \gg n_k^{\alpha/(1+\beta)^2} \gg x^{\alpha/(1+\beta)^2}$, while if $n_k + n_{k-1} < x \leqq n_{k+1}$, we get $A(x) \geqq |A_k| \gg n_{k-1}^{\alpha} \gg x^{\alpha/(1+\beta)^2}$. This completes the proof of the theorem.

We consider next the problem of establishing the existence of an infinite non-averaging set of primes whose counting function grows at least as fast as x^c for some $c > 0$. In order to achieve this we shall need to make use of the following deep result on the distribution of the primes, which we state as a lemma.

LEMMA 2. *If $\theta \geqq 7/12$, the interval $[x, x + x^{\theta}]$ contains at least $cx^{\theta}/\log x$ primes for all sufficiently large x.*

REMARK 4. The bound $\theta \geqq 7/12$ in Lemma 2 is due to Huxley [5] who improved earlier results of Hoheisel, Ingham and Montgomery. See [5] for an account of the history of the problem. In the applications, we can actually get by with the bound $\theta \geqq 3/5$ of Montgomery.

THEOREM 5. *There exists an infinite non-averaging set P of primes whose counting function satisfies*

$$P(x) \gg x^{\alpha/(1+\beta)^2}/(\log x)^2\,.$$

Proof. Note first that since $n_{k-1} \sim (1/m)n_k^{1/(1+\beta)}$ and since $1/(1 + \beta) \geqq 3/5$ $(\beta \leqq 2/3)$, the number of primes in the interval $\{n_k + 1, \cdots, n_k + n_{k-1}\}$ is, by Lemma 2, at least $cn_k^{1/(1+\beta)}/\log n_k$. By Lemma 1, $\{n_k + 1, \cdots, n_k + n_{k-1}\}$ can be partitioned into fewer than $2n_{k-1} \log n_{k-1}/f(n_{k-1})$ non-averaging sets. One of these sets must therefore contain at least $cf(n_{k-1})/(\log n_{k-1})^2$ primes. Let P_k be this set of primes and let $P = \bigcup_{k=1}^{\infty} P_k$. The argument used in Theorem 4 shows that P is non-averaging and that $P(x) \gg x^{\alpha/(1+\beta)^2}/(\log x)^2$.

6. Non-dividing sets. Denote by $g(n)$ the size of a maximal non-dividing subset of $\{1, 2, \cdots, n\}$. Straus [11] proved that if $\{a_1, a_2, \cdots, a_k\}$ is a non-averaging subset of $\{1, 2, \cdots, [n/k]\}$, then $\{n - a_1, n - a_2, \cdots, n - a_k\}$ is a non-dividing set. Thus if $k \leqq f([n/k])$ we have $g(n) \geqq k$. It follows that the following theorem holds:

THEOREM 6. $g(n) \gg n^{\alpha/(1+\alpha)}$.

Our next result is the analogue of Theorem 3 for non-dividing sets.

THEOREM 7. *Let $P = \{p: p \leqq n,\ p \text{ prime}\}$. Then P contains a non-dividing set of size at least $cn^{\alpha/(1+\alpha)}/(\log n)^2$.*

Proof. By Lemma 1 it is possible to partition $\{1, 2, \cdots, [n^{1/(1+\alpha)}]\}$ into fewer than $n^{(1-\alpha)/(1+\alpha)} \log n$ non-averaging sets $A_1, A_2, \cdots, A_k$. By the result of Straus, the sets $B_i = \{n - a_j: a_j \in A_i\}$ are non-dividing. By Lemma 2, the set $\{n - [n^{1/(1+\alpha)}], \cdots, n\}$ contains at least $cn^{1(1+\alpha)}/\log n$ primes. Thus one of the B's must contain at least $cn^{\alpha/(1+\alpha)}/(\log n)^2$ primes, as required.

A simple argument shows that there exist no infinite non-dividing sets of integers. Call a set A quasi-non-dividing if no member of A divides the sum of two or more smaller members of A. We investigate infinite quasi-non-dividing sets. Our first result is the following theorem:

THEOREM 8. *There exists an infinite quasi-non-dividing set A whose counting function satisfies $A(x) \gg x^{1/6}$.*

Proof. It is a simple matter to verify that if $n > 1$ is a positive integer and k is determined by $\binom{k-1}{2} < n \leqq \binom{k}{2}$ then $\{n - k + 1, \cdots, n - 1, n\}$ is a quasi-non-dividing set. Thus, if $h(n)$ denotes the size of a maximal quasi-non-dividing subset of $\{1, 2, \cdots, n\}$, then $h(n) \geqq cn^{1/2}$. Also it is an easy consequence of a result of Szemeredi [12] that $h(n) \leqq cn^{1/2}$.

Let $m > 1$ be a positive integer and let A_1 be a maximal quasi-non-dividing subset of $\{1, 2, \cdots, m\}$. Suppose we have defined sets $A_1, A_2, \cdots, A_r$. Let $t_r = \sum_{a \in \cup A_i} a$, and let p_r be the least prime exceeding t_r. Let A^*_{r+1} be a maximal quasi-non-dividing subset of $\{1, 2, \cdots, t_r\}$ and let $A_{r+1} = \{p_r a: a \in A^*_{r+1}\}$. Put $A = \bigcup_{r=1}^{\infty} A_k$. It is now a simple matter to verify that A is quasi-non-dividing. Moreover, the observation made in the first paragraph together with the fact that, for large r, $p_r \sim t_r$, enables one to show in a straightforward way that $A(x) \gg x^{1/6}$. We suppress these details.

Our final theorem establishes the existence of a reasonably dense quasi-non-dividing set of primes.

THEOREM 9. *There exists an infinite quasi-non-dividing set P of primes whose counting function satisfies* $P(x) \gg x^{\alpha^2/8(1+\alpha)^2}/(\log x)^2$.

Proof. Let m be a large positive integer and let $n_1 = m$. For $k \geqq 2$, let $n_k = [n_{k-1}^{4(1+1/\alpha)}]$. Let P_1 be a maximal non-dividing set of primes in $\{1, 2, \cdots, n_1\}$. Suppose that we have defined $P_1, P_2, \cdots, P_{k-1}$. By Lemma 1, it is possible to partition $\{1, 2, \cdots, [n_k^{1/(1+\alpha)}]\}$ into $s_k \ll n_k^{(1-\alpha)/(1+\alpha)} \log n_k$ non-averaging sets $A_1^{(k)}, \cdots, A_{s_k}^{(k)}$. The sets $B_j^{(k)} = \{n_k - a_i : a_i \in A_j^{(k)}\}$ are then non-dividing sets which cover $\{n_k - [n_k^{1/(1+\alpha)}], \cdots, n_k - 2, n_k - 1\} = I_k$. The primes in I_k, of which, by Lemma 2, there are $t_k \gg n_k^{1/(1+\alpha)}/\log n_k$ in number, are distributed over the $\phi(n_{k-1}^2)$ reduced residue classes mod n_{k-1}^2. Thus one of the B's must contain a set P_k of primes of size at least $[t_k/s_k\phi(n_{k-1}^2)] \gg n_k^{(\alpha/(\alpha+1))/2}/(\log n_k)^2$, and which all belong to the some residue class modulo n_{k-1}^2. Let $P = \bigcup_{k=1}^{\infty} P_k$.

We now show that P is quasi-non-dividing. Suppose there are primes $p_0, p_1, \cdots, p_t \in P$ such that $p_0 < p_1 < \cdots < p_t$ and $p_0 + p_1 + \cdots + p_{t-1} = mp_t$. Let $p_t \in P_k$. If $p_{t-1} \notin P_k$ we get $p_0 + p_1 + \cdots + p_{t-1} < tp_{t-1} < tn_{k-1} \leqq n_{k-1}^2 < n_k - [n_k^{1/(1+\alpha)}] \leqq p_t$, which is a contradiction. Thus $p_{t-1} \in P_k$. Determine r, $1 \leqq r \leqq t-1$ such that $p_r, p_{r+1}, \cdots, p_{t-1} \in P_k$ and $p_0, p_1, \cdots, p_{r-1} \notin P_k$. It then follows easily that $m = t - r$ and hence that

$$p_0 + p_1 + \cdots + p_{r-1} = (t-r)p_t - (p_r + p_{r+1} + \cdots + p_{t-1}) . \tag{15}$$

Since $p_r, p_{r+1}, \cdots, p_t$ all belong to the same residue class modulo n_{k-1}^2, the right side of (15) is divisible by n_{k-1}^2. However, $p_0 + \cdots + p_{r-1} < rn_{k-1} < n_{k-1}^2$ and this is a contradiction. Thus P is quasi-non-dividing. Furthermore, one may easily check that $P(x) \gg x^{\alpha^2/8(1+\alpha)^2}/(\log x)^2$. The details we suppress. This completes the proof of the theorem.

REFERENCES

1. H. L. Abbott, *On a conjecture of Erdös and Straus on non-averaging sets of integers,* Proceedings of the Fifth British Combinatorial Conference, (to appear).

2. H. L. Abbott, A. C. Liu and J. Riddell, *On sets of integers not containing arithmetic progressions of prescribed length,* J. Australian Math. Soc., **28** (1974), 188-193.

3. F. A. Behrend, *On sets of integers which contain no three terms in arithmetical progression,* Proc. Nat. Acad. Sci. U.S.A., **32** (1946), 331-332.

4. P. Erdös and E. G. Straus, *Non-averaging sets II,* Combinatorial Theory and its Applications, Vol. II, Colloquia Mathematica Societatis Janos Bolyai, **4** (1970), 405-411.
5. M. N. Huxley, *The Distribution of Prime Numbers,* Oxford Mathematical Monographs, Part IV.
6. G. G. Lorentz, *On a problem in additive number theory,* Proc. Amer. Math. Soc., **5** (1954), 838-841.
7. L. Moser, *On a theorem of van der Waerden,* Canad. Math. Bull., **3** (1960), 23-25.
8. J. Riddell, *On sets of integers containing no l terms in arithmetic progression,* Neiuw. Arch. voor Wisk., (3), **17** (1969), 204-209.
9. J. Spencer, *Puncture sets,* J. Combinatorial Theory, **A17** (1974), 329-336.
10. S. K. Stein, *Two combinatorial covering theorems,* J. Combinatorial Theory, **A16** (1974), 391-397.
11. E. G. Straus, *Non-averaging sets,* Proceedings of Symposia in Pure Mathematics, Vol. XIX, American Mathematical Society, (1971), 215-222.
12. E. Szemerédi, *On a conjecture of Erdös and Heilbronn,* Acta Arithmetica, **17** (1970), 227-229.

Received April 14, 1976.

UNIVERSITY OF ALBERTA
EDMONTON, ALBERTA
CANADA