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A MULTIPLE SERIES TRANSFORMATION OF THE VERY WELL POISED ${}_{2k+4}\Psi_{2k+4}$

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A multiple series generalization of the q-analog of Whipple's theorem is derived for ${}_{2k+4}\Psi_{2k+4}$ by applying recent analytical techniques of Askey and Ismail to Andrews' multiple series transformation of a well poised ${}_{2k+4}\Phi_{2k+3}$.

1. Introduction. The bilateral basic hypergeometric function is

$$ {}_m\Psi_n\left[\begin{matrix} a_1, a_2, \cdots, a_m; q, t \\ b_1, b_2, \cdots, b_n \end{matrix}\right] = \sum_{j=-\infty}^{\infty} \frac{(a_1)_j(a_2)_j \cdots (a_m)_j t^j}{(b_1)_j(b_2)_j \cdots (b_n)_j} , $$

where

$$ (a; q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n) , \qquad |q| < 1 \tag{1.1} $$

and

$$ (a)_j = (a; q)_j = (a)_\infty (aq^j)_\infty^{-1} , \tag{1.2} $$

or

$$ (a)_{-n} = \left(1 - \frac{a}{q^n}\right)^{-1} \cdots \left(1 - \frac{a}{q}\right)^{-1} = (-a)^{-n} q^{n(n+1)/2} (q/a)_n^{-1} . \tag{1.3} $$

Thus

$$ \begin{aligned} &{}_m\Psi_n\left[\begin{matrix} a_1, \cdots, a_m; q, t \\ b_1, \cdots, b_n \end{matrix}\right] \\ &= \sum_{j=0}^{\infty} \frac{(a_1)_j \cdots (a_m)_j t^j}{(b_1)_j \cdots (b_n)_j} + \sum_{j=1}^{\infty} \frac{\left(\frac{q}{b_1}\right)_j \cdots \left(\frac{q}{b_n}\right)_j}{\left(\frac{q}{a_1}\right)_j \cdots \left(\frac{q}{a_m}\right)_j} \left(\frac{b_1 \cdots b_n}{a_1 \cdots a_m t}\right)^j (-1)^{j(m-n)} \\ &\quad \times q^{j(j+1)(m-n)/2} . \end{aligned} \tag{1.4} $$

Hence we see that to insure convergence we must require $n \leqq m$. Also $b_h \neq q^{-N}$, $a_h \neq q^{N+1}$ for any nonnegative integer N. Finally if $n < m$, we need only in addition require $|t| < 1$; however, if $n = m$, we need also

$$ \left|\frac{b_1 \cdots b_n}{a_1 \cdots a_m}\right| < |t| < 1 . $$

Bailey's sum of the very well poised ${}_6\Psi_6$ is

$$(1.5)\qquad {}_6\Psi_6\left[\begin{matrix} q\sqrt{a},\ -q\sqrt{a},\ b,\ c,\ d,\ e \\ \sqrt{a},\ -\sqrt{a},\ \frac{aq}{b},\ \frac{aq}{c},\ \frac{aq}{d},\ \frac{aq}{e};\ q,\ \frac{a^2q}{bcde}\end{matrix}\right]$$

$$= \Pi\left[\begin{matrix} aq,\ \frac{aq}{bc},\ \frac{aq}{bd},\ \frac{aq}{be},\ \frac{aq}{cd},\ \frac{aq}{ce},\ \frac{aq}{de},\ q,\ \frac{q}{a} \\ \frac{q}{b},\ \frac{q}{c},\ \frac{q}{d},\ \frac{q}{e},\ \frac{aq}{b},\ \frac{aq}{c},\ \frac{aq}{d},\ \frac{aq}{e},\ \frac{a^2q}{bcde}\end{matrix}\right]$$

where

$$\Pi\left[\begin{matrix}\alpha_1,\ \cdots,\ \alpha_r \\ \beta_1,\ \cdots,\ \beta_s\end{matrix}\right] = \frac{(\alpha_1)_\infty \cdots (\alpha_r)_\infty}{(\alpha_1)_\infty \cdots (\alpha_s)_\infty}\,.$$

The identity (1.5) is probably the most general summation identity known for bilateral basic hypergeometric series. Andrews [1, §3] deduces many important diverse results in number theory from (1.5). Other q-series identities that follow from the ${}_6\Psi_6$ summation are given by Slater [11].

There are five known proofs of (1.5). Bailey's original proof [6, §4] relies on ingenious combinations of various transformation formulas he had developed for ordinary and basic hypergeometric series. Slater [12] uses an analog of the Barnes-type integral, Lakin [12] combines a q-difference equation technique with Carlson's theorem on entire functions, and Andrews [1, §3] provides a more elementary proof which utilizes q-difference equations together with the uniqueness of a Laurent series expansion about the origin. In obtaining (1.5) directly from the ${}_6\Phi_5$ summation formula, Askey and Ismail [5] have recently given the most elementary proof of all.

The ${}_6\Phi_5$ summation formula is:

$$(1.6)\qquad {}_6\Phi_5\left[\begin{matrix} a,\ q\sqrt{a},\ -q\sqrt{a},\ b,\ c,\ d \\ \sqrt{a},\ -\sqrt{a},\ \frac{aq}{b},\ \frac{aq}{c},\ \frac{aq}{d};\ q,\ \frac{aq}{bcd}\end{matrix}\right]$$

$$= \Pi\left[\begin{matrix} aq,\ \frac{aq}{bc},\ \frac{aq}{bd},\ \frac{aq}{cd} \\ \frac{aq}{b},\ \frac{aq}{c},\ \frac{aq}{d},\ \frac{aq}{bcd}\end{matrix}\right]$$

where

$${}_m\Phi_n\left[\begin{matrix}a_1,\ \cdots,\ a_m \\ b_1,\ \cdots,\ b_n\end{matrix};\ q,\ t\right] = \sum_{j=0}^{\infty}\frac{(a_1)_j(a_2)_j \cdots (a_m)_j t^j}{(b_1)_j(b_2)_j \cdots (b_n)_j(q)_j}\,.$$

To the end of proving ${}_6\Psi_6$ Askey and Ismail first make the key observation that both sides of (1.5) are analytic functions of $z = a/e$ is a disk of positive radius about the origin. They then show that the ${}_6\Phi_5$ summation formula in (1.6) is equivalent to the statement that these two analytic functions agree when $z = q^m$, $m = 0, 1, 2, \cdots$. Thus they must be identically equal since 0 is an *interior* point of the domain of analyticity. The identity in (1.5) follows.

This proof is motivated by the fact that when $e = q$ the series (1.5) becomes the series in (1.6) since $1/(q; q)_n = 0$ when $n = -1, -2, \cdots$. Ismail [8] has used the above analytical method to extend another result for a power series to a Laurent series.

In this paper we make use of Askey and Ismail's proof of (1.5) and Andrews' [2] transformation of a terminating well poised ${}_{2k+4}\Phi_{2k+3}$ to prove:

THEOREM 1.7.

$$
{}_{2k+4}\Psi_{2k+4}
\times\left[\begin{matrix} q\sqrt{a}, -q\sqrt{a}, b_1, c_1, \cdots, b_{k-1}, c_{k-1}, b_k, c_k, e, f \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{b_1}, \frac{aq}{c_1}, \cdots, \frac{aq}{b_{k-1}}, \frac{aq}{c_{k-1}}, \frac{aq}{b_k}, \frac{aq}{c_k}, \frac{aq}{e}, \frac{aq}{f} \end{matrix}; q, \frac{a^{k+1}q^k}{b_1\cdots b_k c_1\cdots c_k ef}\right]
$$

$$
(1.8)\qquad = \Pi\left[\begin{matrix} aq, \frac{aq}{b_kc_k}, \frac{aq}{b_ke}, \frac{aq}{b_kf}, \frac{aq}{c_ke}, \frac{aq}{c_kf}, \frac{aq}{ef}, q, \frac{q}{a} \\ \frac{q}{b_k}, \frac{q}{c_k}, \frac{q}{e}, \frac{q}{f}, \frac{aq}{b_k}, \frac{aq}{c_k}, \frac{aq}{e}, \frac{aq}{f}, \frac{a^2q}{b_kc_kef} \end{matrix}\right]
$$

$$
(1.9)\qquad \times\Pi\left[\begin{matrix} \frac{b_1}{a}, \cdots, \frac{b_{k-1}}{a} \\ \frac{b_1}{f}, \cdots, \frac{b_{k-1}}{f} \end{matrix}\right] \times \Pi\left[\begin{matrix} b_1, \cdots, b_{k-1} \\ \frac{fb_1}{a}, \cdots, \frac{fb_{k-1}}{a} \end{matrix}\right]
$$

$$
(1.10)\qquad \times\Pi\left[\begin{matrix} \frac{qf}{c_1}, \cdots, \frac{qf}{c_{k-1}} \\ \frac{aq}{c_1}, \cdots, \frac{aq}{c_{k-1}} \end{matrix}\right] \times \Pi\left[\begin{matrix} \frac{qa}{fc_1}, \cdots, \frac{qa}{fc_{k-1}} \\ \frac{q}{c_1}, \cdots, \frac{q}{c_{k-1}} \end{matrix}\right]
$$

$$
(1.11)\qquad \times \sum_{m_1,\cdots,m_{k-1}\geq 0} \frac{(qa/b_1c_1)_{m_1}(aq/b_2c_2)_{m_2}\cdots(aq/b_{k-1}c_{k-1})_{m_{k-1}}}{(q)_{m_1}(q)_{m_2}\cdots(q)_{m_{k-1}}}
$$

$$
\times\frac{(b_2f/a)_{m_1}(c_2f/a)_{m_1}(b_3f/a)_{m_1+m_2}(c_3f/a)_{m_1+m_2}}{(qf/b_1)_{m_1}(qf/c_1)_{m_1}(qf/b_2)_{m_1+m_2}(qf/c_2)_{m_1+m_2}}\cdots
$$

$$
\times\frac{(b_kf/a)_{m_1+\cdots+m_{k-1}}\cdot(c_kf/a)_{m_1+\cdots+m_{k-1}}\cdot(ef/a)_{m_1+\cdots+m_{k-1}}}{(qf/b_{k-1})_{m_1+\cdots+m_{k-1}}\cdot(qf/c_{k-1})_{m_1+\cdots+m_{k-1}}\cdot(b_kc_kef/a^2)_{m_1+\cdots+m_{k-1}}}
$$

$$
\times\left(\frac{aq}{b_2c_2}\right)^{m_1}\left(\frac{aq}{b_3c_3}\right)^{m_1+m_2}\cdots\left(\frac{aq}{b_{k-1}c_{k-1}}\right)^{m_1+m_2+\cdots+m_{k-2}}\cdot q^{m_1+\cdots+m_{k-1}},
$$

where

$$\left(\frac{aq}{b_i c_i}\right) = q^{-N_i}, \tag{1.12}$$

$1 \leqq i \leqq k-1$, *N_i a fixed positive integer.*

Note that for $k = 1$, Theorem 1.7 reduces to Bailey's sum of the very well poised ${}_6\Psi_6$ given in (1.5).

The condition given by (1.12) for the parameters b_i and c_i only applies when $k \geqq 2$. In these cases the series on the left hand side of (1.8) does not terminate while the series on the right hand side in (1.11) does. When the series in (1.11) does not terminate there are additional terms on the right hand side. For $k = 2$ M. Jackson [9] has proven the nonterminating form of Theorem 1.7.

Andrews [1, §3] has introduced the Laurent series $K_{\lambda,k,i}(a_0, \cdots, a_\lambda; z; q) = K_{\lambda,k,i}((a); z; q)$ which generalizes bilateral basic hypergeometric series. As summarized in [4, chapter 7] the function $K_{\lambda,k,i}((a); z; q)$ plays a key role in the proof of numerous partition identities of Rogers-Ramanujan type.

It turns out that the sum on the left hand side of (1.8) is $K_{2k+1,k,1}((a); z; q)$. This suggests it may be possible to discover a multiple series transformation for $K_{\lambda,k,i}((a); z; q)$ that is similar to Theorem 1.7. Furthermore, $K_{\lambda,k,i}((a); z; q)$ should be regarded as a function of z/a_λ rather than z.

2. Proof of Theorem 1.7.

In order to prove Theorem 1.7 we need the following result of Andrews [2, §2] which generalizes the terminating case of the ${}_6\Phi_5$ summation in (1.6), as well as Watson's [13] q-analog of Whipple's theorem.

THEOREM 2.1 (*Andrews*). *For $k \geqq 1$ and N a nonnegative integer*:

$$\begin{aligned}
&{}_{2k+4}\Phi_{2k+3}\left[\begin{matrix} a,\ q\sqrt{a},\ -q\sqrt{a},\ b_1, c_1, \cdots, b_{k-1}, c_{k-1}, b_k, c_k, q^{-N} \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{b_1}, \frac{aq}{c_1}, \cdots, \frac{aq}{b_{k-1}}, \frac{aq}{c_{k-1}}, \frac{aq}{b_k}, \frac{aq}{c_k}, aq^{N+1} \end{matrix}; q, \frac{a^k q^{k+N}}{b_1 \cdots b_k c_1 \cdots c_k}\right] \\
&= \frac{(aq)_N (aq/b_k c_k)_N}{(aq/b_k)_N (aq/c_k)_N} \sum_{m_1, \cdots, m_{k-1} \geqq 0} \frac{(aq/b_1 c_1)_{m_1} \cdots (aq/b_{k-1}c_{k-1})_{m_{k-1}}}{(q)_{m_1}(q)_{m_2} \cdots (q)_{m_{k-1}}} \\
&\times \frac{(b_2)_{m_1}(c_2)_{m_1}(b_3)_{m_1+m_2}(c_3)_{m_1+m_2}}{(aq/b_1)_{m_1}(aq/c_1)_{m_1}(aq/b_2)_{m_1+m_2}(aq/c_2)_{m_1+m_2}} \cdots \frac{(b_k)_{m_1+\cdots+m_{k-1}}}{(aq/b_{k-1})_{m_1+\cdots+m_{k-1}}} \\
&\times \frac{(c_k)_{m_1+\cdots+m_{k-1}}}{(aq/c_{k-1})_{m_1+\cdots+m_{k-1}}} \cdot \frac{(q^{-N})_{m_1+\cdots+m_{k-1}}}{(b_k c_k q^{-N}/a)_{m_1+\cdots+m_{k-1}}}
\end{aligned} \tag{2.2}$$

$$\times\left(\frac{aq}{b_2c_2}\right)^{m_1}\left(\frac{aq}{b_3c_3}\right)^{m_1+m_2}\cdots\left(\frac{aq}{b_{k-1}c_{k-1}}\right)^{m_1+\cdots+m_{k-2}}\cdot q^{m_1+m_2\cdots+m_{k-1}}\,.$$

Observe that if we set $e=q^{-N}$ in (1.14) then,

$$\frac{(aq)_N(aq/b_kc_k)_N}{(aq/b_k)_N(aq/c_k)_N}=\Pi\begin{bmatrix} aq,\ \dfrac{aq}{c_ke},\ \dfrac{aq}{b_ke},\ \dfrac{aq}{b_kc_k} \\ \dfrac{aq}{b_k},\ \dfrac{aq}{c_k},\ \dfrac{aq}{e},\ \dfrac{aq}{b_kc_ke}\end{bmatrix}. \tag{2.3}$$

Noting (2.3) it is not hard to see that both sides of (2.2) are analytic functions of $z=1/e$ in a disk of positive radius about the origin provided that

$$\frac{aq}{b_ic_i}=q^{-N_i}\,,\qquad 1\leqq i\leqq k-1\,. \tag{2.4}$$

Condition (2.4) terminates the sum on the right hand side of (2.2) but not the sum of the left. If the right hand sum did not terminate then 0 would not be an interior point of the domain of analyticity of the above two functions.

The analytic functions representing both sides of (2.2) are equal when $z=q^N$, $N=1,2,\cdots$. Hence, since 0 is an interior point of the domain of analyticity, both sides of (2.2) are equal when q^{-N} is replaced by the parameter e. That is, we have the following extension of Theorem 2.1 in which the left hand side converges, but does not terminate, and the sum on the right hand side terminates:

THEOREM 2.5. *For* $k\geqq 1$ *and*

$$aq/b_ic_i=q^{-N_i}\,,\qquad 1\leqq i\leqq k-1\,,$$

N_i *a positive integer,*

$$\begin{aligned}&{}_{2k+4}\Phi_{2k+3}\begin{bmatrix} a,\ q\sqrt{a},\ -q\sqrt{a},\ b_1,\ c_1,\ \cdots,\ b_k,\ c_k,\ e \\ \sqrt{a},\ -\sqrt{a},\ \dfrac{aq}{b_1},\ \dfrac{aq}{c_1},\ \cdots,\ \dfrac{aq}{b_k},\ \dfrac{aq}{c_k},\ \dfrac{aq}{e}\end{bmatrix};q,\frac{a^kq^k}{b_1\cdots b_kc_1\cdots c_ke}\Bigg] \\ &=\Pi\begin{bmatrix} aq,\ \dfrac{aq}{c_ke},\ \dfrac{aq}{b_ke},\ \dfrac{aq}{b_kc_k} \\ \dfrac{aq}{b_k},\ \dfrac{aq}{c_k},\ \dfrac{aq}{e},\ \dfrac{aq}{b_kc_ke}\end{bmatrix} \\ &\times\sum_{m_1,\cdots,m_{k-1}\geqq 0}\frac{(aq/b_1c_1)_{m_1}(aq/b_2c_2)_{m_2}\cdots(aq/b_{k-1}c_{k-1})_{m_{k-1}}}{(q)_{m_1}(q)_{m_2}\cdots(q)_{m_{k-1}}} \\ &\times\frac{(b_2)_{m_1}(c_2)_{m_1}(b_3)_{m_1+m_2}(c_3)_{m_1+m_2}\cdots(b_k)_{m_1+\cdots+m_{k-1}}}{(aq/b_1)_{m_1}(aq/c_1)_{m_1}(aq/b_2)_{m_1+m_2}(aq/c_2)_{m_1+m_2}\cdots(aq/b_{k-1})_{m_1+\cdots+m_{k-}}}\end{aligned} \tag{2.6}$$

$$\times \frac{(c_k)_{m_1+\cdots+m_{k-1}}}{(aq/c_{k-1})_{m_1+\cdots+m_{k-1}}} \cdot \frac{(e)_{m_1+\cdots+m_{k-1}}}{(b_kc_ke/a)_{m_1+\cdots+m_{k-1}}}$$
$$\times \left(\frac{aq}{b_2c_2}\right)^{m_1}\left(\frac{aq}{b_3c_3}\right)^{m_1+m_2}\cdots\left(\frac{aq}{b_{k-1}c_{k-1}}\right)^{m_1+\cdots+m_{k-2}} \cdot q^{m_1+\cdots+m_{k-1}} .$$

Making use of Theorem 2.5 and Askey and Ismail's proof of (1.5) we now prove Theorem 1.7.

Just as in the proof of Theorem 2.5 it is not hard to see that both sides of the identity in Theorem 1.7 are analytic functions of $z = a/f$ in a disk of poitive radius about the origin provided that (2.4) holds. To complete the proof of Theorem 1.7 we show that these two analytic functions are equal when $z = q^m$, $m = 1, 2, \cdots$. This will be accomplished once we show that Theorem 1.7 is true with $f = aq^{-m}$. (The key idea here is the "right" choice of the variable z. Earlier attempts at this type of proof of (1.5) were made with the variable $z = a$.)

We are now ready to use Theorem 2.5 to show that Theorem 1.7 is true with $f = aq^{-m}$.

Observing that

$$\frac{(q\sqrt{a})_n(-q\sqrt{a})_n}{(\sqrt{a})_n(-\sqrt{a})_n} = (1 - aq^{2n})/(1 - a) ,$$

we see that the left hand side of (1.8) with $f = aq^{-m}$ is:

$$= \sum_{n=-m}^{\infty} \frac{(1 - aq^{2n})}{(1 - a)} \cdot \frac{(b_1)_n}{\left(\frac{aq}{b_1}\right)_n} \cdot \frac{(c_1)_n}{\left(\frac{aq}{c_1}\right)_n} \cdots \frac{(b_k)_n}{\left(\frac{aq}{b_k}\right)_n}$$
$$\times \frac{(c_k)_n}{\left(\frac{aq}{c_k}\right)_n} \cdot \frac{(e)_n}{\left(\frac{aq}{e}\right)_n} \cdot \frac{(aq^{-m})_n}{(q^{m+1})_n}\left(\frac{a^kq^{k+m}}{b_1\cdots b_kc_1\cdots c_ke}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{(1 - aq^{2n-2m})}{(1 - a)} \cdot \frac{(b_1)_{n-m}}{\left(\frac{aq}{b_1}\right)_{n-m}} \cdot \frac{(c_1)_{n-m}}{\left(\frac{aq}{c_1}\right)_{n-m}} \cdots \frac{(b_k)_{n-m}}{\left(\frac{aq}{b_k}\right)_{n-m}} \cdot \frac{(c_k)_{n-m}}{\left(\frac{aq}{c_k}\right)_{n-m}}$$

$$\times \frac{(e)_{n-m}}{\left(\frac{aq}{e}\right)_{n-m}} \cdot \frac{(aq^{-m})_{n-m}}{(q^{m+1})_{n-m}}\left(\frac{a^kq^{k+m}}{b_1\cdots b_kc_1\cdots c_ke}\right)^{n-m} . \tag{2.8}$$

Note that (1.2) and (1.3) imply that

$$(a; q)_{n-m} = (aq)_{-m}(aq^{-m}; q)_n . \tag{2.9}$$

For example,

$$(aq^{-m})_{n-m} = (aq^{-m})_{-m}(aq^{-2m})_n \tag{2.10}$$

and

$$(q^{m+1})_{n-m} = (q^{m+1})_{-m}(q; q)_n \,. \tag{2.11}$$

Using (2.9), (2.10), and (2.11) we rewrite (2.8) as:

$$\frac{(b_1)_{-m}}{\left(\frac{aq}{b_1}\right)_{-m}} \cdot \frac{(c_1)_{-m}}{\left(\frac{aq}{c_1}\right)_{-m}} \cdots \frac{(b_k)_{-m}}{\left(\frac{aq}{b_k}\right)_{-m}} \cdot \frac{(c_k)_{-m}}{\left(\frac{aq}{c_k}\right)_{-m}} \cdot \frac{(e)_{-m}}{\left(\frac{aq}{e}\right)_{-m}} \cdot \frac{(aq^{-m})_{-m}}{(q^{m+1})_{-m}}$$

$$\times \frac{(1-aq^{-2m})}{(1-a)}\left(\frac{b_1 \cdots b_k c_1 \cdots c_k e}{a^k q^{k+m}}\right)^m \tag{2.12}$$

$$\times \sum_{n=0}^{\infty} \frac{(1-aq^{-2m}q^{2n})}{(1-aq^{-2m})} \cdot \frac{(b_1/q^m)_n}{\left(\frac{aq}{b_1 q^m}\right)_n} \cdot \frac{(c_1/q^m)_n}{\left(\frac{aq}{c_1 q^m}\right)_n} \cdots \frac{(b_k/q^m)_n}{\left(\frac{aq}{b_k q^m}\right)_n} \cdot \frac{(c_k/q_m)_n}{\left(\frac{aq}{c_k q^m}\right)_n}$$

$$\times \frac{(e/q^m)_n}{\left(\frac{aq}{eq^m}\right)_n} \cdot \frac{(aq^{-2m})_n}{(q)_n}\left(\frac{a^k q^{k+m}}{b_1 \cdots b_k c_1 \cdots c_k c}\right)^n . \tag{2.13}$$

Note that the sum in (2.13) is the sum on the left hand side of (2.6) with a replaced by aq^{-2m}, b_i by $b_i q^{-m}$, c_i by $c_i q^{-m}$, and e by eq^{-m}. In addition we have,

$$\frac{aq^{-2m}q}{b_i q^{-m} c_i q^{-m}} = \frac{aq}{b_i c_i} \cdot \frac{q^{-2m}}{q^{-2m}} = \frac{aq}{b_i c_i} = q^{-N_i} .$$

Thus we can replace the sum in (2.13) by the right hand side of (2.6) with a replaced by aq^{-2m}, b_i by $b_i q^{-m}$, c_i by $c_i q^{-m}$, and e by eq^{-m}. When this is done the product of the products in (2.12) and the sum in (2.13) become:

$$\frac{(b_1)_{-m}}{\left(\frac{aq}{b_1}\right)_{-m}} \cdot \frac{(c_1)_{-m}}{\left(\frac{aq}{c_1}\right)_{-m}} \cdots \frac{(b_k)_{-m}}{\left(\frac{aq}{b_k}\right)_{-m}} \cdot \frac{(c_k)_{-m}}{\left(\frac{aq}{c_k}\right)_{-m}} \cdot \frac{(e)_{-m}}{\left(\frac{aq}{e}\right)_{-m}} \cdot \frac{(aq^{-m})_{-m}}{(q^{m+1})_{-m}} \tag{2.14}$$

$$\times \left(\frac{b_1 \cdots b_k c_1 \cdots c_k e}{a^k q^{k+m}}\right)^m \tag{2.15}$$

$$\times \frac{(1-aq^{-2m})}{(1-a)}(aq/q^{2m})_\infty \tag{2.16}$$

$$\times \left(\frac{aq}{c_k e}\right)_\infty \left(\frac{aq}{b_k e}\right)_\infty \left(\frac{aq}{b_k c_k}\right)_\infty \Big/ \left(\frac{aqq^m}{b_k c_k e}\right)_\infty \tag{2.17}$$

$$\times \left(\frac{aq}{b_k q^m}\right)_\infty^{-1} \left(\frac{aq}{c_k q^m}\right)_\infty^{-1} \left(\frac{aq}{eq^m}\right)_\infty^{-1} \tag{2.18}$$

$$\times \sum_{m_1,\cdots,m_{k-1} \geq 0} \frac{(aq/b_1c_1)_{m_1}(aq/b_2c_2)_{m_2} \cdots (aq/b_{k-1}c_{k-1})_{m_{k-1}}}{(q)_{m_1}(q)_{m_2} \cdots (q)_{m_{k-1}}} \tag{2.19}$$

$$\times \frac{(b_2/q^m)_{m_1}}{(aq/b_1q^m)_{m_1}} \cdot \frac{(c_2/q^m)_{m_1}}{(aq/c_1q^m)_{m_1}} \cdot \frac{(b_3/q^m)_{m_1+m_2}}{(aq/b_2q^m)_{m_1+m_2}} \cdot \frac{(c_3/q^m)_{m_1+m_2}}{(aq/c_2q^m)_{m_1+m_2}}$$

$$\cdots \frac{(b_k/q^m)_{m_1+\cdots+m_{k-1}}}{(aq/b_{k-1}q^m)_{m_1+\cdots+m_{k-1}}} \cdot \frac{(c_k/q^m)_{m_1+\cdots+m_{k-1}}}{(aq/c_{k-1}q^m)_{m_1+\cdots+m_{k-1}}}$$

$$\times \frac{(e/q^m)_{m_1+\cdots+m_{k-1}}}{(b_kc_ke/aq^m)_{m_1+\cdots+m_{k-1}}} \cdot q^{m_1+\cdots+m_{k-1}}$$

$$\times \left(\frac{aq}{b_2c_2}\right)^{m_1} \cdot \left(\frac{aq}{b_3c_3}\right)^{m_1+m_2} \cdots \left(\frac{aq}{b_{k-1}c_{k-1}}\right)^{m_1+\cdots+m_{k-2}} .$$

It is not hard to see that the sum in (2.19) is the sum in (1.11) with $f = aq^{-m}$. All that is left in the proof of Theorem 1.7 is to show that the product of the terms in (2.14) through (2.18) is nothing but the products in (1.8), (1.9), and (1.10) with $f = aq^{-m}$.

When the products in (2.14) are simplified using (1.3) and then combined with (2.15) we obtain:

(2.20) $$\frac{(b_1/a)_m(b_2/a)_m \cdots (b_{k-1}/a)_m}{(q/c_1)_m(q/c_2)_m \cdots (q/c_{k-1})_m}$$

(2.21) $$\times \frac{(c_1/a)_m(c_2/a)_m \cdots (c_{k-1}/a)_m}{(q/b_1)_m(q/b_2)_m \cdots (q/b_{k-1})_m} \cdot \frac{1}{(b_1 \cdots b_{k-1})^m} \cdot \frac{1}{(c_1 \cdots c_{k-1})^m}$$

(2.22) $$\times \frac{(b_k/a)_m(c_k/a)_m(e/a)_m a^{mk} q^{m^2+(k+2)m}}{(b_kc_ke)^m}$$

(2.23) $$\times (q^{m+1}/a)_m^{-1} = (-1)^m a^m q^{-1/2m(3m+1)} \cdot \left(1-\frac{a}{q^{2m}}\right)^{-1} \cdots \left(1-\frac{a}{q^{m+1}}\right)^{-1}$$

(2.24) $$\times (1/q^m)_m = (-1)^m q^{-1/2m(m+1)}$$

(2.25) $$\times (q)_m$$

(2.26) $$\times (q/b_k)_m^{-1}(q/c_k)_m^{-1}(q/e)_m^{-1} .$$

Observe that

$$b_i^m(q/b_i)_m = (-1)^m q^{1/2m(m+1)} \cdot (b_i/q^m)_m$$

and

$$c_i^{-m}(c_i/a)_m = (-1)^m a^{-m} q^{1/2m(m-1)} \cdot (aq/c_iq^m)_m .$$

Thus (2.21) can be rewritten as:

(2.27) $$\frac{(aq/c_1q^m)_m \cdots (aq/c_{k-1}q^m)_m}{(b_1/q^m)_m \cdots (b_{k-1}/q^m)_m}$$

(2.28) $$\times (aq)^{-(k-1)m} .$$

After some algebraic simplification we find that (2.16) can be rewritten as

$$(2.29)\qquad (aq)_\infty (q/a)_m$$

$$(2.30)\qquad \times (-1)^m a^m q^{-1/2m(m+1)} \cdot \left(1-\frac{a}{q^{2m}}\right)\cdots\left(1-\frac{a}{q^{m+1}}\right),$$

and that (2.18) can be rewritten as

$$(2.31)\qquad (aq/b_k)_\infty^{-1}(aq/c_k)_\infty^{-1}(aq/e)_\infty^{-1}$$

$$(2.32)\qquad \times \frac{(-1)^m (aq)^{-3m} q^{3/2m(m+1)} (b_k c_k e)^m}{(b_k/a)_m (c_k/a)_m (e/a)_m}.$$

The product of the terms in (2.17), (2.25), (2.26), (2.29) and (2.31) is:

$$(2.33)\qquad \frac{(aq)_\infty (aq/c_k e)_\infty (aq/b_k e)_\infty (aq/b_k c_k)_\infty}{(aq/b_k)_\infty (aq/c_k)_\infty (aq/e)_\infty (aq\cdot q^m/b_k c_k e)_\infty}$$

$$(2.34)\qquad \times \frac{(q/a)_m (q)_m}{(q/b_k)_m (q/c_k)_m (q/e)_m}.$$

The product of the terms in (2.20) and (2.7) is:

$$(2.35)\qquad \frac{\{(b_1/a)_m \cdots (b_{k-1}/a)_m\}\cdot\{(aq/c_1 q^m)_m \cdots (aq/c_{k-1}q^m)_m\}}{\{(b_1/q^m)_m \cdots (b_{k-1}/q^m)_m\}\cdot\{(q/c_1)_m \cdots (q/c_{k-1})_m\}}.$$

A routine algebraic simplification shows that the product of the terms in (2.22), (2.32), (2.23), (2.30), (2.24), and (2.28) is 1.

Using (1.2) to rewrite the products in (2.34) it is not hard to see that the product of terms in (2.33) and (2.34) is simply (1, 8) with $f = aq^{-m}$.

Similarly, (1.2) implies that (2.35) is the product of (1.9) and (1.10) with $f = aq^{-m}$.

This completes the proof of Theorem 1.7.

3. Applications. As an application of Theorem 1.7 we give a new proof of the following partition identity due to Andrews [3, §6].

THEOREM 3.1 (*Andrews*).

$$\frac{1}{(\alpha_1)_\infty(\alpha_2)_\infty\cdots(\alpha_{k-1})_\infty}$$

$$(3.2)\qquad = \sum_{m_1,\cdots,m_{k-1}\geq 0} \frac{q^{\sigma_1(m_1,\cdots,m_{k-1})^2-\sigma_2(m_1,\cdots,m_{k-1})-\sigma_1(m_1,\cdots,m_{k-1})}}{(q)_{m_1}(q)_{m_2}\cdots(q)_{m_{k-1}}}$$

$$\times \frac{\alpha_1^{m_1}\alpha_2^{m_2}\cdots\alpha_{k-1}^{m_{k-1}}}{(\alpha_1)_{m_1}(\alpha_2)_{m_1+m_2}\cdots(\alpha_{k-1})_{m_1+\cdots+m_{k-1}}},$$

where $\sigma_i(m_1, \cdots, m_{k-1})$ *is the* ith *elementary symmetric function of the* $m_1, \cdots, m_{k-1}$.

Recalling MacMahon's formula for $\pi_k(m)$ the number of plane partitions of m with k rows [10; p. 243], Andrews [3, §6] has observed that if α_i is replaced by q^i in (3.2) then,

$$\sum_{m=0}^{\infty}\pi_k(m)q^m = \sum_{m_1,\cdots,m_k\geq 0}\frac{q^{\sigma_1(m_1,\cdots,m_k)^2-\sigma_2(m_1,\cdots,m_k)+m_2+2m_3+\cdots+(k-1)m_k}}{(q)_{m_1}\cdots(q)_{m_k}(q)_{m_1}(q^2)_{m_1+m_2}\cdots(q^k)_{m_1+\cdots+m_k}}.$$

In order to prove (3.2) we choose the parameters in Theorem 1.7 as follows:

$$\begin{aligned} &b_i = \alpha_i q^{2N}\\ &c_i = aq/\alpha_i q^N\\ &f = aq^{-N}\\ &b_k,\ c_k,\ e \text{ are nonzero constants independent of } N. \end{aligned} \tag{3.3}$$

This choice of parameters satisfies the terminating condition in (1.12) since $aq/b_ic_i = q^{-N}$.

Observe that

$$\lim_{N\to\infty} b_i = 0 \quad\text{and}\quad \lim_{N\to\infty} c_i = \infty \tag{3.4}$$

since $|q| < 1$.

Now b_i and c_i contribute

$$\begin{aligned} &\frac{(b_i)_n(c_i)_n}{(aq/b_i)_n(aq/c_i)_n}\cdot\frac{1}{(b_ic_i)^n}\\ &\quad = \frac{1}{(aq)^n}\cdot\frac{(b_i)_n}{(aq/c_i)_n}\cdot\frac{(1-1/c_i)\cdots(q^{n-1}-1/c_i)}{(1-b_i/aq)\cdots(q^{n-1}-b_i/aq)} \end{aligned} \tag{3.5}$$

to the sum on the left hand side of (1.8).

If we fix n and let $N\to\infty$ then (3.4) implies that (3.5) tends to

$$\frac{1}{(aq)^n}\cdot\frac{q^{\binom{n}{2}}}{q^{\binom{n}{2}}} = \frac{1}{(aq)^n}. \tag{3.6}$$

An appeal to Tannery's Theorem [7; 49] immediately implies that the sum on the left hand side of (1.8) converges to:

$$\begin{aligned} &{}_6\Psi_6\left[\begin{matrix} q\sqrt{a},\ -q\sqrt{a},\ b_k,\ c_k,\ e,\ f\\ \sqrt{a},\ -\sqrt{a},\ aq/b_k,\ aq/c_k,\ aq/e,\ aq/f\end{matrix};q;\frac{a^{k+1}q^k}{(aq)^{k-1}\cdot b_kc_kef}\right]\\ &\quad = {}_6\Psi_6\left[\begin{matrix} q\sqrt{a},\ -q\sqrt{a},\ b_k,\ c_k,\ e,\ f\\ \sqrt{a},\ -\sqrt{a},\ aq/b_k,\ aq/c_k,\ aq/e,\ aq/f\end{matrix};q;\frac{a^2q}{b_kc_kef}\right]\end{aligned}$$

$$(3.7)\qquad = \prod \begin{bmatrix} aq, \dfrac{aq}{b_k c_k}, \dfrac{aq}{b_k e}, \dfrac{aq}{b_k f}, \dfrac{aq}{c_k e}, \dfrac{aq}{c_k f}, \dfrac{aq}{ef}, q, \dfrac{q}{a} \\ \dfrac{q}{b_k}, \dfrac{q}{c_k}, \dfrac{q}{e}, \dfrac{q}{f}, \dfrac{aq}{b_k}, \dfrac{aq}{c_k}, \dfrac{aq}{e}, \dfrac{aq}{f}, \dfrac{a^2 q}{b_k c_k ef} \end{bmatrix},$$

by Bailey's ${}_6\Psi_6$ summation in (1.5).

Note that the products in (3.7) are exactly the same as those in (1.8), and thus may be cancelled from both sides on (1.8).

When the parameters in (3.3) are used and we let $N \to \infty$, an application of Tannery's Theorem for products [7; §49] implies that the products in (1.9) and (1.10) become:

$$(3.8)\qquad (\alpha_1)_\infty (\alpha_2)_\infty \cdots (\alpha_{k-1})_\infty .$$

When the parameters in (3.3) are substituted into the sum in (1.11) we obtain:

$$(3.9)\qquad \sum_{m_1,\cdots,m_{k-1}\geqq 0} \frac{(q^{-N})_{m_1}(q^{-N})_{m_2}\cdots(q^{-N})_{m_{k-1}}\cdot q^{m_1+\cdots+m_{k-1}}}{(q)_{m_1}(q)_{m_2}\cdots(q)_{m_{k-1}}(\alpha_1)_{m_1}(\alpha_2)_{m_1+m_2}\cdots(\alpha_{k-1})_{m_1+\cdots+m_{k-1}}}$$
$$\times \frac{(b_k q^{-N})_{m_1+\cdots+m_{k-1}}(c_k q^{-N})_{m_1+\cdots+m_{k-1}}(eq^{-N})_{m_1+\cdots+m_{k-1}}}{(aq^{1-3N}/\alpha_{k-1})_{m_1+\cdots+m_{k-1}}(b_k c_k e q^{-N}/a)_{m_1+\cdots+m_{k-1}}}$$
$$\times \frac{(aq^{1-2N}/\alpha_2)_{m_1}(aq^{1-2N}/\alpha_3)_{m_1+m_2}\cdots(aq^{1-2N}/\alpha_{k-1})_{m_1+\cdots+m_{k-2}}}{(aq^{1-3N}/\alpha_1)_{m_1}(aq^{1-3N}/\alpha_2)_{m_1+m_2}\cdots(aq^{1-3N}/\alpha_{k-2})_{m_1+\cdots+m_{k-2}}}$$
$$\times (q^{-N})^{m_1+(m_1+m_2)+\cdots+(m_1+\cdots+m_{k-2})}$$
$$\times (\alpha_2 q^N)_{m_1}(\alpha_3 q^N)_{m_1+m_2}\cdots(\alpha_{k-1}q^N)_{m_1+\cdots+m_{k-2}} .$$

By making use of the relation

$$(a;q)_m = (-1)^m a^m \left(1-\frac{1}{a}\right)\left(q-\frac{1}{a}\right)\cdots\left(q^{m-1}-\frac{1}{a}\right)$$

to rewrite the products involving q^{-N}, we find after some algebraic simplification that when $N \to \infty$ the general term of the sum in (3.9) becomes:

$$\frac{q^{\sigma_1(m_1,\cdots,m_{k-1})^2-\sigma_2(m_1,\cdots,m_{k-1})-\sigma_1(m_1,\cdots,m_{k-1})}}{(q)_{m_1}(q)_{m_2}\cdots(q)_{m_{k-1}}} \times \frac{\alpha_1^{m_1}\alpha_2^{m_2}\cdots\alpha_{k-1}^{m_{k-1}}}{(\alpha_1)_{m_1}(\alpha_2)_{m_1+m_2}\cdots(\alpha_{k-1})_{m_1+\cdots+m_{k-1}}} .$$

An application of Tannery's Theorem now implies that when $N \to \infty$, the sum in (3.9) converges to the sum in (3.2).

Putting everything together we find that 1 equals the products in (3.8) times the sum in (3.2). This finishes the proof of Theorem 3.1.

Note that we have obtained Theorem 3.1 as a limiting case of Theorem 1.7.

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