SOME EXPANSIONS INVOLVING BASIC HYPERGEOMETRIC FUNCTIONS OF TWO VARIABLES

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In this paper we obtain expansions of q-Appell type of functions of higher dimensions. These expansions are different in nature from the ones studies thus far. Transformations and reducibility of basic double hypergeometric functions are also discussed.

Burchnall and Chaundy [5, 6] has made a systematic study of the expansions of the Appell functions. Later on Jackson [8, 9] defined the q-analogue of Appell functions and made a parallel study by obtaining q-analogues of most of the results of Burchnall and Chaundy. Jackson [8; p. 78] had pointed out that it does not seem possible to obtain simple extensions of the expansions (46)–(51) of Burchnall and Chaundy [5]. In §3 of this paper we give q-analogues of five of the results, viz., (46)–(49) and (51) of Burchnall and Chaundy [5] cited above.

It may be remarked that Andrews [1] had proved that the q-analogue of Appell's function $F^{(1)}$ defined by Jackson [8] is infact reducible to the basic hypergeometric series $_3\phi_2$. We show in §4 that some higher dimensional analogues of double hypergeometric functions could also be reduced to basic hypergeometric series and use the reduction formula for obtaining some interesting transformations for double hypergeometric functions. In this sequal we also derive q-analogues of some of the well known transformations of Appell functions and discuss their reducibility.

2. **Definitions and notations.** If we let

$$[a; q]_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1}) ,$$

$$[a; q]_0 = 1 \quad \text{and} \quad [a; q]_\infty = \prod_{r=0}^{\infty} (1 - aq^r) ,$$

then we may define the basic hypergeometric series as

$$_{p+1}\phi_{p+r}\left[\begin{matrix} a_1, a_2, \cdots, a_{p+1}; q; x \\ b_1, b_2, \cdots, b_{p+r} \end{matrix}\right]$$

$$= \sum_{n=0}^{\infty} \frac{[a_1; q]_n \cdots [a_{p+1}; q]_n x^n (-)^{nr} q^{rn/2(n-1)}}{[q; q]_n [b_1; q]_n \cdots [b_{p+r}; q]_n} , \quad |q| < 1 ,$$

where the series $_{p+1}\phi_{p+r}(x)$ converges for all positive integral values of r and for all x, except when $r = 0$, it converges only for $|x| < 1$.

Further to simplify writing we will abbreviate $[a; q]_n$ by $[a]_n$ whenever there is no confusion regarding the base q.

Next, q-analogues of Appell functions $F^{(1)}$, $F^{(2)}$, $F^{(3)}$, and $F^{(4)}$ were defined by Jackson [8] as follows:

$$\phi^{(1)}[a: b, b'; c; x, y; q] = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{[a]_{m+n}[b]_m[b']_n x^m y^n}{[q]_m[q]_n[c]_{m+n}},$$

$$\phi^{(2)}[a: b, b'; c, c'; x, y; q] = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{[a]_{m+n}[b]_m[b']_n x^m y^n}{[q]_m[q]_n[c]_m[c']_n},$$

$$\phi^{(3)}[a, b; a', b'; c; x, y; q] = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{[a]_m[b]_m[a']_n[b']_n x^m y^n}{[q]_m[q]_n[c]_{m+n}},$$

$$\phi^{(4)}[a, b; c, c'; x, y; q] = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{[a]_{m+n}[b]_{m+n} x^m y^n}{[q]_m[q]_n[c]_m[c']_n}.$$

Lastly, we define the generalized basic hypergeometric function of two variables as:

$$\phi \begin{bmatrix} (a_p): (b_s); (c_r) & x, y; q \\ (d_t): (e_u); (f_v) & i; j; k \end{bmatrix}$$
$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{[(a_p)]_{m+n}[(b_s)]_m[(c_r)]_n x^m y^n q^{im/2(m-1)+jn/2(n-1)+kmn}}{[q]_m[q]_n[(d_t)]_{m+n}[(e_u)]_m[(f_v)]_n}.$$

3. In this section we give the q-analogues of the formulae of Burchnall and Chaundy (46, 47, 48, 49 and 51 of [5]):

(3.1)
$$_3\phi_2 \begin{bmatrix} a, b, \dfrac{-abx}{cy}; \dfrac{cy}{ab} \\ c, 0 \end{bmatrix} = \sum_{r=0}^{\infty} \frac{[a]_r[b]_r\left[\dfrac{c}{a}\right]_r(-xy)^r q^{r(r-1)}}{[q]_r[c]_{2r}[y]_r}$$
$$\times \phi \begin{bmatrix} aq^r: bq^r; bq^r & xq^r, \dfrac{cy}{ab}; q \\ cq^{2r}: yq^r; - & 1; -; - \end{bmatrix},$$

(3.2)
$$\phi \begin{bmatrix} a: b; b & \dfrac{cx}{ab}, \dfrac{cy}{ab}; q \\ c: -x; - & 1; -; - \end{bmatrix} = \sum_{r=0}^{\infty} \frac{[a]_r[b]_r\left[\dfrac{c}{a}\right]_r(cxy)^r q^{r/2(r-1)}}{[q]_r[c]_{2r}[-x]_r(ab)^r}$$
$$\times {}_3\phi_2 \begin{bmatrix} aq^r, bq^r, \dfrac{-x}{y}q^r; \dfrac{cy}{ab} \\ cq^{2r}, -xq^r \end{bmatrix},$$

(3.3)
$$_3\phi_2 \begin{bmatrix} a, b, \dfrac{-x}{y}; y \\ c, 0 \end{bmatrix} = \sum_{r=0}^{\infty} \frac{[a]_r[b]_r\left[\dfrac{ab}{c}\right]_r(cxy)^r q^{r/2(5r-3)}}{[q]_r[c]_{2r}[y]_r}$$

$$\times \phi \begin{bmatrix} -: aq^r, bq^r; aq^r, bq^r \\ cq^{2r}: yq^r; - \end{bmatrix} \begin{matrix} xq^{2r}, yq^r; q \\ 1; -; 1 \end{matrix} \Bigg],$$

(3.4)
$$\phi \begin{bmatrix} -: a, b; a, b \\ c: y; - \end{bmatrix} \begin{matrix} x, y; q \\ 1; -; 1 \end{matrix} \Bigg] = \sum_{r=0}^{\infty} \frac{[a[_r[b]_r\left[\dfrac{c}{ab}\right]_r (abxy)^r q^{r/2(5r-3)}}{[q]_r[c]_{2r}[y]_r}$$

$$\times \, {}_3\phi_2 \begin{bmatrix} aq^r, bq^r, \dfrac{-xq^r}{y}; yq^r \\ cq^{2r}, 0 \end{bmatrix}$$

and

(3.5)
$$_2\phi_1 \begin{bmatrix} a, b; \dfrac{cy}{ab} \\ c \end{bmatrix} {}_2\phi_2 \begin{bmatrix} a, b; \dfrac{-cx}{ab} \\ c, -x \end{bmatrix} = \sum_{r=0}^{\infty} \frac{[a]_r[b]_r\left[\dfrac{c}{a}\right]_r\left[\dfrac{c}{b}\right]_r (cxy)^r q^{r/2(r-1)}}{[q]_r[c]_r[c]_{2r}[-x]_r(ab)^r}$$

$$\times \, {}_3\phi_2 \begin{bmatrix} aq^r, bq^r, \dfrac{-x}{y}q^r; \dfrac{cy}{ab} \\ cq^{2r}, -xq^r \end{bmatrix}.$$

Proof of (3.1). In view of the q-analogue of Gauss' summation theorem [12; 3.3.2.5] we have

(3.6)
$$\frac{\left[\dfrac{c}{b}\right]_n\left[\dfrac{c}{b}\right]_m q^{mn}}{\left[\dfrac{c}{b}\right]_{m+n}} = {}_2\phi_1 \begin{bmatrix} q^{-m}, q^{-n}; \dfrac{bq}{c} \\ \dfrac{b}{c}q^{1-m-n} \end{bmatrix},$$

multiplying both sides of (3.6) by

$$\frac{[a]_m[b]_m\left[\dfrac{c}{a}\right]_n\left[\dfrac{c}{b}\right]_{m+n} x^m y^n q^{m/2(m-1)}}{[q]_m[q]_n[c]_{m+n}[y]_m\left[\dfrac{c}{b}\right]_m}$$

and summing with respect to m and n from 0 to ∞, we get:

$$\sum_{m=0}^{\infty} \frac{[a]_m[b]_m x^m q^{m/2(m-1)}}{[q]_m[c]_m[y]_m} {}_2\phi_1 \begin{bmatrix} \dfrac{c}{a}, \dfrac{c}{b}; yq^m \\ cq^m \end{bmatrix}$$

(3.7)
$$= \sum_{r=0}^{\infty} \frac{[a]_r[b]_r\left[\dfrac{c}{a}\right]_r (-xy)^r q^{r(r-1)}}{[q]_r[c]_{2r}[y]_r}$$

$$\times \sum_{m=0}^{\infty} \frac{[aq^r]_m[bq^r]_m x^m q^{m/2(m+2r-1)}}{[q]_m[cq^{2r}]_m[yq^r]_m} {}_2\phi_1 \begin{bmatrix} \dfrac{c}{a}q^r, \dfrac{c}{b}q^{r+m}; y \\ cq^{m+2r} \end{bmatrix}.$$

Using the transformation [11]:

(3.8)
$$
{}_2\phi_1\left[\begin{matrix} a, b; x \\ e \end{matrix}\right] = \frac{\left[\dfrac{abx}{e}\right]_\infty}{[x]_\infty}\,{}_2\phi_1\left[\begin{matrix} \dfrac{e}{a}, \dfrac{e}{b}; \dfrac{abx}{e} \\ e \end{matrix}\right],
$$

to transform the inner ${}_2\phi_1$ on both sides of (3.7) and simplifying, we get (3.1).

Proof of (3.2). The q-analogue of Vandermonde's theorem [12; 3.3.2.7] may be rewritten as:

(3.9)
$$
\frac{\left[\dfrac{c}{b}\right]_{m+n}}{\left[\dfrac{c}{b}\right]_m\left[\dfrac{c}{b}\right]_n} = q^{mn}\,{}_2\phi_1\left[\begin{matrix} q^{-m}, q^{-n}; q \\ \dfrac{c}{b} \end{matrix}\right].
$$

On multiplying both sides of (3.9) by

$$
\frac{[a]_m[b]_m\left[\dfrac{c}{a}\right]_n\left[\dfrac{c}{b}\right]_n (cx)^m y^n q^{m/2(m-1)}}{[q]_m[q]_n[c]_{m+n}[-x]_m(ab)^m}
$$

and summing with respect to m and n from 0 to ∞, we have:

$$
\sum_{m=0}^{\infty} \frac{[a]_m[b]_m(cx)^m q^{m/2(m-1)}}{[q]_m[c]_m[-x]_m(ab)^m}\,{}_2\phi_1\left[\begin{matrix} \dfrac{c}{a}, \dfrac{c}{b}q^m; y \\ cq^m \end{matrix}\right]
$$

(3.10)
$$
= \sum_{r=0}^{\infty} \frac{[a]_r[b]_r\left[\dfrac{c}{a}\right]_r (cxy)^r q^{r/2(r-1)}}{[q_r][c]_{2r}[-x]_r(ab)^r} \sum_{m=0}^{\infty} \frac{[aq^r]_m[bq^r]_m(cx)^m q^{m/2(m-1+2r)}}{[q]_m[cq^{2r}]_m[-xq^r]_m(ab)^m}
$$
$$
\times\,{}_2\phi_1\left[\begin{matrix} \dfrac{c}{a}q^r \dfrac{c}{b}q^r; yq^m \\ cq^{m+2r} \end{matrix}\right],
$$

transforming the two ${}_2\phi_1$ in (3.10), using (3.8), we get:

$$
\phi\left[\begin{matrix} a: b; b \\ c: -x; - \end{matrix}\;\middle|\;\begin{matrix} \dfrac{cx}{ab}, \dfrac{cy}{ab}; q \\ 1; -; - \end{matrix}\right] = \sum_{r=0}^{\infty} \frac{[a]_r[b]_r\left[\dfrac{c}{a}\right]_r (cxy)^r q^{r/2(r-1)}}{[q]_r[c]_{2r}[-x]_r(ab)^r}
$$

$$
\times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{[aq^r]_{m+n}[bq^r]_{m+n}[y]_m(cx)^m(cy)^n q^{m/2(m+2r-1)}}{[q]_m[q]_n[cq^{2r}]_{m+n}[-xq^r]_m(ab)^{m+n}}
$$

(3.11)
$$
= \sum_{r=0}^{\infty} \frac{[a]_r[b]_r\left[\dfrac{c}{a}\right]_r (cxy)^r q^{r/2(r-1)}}{[q]_r[c]_{2r}[-x]_r(ab)^r} \sum_{n=0}^{\infty} \frac{[aq^r]_n[bq^r]_n(cy)^n}{[q]_n[cq^{2r}]_n(ab)^n}
$$
$$
\times\,{}_2\phi_1\left[\begin{matrix} q^{-n}, y; \dfrac{-x}{y}q^{r+n} \\ -xq^r \end{matrix}\right].
$$

(3.11) gives (3.2), on summing the inner most series by the q-analogue of Gauss' theorem.

Proof of (3.3).   On the other hand if we start with the q-analogue of the Saalschütz summation theorem [12; 3.3.2.2]

$$(3.12) \qquad \frac{[b]_{m+n}\left[\dfrac{c}{a}\right]_m\left[\dfrac{c}{a}\right]_n}{[b]_m[b]_n\left[\dfrac{c}{a}\right]_{m+n}} = {}_3\phi_2\left[\begin{array}{c} q^{-m},\, q^{-n},\, \dfrac{ab}{c};\, q \\[2mm] b,\, \dfrac{a}{c}q^{1-m-n} \end{array}\right]$$

and proceed as in the proof of (3.1), we obtain:

$$\sum_{m=0}^{\infty} \frac{[a]_m[b]_m x^m q^{m/2(m-1)}}{[q]_m[c]_m[by]_m}\, {}_2\phi_2\left[\begin{array}{c} bq^m,\, \dfrac{c}{a};\, ayq^m \\[2mm] cq^m,\, byq^m \end{array}\right]$$

$$= \sum_{r=0}^{\infty} \frac{[a]_r[b]_r\left[\dfrac{ab}{c}\right]_r (cxy)^r q^{r/2(5r-3)}}{[q]_r[c]_{2r}[by]_{2r}} \sum_{m=0}^{\infty} \frac{[aq^r]_m[bq^r]_m x^m q^{m/2(m+4r-1)}}{[q]_m[cq^{2r}]_m[byq^{2r}]_m}$$

$$\times\, {}_2\phi_2\left[\begin{array}{c} bq^r,\, \dfrac{c}{a}q^{m+r};\, ayq^{2r+m} \\[2mm] cq^{m+2r},\, byq^{m+2r} \end{array}\right].$$

Transforming the inner ${}_2\phi_2$ on both sides by the following formula of Jackson [7]:

$$(3.13) \qquad {}_2\phi_2\left[\begin{array}{c} a,\, b;\, \dfrac{cz}{b} \\[2mm] c,\, az \end{array}\right] = \frac{[z]_\infty}{[az]_\infty}\, {}_2\phi_1\left[\begin{array}{c} a,\, \dfrac{c}{b};\, z \\[2mm] c \end{array}\right],$$

we get (3.3) on some reduction.

Proof of (3.4).   To accomplishes (3.4) we need only to start with

$$(3.14) \qquad \frac{\left[\dfrac{c}{b}\right]_{m+n}[a]_m[a]_n}{\left[\dfrac{c}{b}\right]_m\left[\dfrac{c}{b}\right]_n[a]_{m+n}} = {}_3\phi_2\left[\begin{array}{c} q^{-m},\, q^{-n},\, \dfrac{c}{ab};\, q \\[2mm] \dfrac{c}{b},\, \dfrac{1}{a}q^{1-m-n} \end{array}\right],$$

instead of (3.12) and proceed along the lines of proof of (3.3).

Proof of (3.5).   Lastly, starting with (3.9) with $b = 1$, multiplying both sides by $[a]_m[b]_m[c/a]_n[c/b]_n(cx)^m y^n q^{m/2(m-1)}/[q]_m[q]_n[c]_{m+n}[-x]_m(ab)^m$ and summing with respect to m and n from 0 to ∞, we get

$${}_2\phi_1\left[\begin{array}{c} \dfrac{c}{a},\, \dfrac{c}{b};\, y \\[2mm] c \end{array}\right]\, {}_2\phi_2\left[\begin{array}{c} a,\, b;\, \dfrac{-cx}{ab} \\[2mm] c,\, -x \end{array}\right] = \sum_{r=0}^{\infty} \frac{[a]_r[b]_r\left[\dfrac{c}{a}\right]_r\left[\dfrac{c}{b}\right]_r (cxy)^r q^{r/2(r-1)}}{[q]_r[c]_r[c]_{2r}[-x]_r(ab)^r}$$

$$\times \sum_{m=0}^{\infty} \frac{[aq^r]_m[bq^r]_m(cx)^m q^{m/2(m+2r-1)}}{[q]_m[cq^{2r}]_m[-xq^r]_m(ab)^m} {}_2\phi_1\begin{bmatrix} \dfrac{c}{a}q^r, \dfrac{c}{b}q^r; yq^m \\ cq^{2r+m} \end{bmatrix}.$$

Transforming the ${}_2\phi_1$ by (3.8), we get:

$${}_2\phi_1\begin{bmatrix} a, b; \dfrac{cy}{ab} \\ c \end{bmatrix} {}_2\phi_2\begin{bmatrix} a, b; \dfrac{-cx}{ab} \\ c, -x \end{bmatrix} = \sum_{r=0}^{\infty} \frac{[a]_r[b]_r\left[\dfrac{c}{a}\right]_r\left[\dfrac{c}{b}\right]_r (cxy)^r q^{r/2(r-1)}}{[q]_r[c]_r[c]_{2r}[-x]_r(ab)^r}$$

$$\times \sum_{m=0}^{\infty} \frac{[aq^r]_m[bq^r]_m[y]_m(cx)^m q^{m/2(m+2r-1)}}{[q]_m[cq^{2r}]_m[-xq^r]_m(ab)^m} {}_2\phi_1\begin{bmatrix} aq^{r+m}, bq^{r+m}; \dfrac{cy}{ab} \\ cq^{m+2r} \end{bmatrix}$$

(3.15)

$$= \sum_{r=0}^{\infty} \frac{[a]_r[b]_r\left[\dfrac{c}{a}\right]_r\left[\dfrac{c}{b}\right]_r (cxy)^r q^{r/2(r-1)}}{[q]_r[c]_r[c]_{2r}[-x]_r(ab)^r} \sum_{n=0}^{\infty} \frac{[aq^r]_n[bq^r]_n(cy)^n}{[q]_n[cq^{2r}]_n(ab)^n}$$

$$\times {}_2\phi_1\begin{bmatrix} q^{-n}, y; \dfrac{-x}{y}q^{r+n} \\ -xq^r \end{bmatrix}.$$

(3.15) gives (3.5), on summing the inner most series by the q-analogue of Gauss' theorem.

4. In this section we would prove q-analogues of the transformations between the various Appell functions. In fact we begin by proving:

(4.1) $$\frac{[a]_\infty[bx]_\infty}{[c]_\infty} {}_3\phi_2\begin{bmatrix} \dfrac{c}{a}, x, y; a \\ bx, b'y \end{bmatrix} = \left[\dfrac{abx}{c}\right]_\infty \phi\begin{bmatrix} \dfrac{c}{a}: \dfrac{c}{bb'}; b', \dfrac{b'y}{x} \Big| \dfrac{abx}{c}, \dfrac{ax}{b'}; q \\ c: -; b'y \quad\quad\Big| -; -; - \end{bmatrix}$$

(4.2) $$= [x]_\infty \phi\begin{bmatrix} -: a, b; \dfrac{c}{a}, b' \Big| x, -ay; q \\ c: -; b'y \quad\quad\Big| -; 1; 1 \end{bmatrix}.$$

Proof of (4.1). If we denote the right hand side of (4.1) by S, then

$$S = \left[\dfrac{abx}{c}\right]_\infty \sum_{n=0}^{\infty} \frac{\left[\dfrac{c}{a}\right]_n[b']_n\left[\dfrac{b'y}{x}\right]_n (ax)^n}{[q]_n[b'y]_n[c]_n(b')^n} {}_2\phi_1\begin{bmatrix} \dfrac{c}{a}q^n, \dfrac{c}{bb'}; \dfrac{abx}{c} \\ cq^n \end{bmatrix}.$$

Transforming the inner ${}_2\phi_1$ by (3.13) and rearranging the series, we get

$$S = \left[\frac{ax}{b'}\right]_\infty \sum_{m=0}^\infty \frac{[a]_m\left[\frac{c}{bb'}\right]_m(-bx)^m q^{m/2(m-1)}}{[q]_m[c]_m\left[\frac{ax}{b'}\right]_m} {}_3\phi_2\left[\begin{array}{c} \frac{c}{a}, \frac{b'y}{x}, b'; \frac{ax}{b'}q^m \\ cq^m, b'y \end{array}\right].$$

Now transforming the inner ${}_3\phi_2$ using:

$$(4.3) \qquad {}_3\phi_2\left[\begin{array}{c} a, b, c; \frac{eg}{abc} \\ e, g \end{array}\right] = \Pi\left[\begin{array}{c} \frac{g}{c}, \frac{eg}{ab} \\ g, \frac{eg}{abc} \end{array}\right]{}_3\phi_2\left[\begin{array}{c} \frac{e}{a}, \frac{e}{b}, c; \frac{g}{c} \\ e, \frac{eg}{ab} \end{array}\right],$$

(which is obtained from [11; (8.3)] by taking the limit as $N \to \infty$) we get,

$$S = \frac{[y]_\infty[ax]_\infty}{[b'y]_\infty} \sum_{m=0}^\infty \frac{[a]_m\left[\frac{c}{bb'}\right]_m(-bx)^m q^{m/2(m-1)}}{[q]_m[c]_m[ax]_m} {}_3\phi_2\left[\begin{array}{c} aq^m, \frac{cx}{b'y}q^m, b'; y \\ cq^m, axq^m \end{array}\right]$$

$$= \frac{[y]_\infty[ax]_\infty}{[b'y]_\infty} \sum_{m=0}^\infty \frac{[a]_m\left[\frac{c}{bb'}\right]_m(-bx)^m q^{m/2(m-1)}}{[q]_m[c]_m[ax]_m} {}_3\phi_2\left[\begin{array}{c} q^{-m}, \frac{b'y}{cx}q^{1-m}, b'; q \\ \frac{bb'}{c}q^{1-m}, 0 \end{array}\right].$$

Next, transforming the inner ${}_3\phi_2$ by [14]

$$(4.4) \qquad {}_2\phi_1\left[\begin{array}{c} a, b; \frac{ec}{ab} \\ e \end{array}\right] = \Pi\left[\begin{array}{c} \frac{e}{a}, \frac{e}{b}; \\ e, \frac{e}{ab} \end{array}\right]{}_3\phi_2\left[\begin{array}{c} a, b, c; q \\ \frac{abq}{e}, 0 \end{array}\right],$$

(provided either a, b or c is of the form q^{-N}, N a nonnegative inner, if only $c = q^{-N}$ then $|ec/ab| < 1$) we have

$$S = \frac{[y]_\infty[ax]_\infty}{[b'y]_\infty} \sum_{m=0}^\infty \frac{[a]_m\left[\frac{c}{b}\right]_m(-bx)^m q^{m/2(m-1)}}{[q]_m[c]_m[ax]_m} {}_2\phi_1\left[\begin{array}{c} q^{-m}, b'; \frac{yq}{bx} \\ \frac{c}{b} \end{array}\right]$$

$$= \frac{[y]_\infty[ax]_\infty}{[b'y]_\infty} \sum_{n=0}^\infty \frac{[a]_n[b']_n y^n}{[q]_n[c]_n[ax]_n} {}_2\phi_2\left[\begin{array}{c} aq^n, \frac{c}{b}q^n; bx \\ cq^n, axq^n \end{array}\right],$$

once again, transforming ${}_2\phi_2$ by (3.13), we get

$$(4.5) \qquad S = \frac{[x]_\infty[y]_\infty}{[b'y]_\infty} \sum_{m=0}^\infty \sum_{n=0}^\infty \frac{[a]_{m+n}[b]_m[b']_n x^m y^n}{[q]_m[q]_n[c]_{m+n}}.$$

Lastly, an appeal to

$$\phi^{(1)}[a:b,\,b';\,c;\,x,\,y;\,q] = \prod \begin{bmatrix} a,\,bx,\,b'y \\ c,\,x,\,y \end{bmatrix}_3\phi_2 \begin{bmatrix} \dfrac{c}{a},\,x,\,y;\,a \\[2mm] bx,\,b'y \end{bmatrix},$$

due to Andrews [1], yields (4.1).

One the other hand in order to prove (4.2), we need only rewrite (4.5) in the form

$$S = \frac{[y]_\infty[x]_\infty}{[b'y]_\infty} \sum_{m=0}^{\infty} \frac{[a]_m[b]_m x^m}{[q]_m[c]_m} {}_2\phi_1 \begin{bmatrix} aq^m,\,b';\,y \\ cq^m \end{bmatrix}$$

and transform the inner $_2\phi_1$ by (3.13).

It may be observed that the $_3\phi_2$ in (4.1) remains unchanged if we interchange the roles of x and y and that of b and b'. This yields us the transformation:

$$\begin{bmatrix} \dfrac{abx}{c} \end{bmatrix}_\infty [b'y]_\infty \phi \begin{bmatrix} \dfrac{c}{a}:\dfrac{c}{bb'};\,b',\,\dfrac{b'y}{x} \,\bigg|\, \dfrac{abx}{c},\,\dfrac{ax}{b'};\,q \\[2mm] c:\,-;\,b'y \qquad\qquad \bigg|\, -;\,-;\,- \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{ab'y}{c} \end{bmatrix}_\infty [bx]_\infty \phi \begin{bmatrix} \dfrac{c}{a}:\dfrac{c}{bb'};\,b,\,\dfrac{bx}{y} \,\bigg|\, \dfrac{ab'y}{c},\,\dfrac{ay}{b};\,q \\[2mm] c:\,-;\,bx \qquad\qquad \bigg|\, -;\,-;\,- \end{bmatrix}$$

which is a q-analogue of a known transformation between two $F^{(1)}$'s [3; §9.4 (4) and (5)] to which it reduces if we replace a, b, b', c by q^a, q^b, $q^{b'}$, q^c and let $q \to 1$. Similarly we could have obtained from (4.2) the transformation

$$(4.6) \qquad \begin{aligned} & \frac{[b'y]_\infty}{[y]_\infty} \phi \begin{bmatrix} -:a,\,b;\,\dfrac{c}{a},\,b' \,\bigg|\, x,\,-ay;\,q \\[2mm] c:\,-;\,b'y \qquad\quad \bigg|\, -;\,1;\,1 \end{bmatrix} \\[3mm] & \qquad = \frac{[bx]_\infty}{[x]_\infty} \phi \begin{bmatrix} -:a,\,b';\,\dfrac{c}{a},\,b \,\bigg|\, y,\,-ax;\,q \\[2mm] c:\,-;\,bx \qquad\quad \bigg|\, -;\,1;\,1 \end{bmatrix}, \end{aligned}$$

which is the q-analogue of yet another known transformation between two $F^{(3)}$'s [3] to which it reduces if we replace a, b, b', c by q^a, q^b, $q^{b'}$, q^c and let $q \to 1$.

Next, we would prove:

$$(4.7) \qquad \frac{[ay]_\infty}{[y]_\infty} \phi \begin{bmatrix} a:b;\,\dfrac{c'}{b'} \,\bigg|\, x,\,-b'y;\,q \\[2mm] ay:\,c;\,c' \,\bigg|\, -;\,1;\,- \end{bmatrix} = \phi^{(2)}[a:b,\,b';\,c,\,c';\,x,\,y;\,q]$$

$$(4.8) \qquad = \frac{[ax]_\infty}{[x]_\infty} \phi \begin{bmatrix} a:b';\,\dfrac{c}{b} \,\bigg|\, y,\,-bx;\,q \\[2mm] ax:\,c';\,c \,\bigg|\, -;\,1;\,- \end{bmatrix},$$

(4.9)
$$\phi\begin{bmatrix} a: b; b', \dfrac{c'x}{b'y} & x, y; q \\ -: c; c' & -; -; - \end{bmatrix} = \frac{[ay]_\infty}{[y]_\infty}\phi\begin{bmatrix} a: \dfrac{c}{b}; \dfrac{c'}{b'}, \dfrac{b'y}{x} & -bx, x; q \\ ay: c; c' & 1; -; - \end{bmatrix},$$

(4.10)
$$\phi\begin{pmatrix} a: b, -b; c, -c & \dfrac{xq}{y}, -y; q \\ -: b^2, ax, \dfrac{q}{y}; c^2, ay & 1; -; -1 \end{pmatrix}$$

$$= \Pi\begin{bmatrix} x, y \\ ax, ay \end{bmatrix}\phi\begin{bmatrix} a, aq: -; \dfrac{q}{x}, \dfrac{q^2}{x} & x^2, \dfrac{x^2y^2}{q^3}; q^2 \\ -: b^2q; c^2q & -; -2; -2 \end{bmatrix}$$

(provided a is of the form q^{-N})
and

(4.11)
$$\phi^{(2)}\begin{bmatrix} \dfrac{cc'}{q}: b, b; c, c'; x, y; q \end{bmatrix} = \phi\begin{bmatrix} \dfrac{cc'}{q}, b: \dfrac{q}{y}; \dfrac{c'x}{q} & \dfrac{-xy}{q}, y; q \\ -: c; c' & -1; -; -1 \end{bmatrix}$$

(provided cc'/q is of the form q^{-N}).

Proof of (4.7–4.8). Rewriting the left hand side in the form

$$\frac{[ay]_\infty}{[y]_\infty}\sum_{m=0}^{\infty}\frac{[a]_m[b]_m x^m}{[q]_m[c]_m[ay]_m}\,{}_2\phi_2\begin{bmatrix} aq^m, \dfrac{c'}{b'}; b'y \\ c', ayq^m \end{bmatrix}$$

and transforming $_2\phi_2$ by (3.13) we get the right hand side of (4.7), for proving (4.8) rearranging the series on the right hand side of (4.7) and transforming the $_2\phi_1$ by (3.13). This reduces to a known transformation between $F^{(2)}$'s [3; §(9.4)]. It may be pointed out that (4.7) is a known result due to Upadhyay [13].

Proof of (4.9). Rewrite the left hand side of (4.9) as:

$$S = \sum_{n=0}^{\infty}\frac{[a]_n[b']_n\left[\dfrac{c'x}{b'y}\right]_n y^n}{[q]_n[c']_n}\,{}_2\phi_1\begin{bmatrix} aq^n, b; x \\ c \end{bmatrix},$$

transforming the $_2\phi_1$ by (3.13) and rearranging the series, we get

$$S = \frac{[ax]_\infty}{[x]_\infty}\sum_{m=0}^{\infty}\frac{[a]_m\left[\dfrac{c}{b}\right]_m(-bx)^m q^{m/2(m-1)}}{[q]_m[c]_m[ax]_m}\,{}_3\phi_2\begin{bmatrix} b', \dfrac{c'x}{b'y}, aq^m; y \\ c', axq^m \end{bmatrix}.$$

Transforming the $_3\phi_2$ by (4.2) and simplifying, we get (4.9). This result is a q-analogue of a known transformation between two $F^{(2)}$'s [3].

Proof of (4.10). If we denote the left hand side of (4.10) by S, then

$$S = \sum_{m \geq 0} \frac{[a]_m [b^2; q^2]_m x^m q^{m/2(m+1)}}{[q]_m [b^2]_m [ax]_m \left[\dfrac{q}{y}\right]_m} y^m \, {}_3\phi_2 \begin{bmatrix} aq^m, c, -c; -yq^{-m} \\ c^2, ay \end{bmatrix}.$$

Transforming the ${}_3\phi_2$ by the formula [10; equation 3.1]:

$$(4.12) \qquad {}_3\phi_2 \begin{bmatrix} a, b, -b; -z \\ b^2, az \end{bmatrix} = \frac{[z]_\infty}{[az]_\infty} {}_2\phi_1 \begin{bmatrix} a, aq; q^2; z^2 \\ b^2 q \end{bmatrix}$$

and rearranging the series, we get

$$(4.13) \qquad S = \frac{[y]_\infty}{[ay]_\infty} \sum_{n \geq 0} \frac{[a]_{2n} y^{2n}}{[q^2; q^2]_n [c^2 q; q^2]_n} {}_3\phi_2 \begin{bmatrix} aq^{2n}, b, -b; -xq^{-2n} \\ b^2, ax \end{bmatrix}.$$

Once again, transforming the ${}_3\phi_2$ by (4.12) and simplifying, (4.13) yields (4.10), which is the q-analogue of a terminating version of [4; (4.7)].

Proof of (4.11). In view of the q-analogue of Gauss' summation theorem, we have

$$(4.14) \qquad \frac{[b]_m [b]_n}{[b]_{m+n}} = q^{-mn} {}_2\phi_1 \left(\begin{matrix} q^{-m}, q^{-n}; \dfrac{q}{b} \\ \dfrac{1}{b} q^{1-m-n} \end{matrix} \right).$$

Multiplying both sides by $[cc'/q]_{m+n}[b]_{m+n} x^m y^n / [q]_m [q]_n [c]_m [c']_n$ (where $cc'/q = q^{-N}$) and summing with respect to m and n, we get

$$\phi^{(2)} \left[\frac{cc'}{q} : b, b; c, c'; x, y; q \right]$$

$$= \sum_{m \geq 0} \sum_{n \geq 0} \sum_{p \geq 0} \frac{\left[\dfrac{cc'}{q}\right]_{m+n+2p} [b]_{m+n+p} x^{m+p} y^{n+p} (-)^p q^{-p/2(p+1+2n+2m)}}{[q]_m [q]_n [q]_p [c]_{m+p} [c']_{n+p} q^{mn}}$$

$$= \sum_{m \geq 0} \sum_{n \geq 0} \sum_{p \geq 0} \frac{\left[\dfrac{cc'}{q}\right]_{m+n+p} [b]_{m+n+p} x^{m+p} y^{n+p} (-c')^p q^{p^2/2-3p/2}}{[q]_m [q]_n [q]_p [c]_m [c']_{n+p} q^{mn+mp}} \sum_{s=0}^{p} \frac{[q^{-p}]_s \left[\dfrac{1}{c'} q^{1-n-p}\right]_s q^s}{[q]_s [cq^m]_s}$$

$$= \sum_{m \geq 0} \sum_{n \geq 0} \frac{\left[\dfrac{cc'}{q}\right]_{m+n} [b]_{m+n} x^m y^n}{[q]_m [q]_n [c]_m [c']_n q^{mn}} {}_1\phi_0 \begin{bmatrix} q^{-n}; c'xq^{n-1} \\ - \end{bmatrix} {}_1\phi_0 \begin{bmatrix} q^{-m}; y \\ - \end{bmatrix},$$

summing the two ${}_1\phi_0$, we get (4.11), which is q-analogue of a terminating version of a formula [3; Ex. 20(i) p. 102]. Lastly, we

give q-analogues of reduction formulae for Appell's double, hypergeometric function:

$$(4.15) \quad \phi\begin{bmatrix} a: b, y; c \\ cy: a; a \end{bmatrix}\begin{matrix} x, -y; q \\ -; 1; - \end{matrix}\end{bmatrix} = \Pi\begin{bmatrix} bx, y \\ x, cy \end{bmatrix} {}_2\phi_2\begin{bmatrix} b, c; xy \\ a, bx \end{bmatrix},$$

$$(4.16) \quad \phi\begin{pmatrix} -: a, b; \dfrac{c}{a}, \dfrac{c}{b} \\ c: \dfrac{aby}{c}; - \end{pmatrix}\begin{matrix} x, \dfrac{aby}{c}; q \\ 1; -; 1 \end{matrix}\end{pmatrix} = \dfrac{[y]_\infty}{\left[\dfrac{aby}{c}\right]_\infty} {}_3\phi_2\begin{bmatrix} a, b, -\dfrac{x}{y}; y \\ c, 0 \end{bmatrix},$$

$$(4.17)$$

$$\phi\begin{bmatrix} -: a, \dfrac{q}{a}; c, \dfrac{q}{c} \\ d: -q; -q \end{bmatrix}\begin{matrix} -x, x; q \\ -; -; - \end{matrix}\end{bmatrix} = {}_6\phi_5\begin{bmatrix} ac, \dfrac{q^2}{ac}, \dfrac{aq}{c}, \dfrac{cq}{a}, 0, 0; q^2; x^2 \\ q, -q, -q^2, d, dq \end{bmatrix}$$

$$- \dfrac{x(a-c)(q-ca)}{ac(1-d)(1-q^2)} \cdot {}_6\phi_5\begin{bmatrix} acq, \dfrac{q^3}{ac}, \dfrac{aq^2}{c}, \dfrac{cq^2}{a}, 0, 0; q^2; x^2 \\ -q^2, q^3, -q^3, dq, dq^2 \end{bmatrix},$$

$$(4.18) \quad \phi\begin{bmatrix} a: b, -b; c, -c \\ -: b^2; c^2 \end{bmatrix}\begin{matrix} x, -x; q \\ -; -; - \end{matrix}\end{bmatrix}$$

$$= {}_6\phi_5\begin{bmatrix} a, aq, bc, -bc, bcq, -bcq; q^2; x^2 \\ b^2q, c^2q, b^2c^2, 0, 0 \end{bmatrix}.$$

Proof of (4.15).   Rewrite the left hand side of (4.15) in the form

$$S = \sum_{m=0}^{\infty} \dfrac{[b]_m[y]_m x^m}{[q]_m[cy]_m} {}_2\phi_2\begin{bmatrix} aq^m, c; y \\ a, cyq^m \end{bmatrix}$$

$$= \dfrac{[y]_\infty}{[cy]_\infty} \sum_{m=0}^{\infty} \dfrac{[b]_m x^m}{[q]_m} {}_2\phi_1\begin{bmatrix} q^{-m}, c; yq^m \\ a \end{bmatrix}$$

$$(4.19) \qquad = \dfrac{[y]_\infty}{[cy]_\infty} \sum_{n=0}^{\infty} \dfrac{[b]_n[c]_n(-xy)^n q^{n/2(n-1)}}{[q]_n[a]_n} {}_1\phi_0[bq^n; -; x].$$

Summing the inner ${}_1\phi_0$ of (4.19) and we get (4.15) which is a q-analoque of [3; §9.5-(7)] to which it reduces if we replace a, b, c by q^a, q^b, q^c respectively and let $q \to 1$.

Proof of (4.16).   Left hand side of (4.16) may be rewritten as:

$$S = \sum_{m=0}^{\infty} \dfrac{[a]_m[b]_m x^m q^{m/2(m-1)}}{[q]_m[c]_m\left[\dfrac{aby}{c}\right]_m} {}_2\phi_1\begin{bmatrix} \dfrac{c}{a}, \dfrac{c}{b}; \dfrac{aby}{c}q^m \\ cq^m \end{bmatrix}$$

$$= \dfrac{[y]_\infty}{\left[\dfrac{aby}{c}\right]_\infty} \sum_{m=0}^{\infty} \dfrac{[a]_m[b]_m x^m q^{m/2(m-1)}}{[q]_m[c]_m} {}_2\phi_1\begin{bmatrix} aq^m, bq^m; y \\ cq^m \end{bmatrix}$$

(4.20)
$$= \frac{[y]_\infty}{\left[\dfrac{aby}{c}\right]_\infty} \sum_{n=0}^\infty \frac{[a]_n[b]_n y^n}{[q]_n[c]_n} {}_1\phi_0\left[q^{-n}; \ -; \ -\frac{x}{y}q^n\right].$$

Summing the inner ${}_1\phi_0$ of (4.20) and we get (4.16). It is a q-analogue of [3; §(9.5)-(5)].

Proof of (4.17). The summation theorem [10]

(4.21)
$$
{}_4\phi_3\left[\begin{array}{c} a, \ \dfrac{q}{a}, \ -q^{-n}, \ q^{-n}; \ q \\[2mm] -q, \ c, \ \dfrac{1}{c}q^{1-2n} \end{array}\right] = \frac{[ac; q^2]_n\left[\dfrac{cq}{a}; q^2\right]_n}{[c]_{2n}},
$$

could be written in the symmetrical form (on replacing c by cq^{-n}):

(4.22)
$$
\sum_{r=0}^n \frac{[a]_r\left[\dfrac{q}{a}\right]_r [c]_{n-r}\left[\dfrac{q}{c}\right]_{n-r}(-)^r}{[q^2; q^2]_r[q^2; q^2]_{n-r}}
$$
$$
= \frac{[acq^{-n}; q^2]_n\left[\dfrac{c}{a}q^{1-n}; q^2\right]_n(-)^n q^{n/2(n+1)}}{c^n[q^2; q^2]_n},
$$

multiplying on both sides of (4.22) by $x^n/[d]_n$ and summing with respect to n form 0 to ∞, we get (4.17) (on separating the even and odd parts on right hand side) on some simplification.

Proof of (4.18). The q-analogue of Watson's theorem [2] can be written in the symmetrical form (on replacing b by $q^{1/2-n}/b$):

(4.23)
$$
\sum_{r=0}^{2n} \frac{[c^2; q^2]_r[b^2; q^2]_{2n-r}(-)^r}{[q]_r[q]_{2n-r}[c^2]_r[b^2]_{2n-r}} = \frac{[b^2c^2; q^2]_{2n}}{[b^2q; q^2]_n[c^2q; q^2]_n[b^2c^2; q^2]_n[q^2; q^2]_n},
$$

multiplying both sides of (4.23) by $[a]_{2n}x^{2n}$ and summing with respect to n from 0 to ∞, we get

(4.24)
$$
\sum_{n=0}^\infty [a]_{2n}x^{2n} \sum_{r=0}^{2n} \frac{[c^2; q^2]_r[b^2; q^2]_{2n-r}(-)^r}{[q]_r[q]_{2n-r}[c^2]_r[b^2]_{2n-r}}
$$
$$
= {}_6\phi_5\left[\begin{array}{c} a, \ aq, \ bc, \ -bc, \ bcq, \ -bcq; \ q^2; \ x^2 \\[1mm] b^2q, \ c^2q, \ b^2c^2, \ 0, \ 0 \end{array}\right].
$$

But, since
$$
\sum_{r=0}^{2n+1} \frac{[c^2; q^2]_r[b^2; q^2]_{2n+1-r}(-)^r}{[q]_r[q]_{2n+1-r}[c^2]_r[b^2]_{2n+1-r}} = 0 ,
$$

therefore (4.24) yields

$$(4.25) \quad \sum_{n=0}^{n} [a]_n x^n \sum_{r=0}^{\infty} \frac{[c^2; q^2]_r [b^2; q^2]_{n-r} (-)^r}{[q]_r [q]_{n-r} [c^2]_r [b^2]_{n-r}}$$

$$= {}_6\phi_5 \begin{bmatrix} a, \ aq, \ bc, \ -bc, \ bcq, \ -bcq; \ q^2; \ x^2 \\ b^2q, \ c^2q, \ b^2c^2, \ 0, \ 0 \end{bmatrix}.$$

Now, rearranging the series in left hand side of (4.25), we get (4.18), (4.17) and (4.18) are the q-analogues of (4.2) and (4.4) of [4] respectively.

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