PACIFIC JOURNAL OF MATHEMATICS
Vol. 91, No. 2, 1980

ON THE HOMOMORPHIC AND ISOMORPHIC EMBEDDINGS OF A SEMIFLOW INTO A RADIAL FLOW

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It is the main purpose of this paper to prove the following two theorems.

THEOREM I. **(Isomorphism) Let $(X, \boldsymbol{R}^+, f)$ be a semiflow on a separable metric space (X, d), having the properties:**

(i) there is an $\omega \in X$ such that, for each neighborhood U of ω, there is a $T \in \boldsymbol{R}^+$ with $f[X, t] \subset U$ for all $t \geqq T$;

(ii) for each $t \in \boldsymbol{R}^+, f(\cdot, t)$ is a homeomorphism of X onto a closed subspace of X.

Then $(X, \boldsymbol{R}^+, f)$ is isomorphic to a radial semiflow on a subset of the Hilbert Cube in l^2.

THEOREM II. **(Homomorphism) If $(X, \boldsymbol{R}^+, f)$ satisfies the hypotheses of Theorem I, with (i) replaced by**

(i′) $\cap \{f[X, t]: t \geqq 0\} = \{\omega\}$ for some $\omega \in X$, then $\{X, \boldsymbol{R}^+, f)$ is homomorphic to a radial semiflow on a subset of the Hilbert Cube C and the subsemiflow induced on $X/\{\omega\}$ is isomorphic to a radial semiflow in C.

1. Introduction.

Let X be a nonempty subset of a normed linear space and suppose that, with $0 < \lambda < 1$,

$$f(x, t) = \lambda^t x \quad ((x, t) \in X \times \boldsymbol{R}) . \tag{1}$$

The triple $(X, \boldsymbol{R}, f)$, with f as above, determines a dynamical system or a flow (cf. [4], [5]) on X such that the semitrajectory of each $x \neq 0$ is a line segment joining x with the origin 0. The terms "a radial flow" or "a radial dynamical system" seem appropriate. Similarly, with $\boldsymbol{R}$ replaced by $\boldsymbol{R}^+$, the nonnegative reals, we refer to $(X, \boldsymbol{R}^+, f)$ as a radial semiflow, or a radial semidynamical system. By a homomorphic (isomorphic) embedding of $(X, \boldsymbol{R}, f)$ into $(Y, \boldsymbol{R}, g)$ we understand a one-one continuous mapping (a homeomorphism) h of X into Y such that

$$h(f(x, t)) = g(h(x), t) \ ((x, t) \in X \times \boldsymbol{R}) .$$

A similar definition applies to semiflows. Thus, to say that $(X, \boldsymbol{R}^+, f)$ is isomorphic to a radial semiflow means that a homeomorphism h of X into a normed linear space exists such that

$$h(f(x, t)) = \lambda^t h(x) \ ((x, t) \in X \times \boldsymbol{R}^+) . \tag{2}$$

In a recent paper L. Janos [3] proved that a semiflow $(X, \boldsymbol{R}^+, f)$

on a compact metric space X is isomorphic to a radial semiflow on a subset X of the Hilbert Cube in l_2 if the transition map f is one-one and $\cap\{f[X, t]: t \geqq 0\}$ is a singleton.

It is the main purpose of this paper to establish the extensions of the above result to the setting of semiflows on a separable metric space which were stated in the opening paragraph.

As can be readily verified, conditions (i) and (i′) are equivalent if X is compact metric. Since (ii) is automatically satisfied for a one-one continuous transition map on a compact metric space, the above mentioned result of Janos follows as a corollary. On the other hand it is quite easy to show that the conclusion of Theorem I cannot be obtained with (i) replaced by (i′).

An analog for discrete semiflows on a compact metric space was used by Janos [3] in the proof of his result. While such an analog was available before (cf. [1], [2]), the corresponding one for separable metric spaces given here (Theorems 2 and 3) is new and of independent interest. Also of some interest is the result asserting that the property of radiality is passed on to a flow by the corresponding property of its discrete subflows (Theorem 1). This result parallels the "Embedding Lemma" of [3].

2. Inheritance of radiality.

THEOREM 1. *Let $(X, \boldsymbol{R}, f)$ be a dynamical system on a nonempty subset X of l_2 with the property that for some λ, $0 < \lambda < 1$, all nonnegative integers n and all $(x, t) \in X \times \boldsymbol{R}$,*

$$f(x, t + n) = \lambda^n f(x, t) . \tag{3}$$

Then $(X, \boldsymbol{R}, f)$ is isomorphic to a radial flow in l_2.

Proof. Applying an idea of M. Bebutov (cf., e.g., [4], p. 333), we consider the integral

$$\int_0^\infty ||f(x, t)||\,dt . \tag{4}$$

This improper integral converges since

$$\sum_{m=0}^\infty \int_m^{m+1} ||f(x, t)||\,dt = \sum_{m=0}^\infty \int_0^1 ||f(x, t + m)||\,dt ;$$

by (3) then

$$\int_0^\infty ||f(x, t)||\,dt = \frac{1}{1 - \lambda}\int_0^1 ||f(x, t)||\,dt \tag{5}$$

proving convergence. Thus $s_x(T) = \int_T^\infty ||f(x, t)||dt$ defines a function from $\boldsymbol{R}$ to $\boldsymbol{R}^+$.

Clearly, this function is continuous and decreasing for x distinct from the origin 0. Its range for all $x \neq 0$ contains $\{1\}$ as it consists of all positive reals. [Indeed $\lim_{T\to\infty} s_x(T) = 0$ by convergence of the integral in (4) and, on the other hand

$$s_x(-n) > \int_{-n}^0 ||f(x, t)||dt \geqq \lambda^{-n}\int_0^1 ||f(x, t)||dt \ ;$$

so that $s_x(T) \to \infty$ as $T \to -\infty$.]

To each $x \in X\backslash\{0\}$ there corresponds a unique t_x such that $s_x(t_x) = 1$. The correspondence $x \to t_x$ which arises in this manner is clearly one-one on trajectories lying in $X\backslash\{0\}$. Also, if $y = f(x, t)$ then, as can be readily seen

$$t_y = t_x - t \ . \tag{6}$$

To prove that t_x is continuous in x — a fact needed in the sequel— let $\{x_n\}$ be a sequence in $X\backslash\{0\}$ with $x_n \to x \neq 0$. Then, we claim,

$$\lim_{n\to\infty}\int_{t_x}^\infty ||f(x_n, t)||dt = 1 \ . \tag{7}$$

Indeed,

$$\begin{aligned}\int_{t_x}^\infty ||f(x_n, t)||dt &= \int_{t_x}^0 ||f(x_n, t)||dt + \sum_{m=0}^\infty \int_m^{m+1} ||f(x_n, t)||dt \\ &= \int_{t_x}^0 ||f(x_n, t)||dt + \frac{1}{1-\lambda}\int_0^1 ||f(x_n, t)||dt \ .\end{aligned}$$

As $f(x_n, t) \to f(x, t)$ uniformly on compact intervals ([4], p. 327), we obtain in the limit, as $n \to \infty$,

$$\int_{t_x}^\infty ||f(x, t)||dt = 1 \ . \tag{8}$$

Thus

$$\int_{t_{x_n}}^{t_x} ||f(x_n, t)||dt = 1 - \int_{t_x}^\infty ||f(x_n, t)||dt \longrightarrow 0 \text{ as } n \longrightarrow \infty \ .$$

However this is only possible if $t_{x_n} \to t_x$.

Let S be the set of all x in X such that $t_x = 0$; i.e.,

$$S = \left\{x \in X \colon \int_0^\infty ||f(x, t)||dt = 1\right\} \ .$$

In view of the preceding discussion S is a closed subset of $X\backslash\{0\}$

and each trajectory, lying in that subset, meets S at exactly one point. (In the usual terminology S is a section in $X\backslash\{0\}$.)

Let $\bar{\sigma}: S \rightarrow \sum$ be a homeomorphism onto a subset of $H \cap C$ where H is the hyperplane $\{x \in l_2: \langle x, e_1\rangle = 1\}$, and C denotes the Hilbert Cube in l_2. [That such a homeomorphism exists follows from the facts that $(x_1, x_2, \cdots) \leftrightarrow (1, x_1, x_2, \cdots)$ sets up an isometry in l_2 and that any separable metric space is homeomorphic with a subset of the Hilbert Cube.] The desired isomorphism is now defined as the mapping σ of X into $\boldsymbol{R} \sum$ obtained by setting

$$\sigma(x) = \lambda^{-t_x}\bar{\sigma}(f(x, t_x)) \ (x \neq 0)$$

$$\text{and if } 0 \in X,\ \sigma(0) = 0 . \tag{9}$$

On $X\backslash\{0\}$ σ is continuous as the composition of t_x, f and $\bar{\sigma}$. If $r > 0$ and $y \in r\Sigma$ then $y = \lambda^t\bar{\sigma}(x) \Leftrightarrow \bar{\sigma}^{-1}(\lambda^{-t}y) = x$ for some $t \in \boldsymbol{R}$; hence (the existence and) continuity of the inverse. To prove continuity at 0 of both σ and σ^{-1} it suffices, in view of the definition (9), to show that a sequence $\{x_n\}$ in $X\backslash\{0\}$ converges to 0 if, and only if, $t_{x_n} \rightarrow -\infty$.

Suppose, first, that $t_{x_n} \rightarrow -\infty$. If $\{x_n\}$ fails to converge to 0 then we may assume that $||x_n|| \geqq \varepsilon$ for some $\varepsilon > 0$. As a result,

$$M = \inf\{||f(x_n, t)||: n = 1, 2, \cdots;\ 0 \leqq t \leqq 1\} > 0 .$$

[Otherwise a sequence $\{t_i\}$ in $[0, 1]$ and a subsequence $\{x_{n_i}\}$ would have to exist such that $t_i \rightarrow t^* \in [0, 1]$ and $f(x_{n_i}, t_i) \rightarrow 0$. But then $f(x_{n_i}, t^*) = (f(x_{n_i}, t^* - t_i + t_i)) = f(f(x_{n_i}, t_i),\ t_i^* - t_i) \rightarrow 0$, implying $x_{n_i} \rightarrow 0$.] Hence by (8), (cf. also (5)),

$$\begin{aligned} 1 = \int_{t_{x_n}}^{\infty} ||f(x_n, t)||dt &= \int_{t_{x_n}}^{-m} ||f(x_n, t)||dt + \frac{\lambda^{-m}}{1-\lambda}\int_0^1 ||f(x_n, t)||dt \\ &\geqq (1-\lambda)^{-1}\lambda^{-m}M, \text{ if } -m-1 \leqq t_{x_n} \leqq -m . \end{aligned}$$

However, for sufficiently large n, $\lambda^{-m}M > 1 - \lambda$, leading to a contradiction.

Next, let $x_n \rightarrow 0$ and assume, for a contradiction, that $t_{x_n} \nrightarrow -\infty$. Then we may assume that $t_{x_n} \geqq -m$, where m is a fixed positive integer. Hence,

$$\begin{aligned} 1 = \int_{t_{x_n}}^{\infty} ||f(x_n, t)||dt &\leqq \int_{-m}^{\infty} ||f(x_n, t)||dt \\ &= (1-\lambda)^{-1}\lambda^{-m}\int_0^1 ||f(x_n, t)||dt . \end{aligned}$$

However, as already observed, $x_n \rightarrow 0 \Rightarrow \int_0^1 ||f(x_n, t)||dt \rightarrow 0$ which is impossible.

Finally, with $f(x, t) = y$, and by a repeated application of (6),

$$\sigma(f(x, t)) = \lambda^{-t_y}\bar{\sigma}(f(y, t_y)) = \lambda^{-t_x+t}\bar{\sigma}(f(x, t + t_y)) = \lambda^t\lambda^{-t_x}\bar{\sigma}(f(x, t_x)) = \lambda^t\sigma(x)$$

showing that $(X, \boldsymbol{R}, f)$ is isomorphic with the radial flow $(\boldsymbol{R}\Sigma, \boldsymbol{R}, g)$, where $g(x, t) = \lambda^t x$.

3. Discrete semiflows. In analogy with the definition of a (continuous) radial semiflow we define a discrete radial semiflow $(X, \boldsymbol{N}^+, f)$ on a nonempty set X in a normed linear space as one in which

$$f(x, n) = \lambda^n x \tag{1'}$$

for some λ, $0 < \lambda < 1$, and all $(x, n) \in X \times \boldsymbol{N}^+$.

A continuous mapping f of a metric space M into itself determines a discrete semiflow $(M, \boldsymbol{N}^+, \bar{f})$ by setting $\bar{f}(x, n) = f^n(x)$. Hence a homeomorphism (a one-one continuous mapping) $h: M \to l_2$ satisfying

$$h(f(x)) = \lambda h(x) \tag{*}$$

with $0 < \lambda < 1$, determines an isomorphic (a homomorphic) embedding into a radial semiflow in l_2; namely one for which $h(\bar{f}(x, n)) = \lambda^n h(x)$.

For a mapping h to satisfy (*) it is clearly necessary and sufficient that the "coordinate functions" $\psi_n(x) = \langle h(x), e_n\rangle$, $(n=1, 2, \cdots)$, where e_n is a member of the standard orthonormal basis, do likewise; i.e.,

$$\psi_n(f(x)) = \lambda\psi_n(x) \; . \tag{10}$$

Further, for h to be one-one it is necessary and sufficient that the family $\{\psi_n\}$ distinguishes points; i.e., if x', x'' are distinct members of X then $\psi_n(x') \neq \psi_n(x'')$ for some $n \in \boldsymbol{N}^+$.

By means of such a family of functions it was shown in [1] that a homeomorphism $h: X \to l_2$ exists satisfying (*) if X is compact metric, $f: X \to X$ is one-one continuous, and

$$\cap \{f^n[X]: n = 1, 2, \cdots\} = \{\omega\} \tag{11}$$

where $\omega \in X$.

If compactness is removed from the hypotheses of the above result then, as simple examples show, the conclusions may no longer be true even if f is a homeomorphism (onto $f[X]$). This is due to the fact that in general h may fail to have a continuous inverse at some points of $h[X]$. Furthermore, the construction of a homeomorphism h such that h^{-1} is continuous at some points, e.g., on $h[X]\backslash\{0\}$, as in the next theorem, seems inadequate. Hence the need for the more refined construction which we produce in the proofs of the following theorems.

3.1. Homomorphisms.

THEOREM 2. *Let X be a separable metric space and f a homeomorphism of X, onto a closed proper subset of X, satisfying* (11). *Then there is a one-one continuous mapping h of X into the Hilbert Cube C in l_2 such that h^{-1} is continuous on $h[X]\backslash\{0\}$ and* (*) *holds.*

Proof. Let $\mathscr{B}$ be a countable base for the topology of $X\backslash f[X]$ and let $\{(U_n, V_n): n = 1, 2, \cdots\}$ be an enumeration of those pairs $(U, V) \in \mathscr{B}\times\mathscr{B}$ for which $\bar{U}\subset V$. Set

$$B_n^{(k)} = \cup\{B(x, r(x)): x\in f^k[V_n]\}\ ,\qquad (n = 1, 2, \cdots;\ k = 0, 1, \cdots)\ ,$$

where $B(x, r(x))$ denotes the open ball about x, of radius $r(x) = 2^{-1}d(x, f^k[X]\backslash f^k[V_n])$; (with this choice of $r(x)$ we have $B_n^{(i)}\cap B_n^{(j)} = \varnothing$ for $i\neq j$).

For $k = 0, 1, \cdots$ let $\alpha_{2n-1}^{(k)}$ be a continuous mapping into $[0, 1]$ such that

$$\alpha_{2n-1}^{(k)}[X\backslash\cup\{B_n^{(i)}: i = 0, 1, \cdots, k\}] = 0$$
$$\alpha_{2n-1}^{(k)}[U_n\cup f[U_n]\cup\cdots\cup f^k[U_n]] = 1\ .$$

Since $B_n^{(i)}$ is disjoint from $f^k[X]$ for $i\leqq k$ we have

$$\alpha_{2n-1}^{(k)}f^{k+1} = 0\ .$$

To define $\alpha_{2^m(2n-1)}^{(k)}$ for integers $m\neq 0$ pick, for $m > 0$, an arbitrary continuous extension of $\bar{\alpha}_{2^{m-1}(n2-1)}^{(k)}$, where

$$\bar{\alpha}_{2^{m-1}(2n-1)}^{(k)} = \begin{cases}\alpha_{2^{m-1}(2n-1)}^{(k)}f^{-1} & \text{on } f[X]\\ 1 \text{ on } U_n\end{cases}$$

and, for other subscripts

$$\alpha_{2^{-m}(2n-1)}^{(k)} = \alpha_{2^{-m+1}(2n-1)}^{(k)}f\ .$$

Thus

$$\alpha_{2^m(2n-1)}^{(k)}f = \alpha_{2^{m-1}(2n-1)}^{(k)}\ (m = 0, \pm 1, \cdots;\ n = 1, 2, \cdots)\ . \tag{12}$$

A countable family of coordinate functions $\psi_n^{(k)}: X \to [0, 1]$ is now obtained by setting

$$\psi_n^{(k)} = (1-\lambda)\sum_{m=-k}^{\infty} \lambda^{m+k}\alpha^{(k)}_{2^m(2n-1)} .$$

By (12), and the fact that $\alpha^{(k)}_{2^{-k}(2n-1)}f = \alpha^{(k)}_{2n-1}f^{k+1} = 0$, we have

$$\begin{aligned}\psi_n^{(k)}f &= (1-\lambda)\sum_{m=-k}^{\infty} \lambda^{m+k}\alpha^{(k)}_{2^m(2n-1)}f \\ &= (1-\lambda)\sum_{m=-k+1} \lambda^{m+k}\alpha^{(k)}_{2^m(2n-1)}f \\ &= \lambda(1-\lambda)\sum_{m=-k+1} \lambda^{m+k-1}\alpha^{(k)}_{2^{m-1}(2n-1)} = \lambda\psi_n^{(k)} ,\end{aligned}$$

showing that (10) holds for all coordinate functions. To define h: $X \to C \subset l_2$ let $\{\psi_j: j = 1, 2, \cdots\}$ be an enumeration of $\{\psi_n^{(k)}\}$ and set

$$h(x) = \left(\frac{\psi_1(x)}{1}, \frac{\psi_2(x)}{2}, \cdots, \frac{\psi_j(x)}{j}, \cdots\right).$$

A standard argument, omitted here, shows that h is continuous. To prove that h is one-one, let x_1, x_2 be distinct points of X and suppose that $x_1 = f^{k_1}(y_1)$, $x_2 = f^{k_2}(y_2)$ with $k_1 \leqq k_2$ and $y_1 \in X\backslash f[X]$. Choose n such that $\alpha^{(0)}_{2n-1}(y_1) = 1$ and $\alpha^{(0)}_{2n-1}(y_2) = 0$. Then

$$\psi_n^{(0)}(y_1) = (1-\lambda)\sum_{m=0}^{\infty}\lambda^m\alpha^{(0)}_{2^m(2n-1)}(y_1) = 1 \Longrightarrow \psi_n^{(0)}(x_1) = \lambda^{k_1}$$

while

$$\psi_n^{(0)}(y_2) = (1-\lambda)\sum_{m=1}^{\infty}\lambda^m\alpha^{(0)}_{2^m(2n-1)}(y_2) \leqq \lambda \Longrightarrow \psi_n^{(0)}(x_n) \leqq \lambda^{k_2+1} .$$

Hence $\psi_n^{(0)}(x_2) \leqq \lambda\psi_n^{(0)}(x_1)$ and $h(x_1) \neq h(x_2)$.

To prove continuity of h^{-1} at $h(x)$, $x \neq 0$, let $\{h(x_n): n=1, 2, \cdots\}$ be a sequence in $h[X]\backslash\{0\}$ converging to $h(x)$ and suppose that $x = f^k(y)$ for some $k \geqq 0$ and $y \in X\backslash f[X]$. Suppose $x_n \nrightarrow x$. Quite clearly a subsequence $\{x_{n_i}\}$ and an $\varepsilon > 0$ must then exist such that $d(x_{n_i}, f^j(y)) \geqq \varepsilon$ for all $i = 1, 2, \cdots$ and $j = 0, 1, \cdots, k$. [If not, then a subsequence $\{x_{n_i}\}$ exists such that $x_{n_i} \nrightarrow x$ but $x_{n_i} \to f^j(y)$ for some $j \neq k$; but then $h(x_{n_i}) \to h(f^j(y)) = \lambda^{j-k}h(x) \neq h(x)$.] Let n be such that each $B_n^{(j)}$ is contained in the open ball about $f^j(y)$, of radius ε and $y \in V_n$. Then

$$\alpha^{(k)}_{2n-1}(x_{n_i}) = 0 \text{ and } \alpha^{(k)}_{2n-1}(f^j(y)) = 1 \ (j = 1, 2, \cdots, k).$$

Hence $\psi_n^{(k)}(y) = 1 \Rightarrow \psi_n^{(k)}(x) = \lambda^k$; while

$$\psi_n^{(k)}(x_{n_i}) = (1-\lambda)\sum_{m=-k}^{\infty}\lambda^{m+k}\alpha_{2^m(2n-1)}^{(k)}(x_{n_i})$$
$$= (1-\lambda)\lambda^k\sum_{m=1}^{\infty}\lambda^m\alpha_{2^m(2n-1)}^{(k)}(x_{n_i}) \leqq \lambda^{k+1} = \lambda\psi_n^{(k)}(x) .$$

Clearly then $||h(x_{n_i}) - h(x)||$ is bounded away from zero, against the assumption that $h(x_n) \to h(x)$.

3.2. Isomorphism. In the next theorem we have the stronger conclusion that h is a homeomorphism. As in the case of Theorem I, where (i) is stronger than (i′), it is necessary to strengthen condition (11) by the discrete counterpart of condition (i). Thus we shall assume that the following hypothesis is satisfied.

There exists a $\omega \in X$ such that for every neighborhood U of ω there is a positive integer N with the property that

$$f^n[X] \subset U \tag{13}$$

for $n \geqq N$.

(A simple argument shows that in a compact metric space (11) and (13) are equivalent.)

THEOREM 3. *Let X be as in Theorem 2 with* (11) *replaced by* (13). *Let $0 < \lambda < 1$. Then a homeomorphism h of X into the Hilbert Cube C in l_2 exists such that $h(f(x)) = \lambda h(x)$.*

Proof. Let B_n be an open ball about ω, of radius ε/n with $\varepsilon > 0$ such that $B_1 \neq X$. By (13) there are integers $N_1 < N_2 < \cdots < N_n < \cdots$ such that

$$f^{N_n}[X] \subset B_n .$$

Let $\alpha_{2n-1}\colon X \to [0, 1]$ be a continuous function such that $\alpha_{2n-1}[f^{N_n}[X]] = 0$ and $\alpha_{2n-1}[X \backslash B_n] = 1$.

Let $\alpha_{2(2n-1)}$ be a continuous extension of $\alpha_{2n-1}f^{-1}$ to the whole of X and, recursively, let $\alpha_{2^m(2n-1)}$ be a continuous extension of $\alpha_{2^{m-1}(2n-1)}f^{-1}$ to the whole of X. Define $\alpha_{2^{-m}(2n-1)}\colon X \to [0, 1]$ $(m = 1, 2, \cdots)$ by setting

$$\alpha_{2^{-m}(2n-1)} = \alpha_{2^{-m+1}}f .$$

This defines a family $\{\alpha_{2^m(2n-1)}\colon m = 0, \pm 1, \cdots; n = 1, 2, \cdots\}$ of continuous mappings into [0, 1] such that for all integers m,

$$\alpha_{2^m(2n-1)}f = \alpha_{2^{m-1}(2n-1)} \tag{14}$$

and

$$\alpha_{2^m(2n-1)} = 0 \ (m \leqq -N_n) . \tag{15}$$

Let now

$$\tilde{\psi}_n(x) = (1-\lambda)\lambda^{N_n-1} \sum_{m=-N_n+1}^{\infty} \lambda^m \alpha_{2^m(2n-1)}(x)$$
$$(n = 1, 2, \cdots ; \ x \in X) .$$

Then $\tilde{\psi}_n$: $X \to [0, 1]$ is continuous and by (14) and (15),

$$\begin{aligned}\tilde{\psi}_n(f(x)) &= (1-\lambda)\lambda^{N_n-1} \sum_{-N_n+1}^{\infty} \lambda^m \alpha_{2^m(2n-1)}(f(x)) \\ &= (1-\lambda)\lambda^{N_n-1} \sum_{-N_n+1}^{\infty} \lambda^m \alpha_{2^{m-1}(2n-1)}(x) \\ &= (1-\lambda)\lambda^{N_n} \sum_{-N_n+1}^{\infty} \lambda^m \alpha_{2^m(2n-1)}(x) = \lambda\tilde{\psi}_n(x)\end{aligned}$$

showing that (10) is satisfied.

To define $h: X \to l_2$ let $\{\psi_j: j = 1, 2, \cdots\}$ be an enumeration of $\{\tilde{\psi}_n\} \cup \{\psi_n^{(k)}\}$ where $\{\psi_n^{(k)}\}$ is as in the proof of Theorem 2. Set

$$h(x) = \left(\frac{\psi_1(x)}{1}, \frac{\psi_2(x)}{2}, \cdots, \frac{\psi_j(x)}{j}, \cdots\right).$$

The continuity of h, the existence of h^{-1} and the continuity of h^{-1} at $h(x) \neq 0$ follows, as in the proof of Theorem 2, from the relevant properties of $\{\psi_n^{(k)}\}$. To prove continuity of h^{-1} at 0 let $\{h(x_n)\}$ be a sequence converging to 0 and suppose that $\{x_n\}$ fails to converge to 0. Then a subsequence $\{x_{n_i}\}$ and a positive integer n exist such that $\{x_{n_i}\}$ is in $X \backslash B_n$. Hence $\alpha_{2n-1}(x_{n_i}) = 1$. $(i = 1, 2, \cdots)$, and therefore $\tilde{\psi}_n(x_{n_i}) \geqq (1-\lambda)\lambda^{N_n-1}$. It follows that $\|h(x_{n_i})\|$ is bounded away from zero, against the assumption that $\{h(x_n)\}$ converges to the origin.

4. Main results.

Proof of Theorem I. Fix λ, $0 < \lambda < 1$, and set $\bar{f}(x) = f(x, 1)$. The mapping $\bar{f}$: $X \to X$ satisfies the hypotheses of Theorem 3. Hence there is a homeomorphism h of X into the Hilbert Cube with $h(\bar{f}(x)) = \lambda h(x)$ or, in terms of discrete semiflows,

$$h(f(x, n)) = h(\bar{f}^n(x)) = \lambda^n(h(x)), \ (x \in X) .$$

Let Y be the set of all y in l_2 with the property that for some nonnegative integer n, $\lambda^n y \in h[X]$, and define g: $Y \times \boldsymbol{R} \to Y$ by setting

$$g(y, t) = \lambda^{-m-n} h(f(h^{-1}(\lambda^n y), t + m)) \tag{16}$$

where m and n are the smallest nonnegative integers such that $\lambda^n y \in h[X]$ and $t + m \geqq 0$.

From the fact that

$$f(x, t + m + m') = f(f(x, t + m), m') \quad (x \in X;\ t + m \geqq 0;\ m' \geqq 0)$$

and

$$h(f(f(x, t + m), m')) = \lambda^{m'} h(f(x, t + m))$$

it follows that

$$\lambda^{-m-m'} h(f(x, t + m + m')) = \lambda^{-m} h(f(x, t + m))$$

showing that (16) holds for any nonnegative integer m with $m + t \geqq 0$ (and not only for the smallest one with the said property). A similar argument, omitted here, shows that (16) remains true with n replaced by n'.

To verify the additivity of g let

$$g(y, t + t') = \lambda^{-m-m'-n} h(f(h^{-1}(\lambda^n y),\ t + t' + m + m'))$$

where $\lambda^n y \in h[X]$, $t + m \geqq 0$, $t' + m' \geqq 0$. We have

$$h(f(h^{-1}(\lambda^n y), t + m)) = \lambda^{m+n} g(y, t) \in h[X]$$

and

$$g(g(y, t), t') = \lambda^{-m-n-m'} h(f(h^{-1}(\lambda^{m+n} g(y, t), t' + m')))$$

showing that

$$g(y, t + t') = g(g(y, t), t') .$$

To sum up, Y is a nonempty subset of l_2, $(Y, \boldsymbol{R}, g)$ is a dynamical system and for $t \in \boldsymbol{R}$, k a nonnegative integer, we obtain (by choosing positive integers n, m with $\lambda^n y \in h[X]$, $t + m \geqq 0$) that

$$\begin{aligned} g(y, t + k) &= \lambda^{-m-n} h(f(h^{-1}(\lambda^n y),\ t + k + m)) \\ &= \lambda^k \lambda^{-m-n} h(f(h^{-1}(\lambda^n y', t + m))) = \lambda^k g(y, t) \end{aligned}$$

so that the hypotheses of Theorem 1 are satisfied. Hence $(Y, \boldsymbol{R}, g)$ is isomorphic to a radial flow in l_2. Furthermore, we may assume that the set S, in the proof of Theorem 1, is disjoint from $h[X]$. (Otherwise it may be replaced by $\lambda^{-n} S$ with n sufficiently large.) It follows that $(X, \boldsymbol{R}^+, f)$ is isomorphic to a radial semiflow on a subset of the Hilbert Cube.

Proof of Theorem II. As in the preceding proof fix λ, $0 < \lambda < 1$ and set $\bar{f}(x) = f(x, 1)$. The mapping $\bar{f}\colon X \to X$ satisfies the hypotheses of Theorem 2. Hence there is a one-one continuous mapping h of X into the Hilbert Cube in l_2 such that $h(\bar{f}(x)) = \lambda h(x)$ with h^{-1} continuous on $h[X]\backslash\{0\}$.

The rest of the preceding proof applies verbatim with the only change that $(Y, \boldsymbol{R}, g)$ as defined there is homomorphic (rather than isomorphic) to a radial flow on a subset of the Hilbert Cube in l_2. Hence $(X, \boldsymbol{R}^+, f)$ is homomorphic to such a semiflow.

The author wishes to record his thanks to the Weizmann Institute of Science, Rehovot, Israel and to Michigan State University, East Lansing, Michigan, for the hospitality and assistance extended to him while on sabbatical leave from Dalhousie University, Halifax, N.S.

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Received August 27, 1979.

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