PACIFIC JOURNAL OF MATHEMATICS
Vol. 91, No. 2, 1980

$|C, 1|$ SUMMABILITY OF SERIES ASSOCIATED WITH FOURIER SERIES

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The purpose of this paper is to prove the following theorem. Suppose that for $u \geq n_0$, $g(u)$ and $d(u)$ are positive functions such that $ud(u)$ is nondecreasing and (i) $\sum n^{-1}g(n)d(n) < \infty$. Then the series $\sum d(n)A_n(x)$ is summable $|C, 1|$, if the following hold:

(1.1) $$\Phi(t) = \int_0^t |\varphi(u)|\, du = O(tg(t^{-1}))\ , \quad t \longrightarrow +0\ ;$$

(1.2) $$\sum n^{-1}d(n)I(n^{-1}) = \sum n^{-1}d(n)\int_{n^{-1}}^{\pi} t^{-1}|\varphi(t)|\, dt < \infty\ .$$

1. The main result. Let $\sum_{n=0}^{\infty} a_n$ be a given infinite series with $\{s_n\}$ as the sequence of its partial sums. The nth $(C, 1)$ mean of $\{s_n\}$ is given by

$$t_n = \sum_{k=0}^{n} s_k/n + 1$$

and the series $\sum_{n=0}^{\infty} a_n$ is said to be $|C, 1|$ summable, if $\sum_{n=1}^{\infty} |t_n - t_{n-1}| < \infty$. It is known that (see [2])

(1.3) $$n(t_n - t_{n-1}) = T_n$$

where T_n is the nth $(C, 1)$ mean of the sequence $\{na_n\}$.

Let $f(t)$ be a Lebesgue integrable periodic function with period 2π and $\sum_{n=0}^{\infty} A_n(t)$ denotes its Fourier series. Then for $k \geqq 1$,

(1.4) $$\pi A_k(x) = \int_0^{\pi} \varphi(t) \cos kt\, dt$$

where $\varphi(t) = f(x + t) + f(x - t) - 2f(x)$. For some positive integer n_0, we write $\sum$ for $\sum_{n=n_0}^{\infty}$. We now turn to the proof of the theorem stated in the first paragraph.

REMARKS. If we take $d(u) = u^{-a}$ and $g(u) = (\log u)^b$, for any $a, b > 0$, then clearly $u\, d(u)$ is nondecreasing for $u \geqq 1$ and (i) holds if $a \leqq 1$. Further, assuming (1.1), we have by integration by parts

(1.5) $$I(n^{-1}) = [t^{-1}\Phi(t)]_{1/n}^{\pi} + \int_{1/n}^{\pi} t^{-2}\Phi(t)\, dt$$

so that $I(n^{-1}) = O[(\log n)^{1+b}]$ and (1.2) follows. Thus, we have

COROLLARY. *Suppose that for any positive b, however, large,*

$$(1.1') \qquad \Phi(t) = \int_0^t |\varphi(u)|\, du = O[t(\log(1/t))^b]\,, \quad t \longrightarrow +0\,.$$

Then the series $\sum n^{-a}A_n(x)$ *with* $n_0 = 1$, *is summable* $|C, 1|$ *for* $0 < a \leqq 1$ *and is absolutely convergent for* $a > 1$.

In order to see the later part of the corollary we notice that

$$\pi A_n(x) = O(\Phi(n^{-1}) + I(n^{-1})) = O[(\log n)^{1+b}]$$

and the consequence follows. Since $\Phi(t) = O(t)$ implies (1.1′) (but not conversely) the result of the corollary is an improvement over that contained in [1, Theorem 1].

Taking $g(u) = [\prod_{r=1}^{k} \log^r u]^{-1}$ and $d(u) = g(u)\{\log^k(u)\}^{-b}$ for any $b > 0$, where $\log^1(u) = \log u$; $\log^2(u) = \log\log u$; $\cdots$; we see that the hypothesis (i) holds. Now assuming (1.1) we have from (1.5), $I(n^{-1}) = O(\log^{k+1} n)$, so that (1.2) holds and we deduce the result of Theorem 2 in [1].

2. Proof of the theorem. In view of (1.3) and (1.4), it follows that in order to prove the theorem it is sufficient to show that

$$(2.1) \qquad J \equiv \sum \{n(n+1)\}^{-1} \left| \int_0^\pi h(n, t)\varphi(t)\, dt \right| < \infty\,,$$

where $h(n, t) = \sum_{k=n_0}^{n} (k - n_0)\, d(k) \cos kt$. Since $k\, d(k)$ is nonnegative, nondecreasing and

$$(k - n_0)\, d(k) = k\, d(k)(1 - (n_0/k))$$

therefore, $(k - n_0)\, d(k)$ is nonnegative, nondecreasing for $k \geqq n_0$. Thus, we have by Abel's lemma

$$(2.2) \qquad h(n, t) = O\left(n\, d(n) \max_{n_0 < r \leqq n} \left|\sum_{k=r}^{n} \cot kt\right|\right) = O(sn\, d(n))$$

where $s = n$ or t^{-1}. Thus,

$$(2.3) \qquad J_1 = \sum \{n(n+1)\}^{-1} \left| \int_0^{1/n} h(n, t)\varphi(t)\, dt \right| = O(\sum d(n)\Phi(n^{-1})) = O(1)\,,$$

by virtue of the hypotheses (i) and (1.1). Using (2.2) with $s = t^{-1}$, we have

$$(2.4) \qquad J_2 = \sum \{n(n+1)\}^{-1} \left| \int_{1/n}^{\pi} h(n, t)\varphi(t)\, dt \right| = O(\sum n^{-1} d(n) I(n^{-1})) = O(1)\,,$$

by virtue of (1.2). Combining (2.3) and (2.4) with (2.1), we complete the proof of the theorem.

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Received June 5, 1979 and in revised form September 18, 1979.

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