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STABLE SEQUENCES IN PRE-ABELIAN CATEGORIES

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In the Pacific Journal of Mathematics, 71 (1977), Richman and Walker gave a natural definition for Ext in an arbitrary pre-abelian category. Their Theorem 4, which states that $(\alpha E)\beta = \alpha(E\beta)$ for an arbitrary sequence E, is in error. We show, however, that $(\alpha E)\beta = \alpha(E\beta)$ does hold for a stable exact sequence. Without Theorem 4, the crucial step in their theory is showing that αE is stable if E is stable. We prove this. Consequently, the theory of Richman and Walker for Ext in a pre-abelian category is valid.

1. Introduction. An additive category with kernels and cokernels is called *pre-abelian.* Richman and Walker [3] developed an additive bifunctor Ext from an arbitrary pre-abelian category to the category of abelian groups. The Ext introduced in [3] coincides with the standard Ext (e.g., see [2]) if the category is, in fact, abelian. This theory is subsequently used by Richman and Walker [4] in the category of valuated groups. The theory of [3] is also used in [1] to examine certain relative homological algebras and to compute certain $\mathrm{Ext}(C, A)$ in the category of finite valuated groups.

However, Theorem 4 [3, p. 523] is incorrect. Without Theorem 4, one needs to prove that the sequence αE is stable if the sequence E is stable. This is our Theorem 2.

We use the terminology and notation of [3]. Thus, we are working working in an arbitrary pre-abelian category. If $f: A \to B$ and $\alpha: A \to A'$, then the pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\xi} \\ A' & \xrightarrow{\beta} & P \end{array}$$

is constructed by setting $P = \mathrm{coker}(f \oplus (-\alpha))\Delta$, where $\Delta: A \to A \oplus A$ is the diagonal map. We say that β is the pushout of f along α. Pullbacks are obtained dually. A *sequence* E is a diagram $A \xrightarrow{f} B \xrightarrow{g} C$ such that $gf = 0$. E is *left exact* if f is the kernel of g, *right exact* if g is the cokernel of f, and *exact* if it is both left and right exact. If $\alpha: A \to A'$, we pushout f along α to construct the sequence αE.

$$\begin{array}{rcccc} E\colon & A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ & {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\varphi} & & \| \\ \alpha E\colon & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C\,. \end{array}$$

The pushout property gives the existence and uniqueness of $g'\colon B' \to C$ such that $g'f' = 0$ and $g'\varphi = g$. We obtain $E\beta$ in the dual manner for $\beta\colon C' \to C$.

Richman and Walker [3, Theorem 4] assert that $(\alpha E)\beta = \alpha(E\beta)$ for an arbitrary sequence E. This is not true, even if E is exact. Consider the category of abelian p-groups with no elements of infinite height. Let B be a direct sum of cyclic groups of order p^n for $n = 1, 2, 3, \cdots$, $\bar{B}$ be the torsion subgroup of the corresponding direct product, and $G[p] = \{g \in G\colon pg = 0\}$. Then we have

$$\begin{array}{rccccc} E\colon & \bar{B}[p] & \xrightarrow{i} & \bar{B} & \xrightarrow{\beta} & \bar{B}/\bar{B}[p] \\ & {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & \| \\ \alpha E\colon & \dfrac{\bar{B}[p]}{B[p]} & \xrightarrow{0} & \dfrac{\bar{B}}{\bar{B}[p]} & = & \dfrac{\bar{B}}{\bar{B}[p]} \\ & \| & & \uparrow{\scriptstyle\beta} & & \uparrow{\scriptstyle\beta} \\ (\alpha E)\beta\colon & \dfrac{\bar{B}[p]}{B[p]} & \xrightarrow{0} & \bar{B} & = & \bar{B} \end{array}$$

where i is the injection map and α and β are the coset maps. The fact that 0 is the pushout of i along α is due to Richman and Walker [3, p. 522]. Note that E is exact, but αE is not left exact. On the other hand, we have

$$\begin{array}{rccccc} \alpha(E\beta)\colon & \dfrac{\bar{B}[p]}{B[p]} & \longrightarrow & \dfrac{\bar{B}[p]}{B[p]} \oplus \bar{B} & \longrightarrow & \bar{B} \\ & {\scriptstyle\alpha}\uparrow & & \uparrow & & \| \\ E\beta\colon & \bar{B}[p] & \longrightarrow & \bar{B}[p] \oplus \bar{B} & \longrightarrow & \bar{B} \\ & \| & & \downarrow{\scriptstyle\nabla(i \oplus 1)} & & \downarrow{\scriptstyle\beta} \\ E\colon & \bar{B}[p] & \xrightarrow{i} & \bar{B} & \xrightarrow{\beta} & \bar{B}/\bar{B}[p] \end{array}$$

where $\nabla\colon \bar{B} \oplus \bar{B} \to \bar{B}$ is the codiagonal map. Hence $(\alpha E)\beta \neq \alpha(E\beta)$.

2. Stable exact sequences. The example motivates following definition.

DEFINITION (Richman and Walker [3, p. 524]). An exact sequence

E is said to be *stable* if αE and $E\beta$ are exact for all maps α and β.

LEMMA 1 (*Richman and Walker* [3, *Theorem* 5]). *If E is right exact, then αE is right exact. If E is left exact, then $E\beta$ is left exact.*

The objective of [3] was to define Ext so that it is a functor. Thus showing that $(\alpha E)\beta = \alpha(E\beta)$ if E is stable is crucial. Now there is always a morphism $\alpha(E\beta) \rightarrow (\alpha E)\beta$. The problem is to get the morphism back. We now construct the morphism $\alpha(E\beta) \rightarrow (\alpha E)\beta$. Consider the diagram

$$\begin{array}{rccccc} E\beta\colon & A & \xrightarrow{f_1} & B_1 & \xrightarrow{g_1} & C' \\ & \| & & \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \beta} \\ E\colon & A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ & {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \lambda} & & \| \\ \alpha E\colon & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C\ . \end{array}$$

Construct $(\alpha E)\beta$:

$$\begin{array}{rccccc} (\alpha E)\beta\colon & A' & \xrightarrow{f_3} & B_3 & \xrightarrow{g_3} & C' \\ & \| & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \beta} \\ \alpha E\colon & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C\ . \end{array}$$

Then $\beta g_1 = g'\lambda\xi$ implies there exists $\delta\colon B_1 \rightarrow B_3$ such that $g_3\delta = g_1$ and $\varphi\delta = \lambda\xi$ (since g_3 is the pullback of g' along β). Thus the diagram

$$\begin{array}{rccccc} E\beta\colon & A & \xrightarrow{f_1} & B_1 & \xrightarrow{g_1} & C' \\ & {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \delta} & & \| \\ (\alpha E)\beta\colon & A' & \xrightarrow{f_3} & B_3 & \xrightarrow{g_3} & C' \end{array}$$

commutes since $g_3(\delta f_1 - f_3\alpha) = g_3\delta f_1 = g_1 f_1 = 0$ and $\varphi(\delta f_1 - f_3\alpha) = \varphi\delta f_1 - \varphi f_3\alpha = \lambda\xi f_1 - f'\alpha = 0$ imply $f_3\alpha = \delta f_1$ (again using the fact that g_3 is a pullback). Now construct $E\beta g_3$ and factor the morphism $(\alpha, \delta, 1)$ through $\alpha(E\beta)$, using the pushout property, to obtain the commutative diagram

$$
\begin{array}{rccccc}
E\beta g_3\colon & A & \xrightarrow{f_0} & X & \xrightarrow{g_0} & B_3 \\
 & \Vert & & \downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle g_3} \\
E\beta\colon & A & \xrightarrow{f_1} & B_1 & \xrightarrow{g_2} & C' \\
 & {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \varphi_1} & & \Vert \\
\alpha(E\beta)\colon & A' & \xrightarrow{f_2} & B_2 & \xrightarrow{g_2} & C' \\
 & \Vert & & \downarrow{\scriptstyle \varphi_2} & & \Vert \\
(\alpha E)\beta\colon & A' & \xrightarrow{f_3} & B_3 & \xrightarrow{g_3} & C'
\end{array}
$$

where $\varphi_2\varphi_1 = \delta$. We now use this diagram to prove

THEOREM 2. *Let $E\colon A \xrightarrow{f} B \xrightarrow{g} C$ be stable exact. Then αE and $E\beta$ are stable exact. Furthermore, $\alpha(E\beta) = (\alpha E)\beta$ for all $\alpha\colon A \to A'$ and $\beta\colon C' \to C$.*

Proof. Now $E\beta$ and αE are exact since E is stable. And since the pull back of a pullback, is a pullback, $(E\beta)\gamma = E(\beta\gamma)$ is exact. Dually $\mu(\alpha E) = (\mu\alpha)E$ is exact. Thus to show that αE is stable requires $(\alpha E)\beta$ to be exact. Dually, the stability of $E\beta$ requires the exactness of $\alpha(E\beta)$. However, $g_2 = \operatorname{coker} f_2$ and $f_3 = \ker g_3$ by Lemma 1. Thus, in order to show that αE is stable, it only remains to show $g_3 = \operatorname{coker} f_3$.

First, we show that φ_2 is a epimorphism. From the diagram, we observe that $g_3\varphi_2\varphi_1\varphi_0 = g_3g_0$, that is, $g_3(\varphi_2\varphi_1\varphi_0 - g_0) = 0$. So there is $\gamma\colon X \to A'$ such that $f_3\gamma = \varphi_2\varphi_1\varphi_0 - g_0$ since $f_3 = \ker g_3$. Then $f_3\gamma f_0 = \varphi_2\varphi_1\varphi_0 f_0 = f_3\alpha$. So $\gamma f_0 = \alpha$ since f_3 is a monomorphism. Now $0 = \varphi_1\varphi_0 f_0 - f_2\alpha = \varphi_1\varphi_0 f_0 - f_2\gamma f_0 = (\varphi_1\varphi_0 - f_2\gamma)f_0$. So there is $\nu\colon B_3 \to B_2$ such that $\nu g_0 = \varphi_1\varphi_0 - f_2\gamma$ since $g_0 = \operatorname{coker} f_0$. Then $\varphi_2\nu g_0 = \varphi_2\varphi_1\varphi_0 - \varphi_2 f_2\gamma = \varphi_2\varphi_1\varphi_0 - f_3\gamma = g_0$. But $\varphi_2\nu g_0 = g_0$ implies $\varphi_2\nu = 1$ since g_0 is a cokernel and hence an epimorphism. Hence, φ_2 is an epimorphism since $\mu\varphi_2 = 0$ implies $\mu = \mu\varphi_2\nu = 0$.

Suppose $\mu f_3 = 0$. Then $\mu\varphi_2 f_2 = 0$. Since $g_2 = \operatorname{coker} f_2$, there is η such that $\eta g_2 = \mu\varphi_2$. And $\eta g_3\varphi_2 = \eta g_2 = \mu\varphi_2$ implies $\eta g_3 = \mu$. And if $\eta' g_3 = \mu$, then $\eta' g_2 = \eta' g_3\varphi_2 = \mu\varphi_2 = \eta g_2$. And $\eta' = \eta$ since g_2 is an epimorphism. Hence $g_3 = \operatorname{coker} f_3$.

Consider the dual diagram

$$\alpha(E\beta) \longrightarrow (\alpha E)\beta \longrightarrow \alpha E \longrightarrow f_3\alpha E\ .$$

Now the dual of the above argument gives φ_2 is a monomorphism and $f_2 = \ker g_2$. Consequently, $E\beta$ is stable.

Recall $\varphi_2\nu = 1$. Then $\varphi_2\nu\varphi_2 = \varphi_2$ implies $\nu\varphi_2 = 1$ since φ_2 is a monomorphism. Hence $(\alpha E)\beta = \alpha(E\beta)$ in the sense that φ_2 is an isomorphism.

Ext in a pre-abelian category can now be pursued as in [3]. That is, the results and proofs of Richman and Walker [3, §3ff.] hold as stated. We conclude with the fact for an exact sequence, stability is equivalent to associativity.

THEOREM 3. *An exact sequence E is stable if and only if* $(\alpha E)\beta = \alpha(E\beta)$ *for all* α *and* β.

Proof. Only associativity implies stability needs to proved. (The following argument is Fred Richman's.) Consider the diagrams

$$\begin{array}{rcccc} E\colon A & \longrightarrow & B & \longrightarrow & C \\ {\scriptstyle\alpha}\big\downarrow & & \big\downarrow & & \big\| \\ \alpha E\colon A' & \longrightarrow & K & \xrightarrow{\varphi} & C \\ \big\| & & \big\uparrow & & \big\uparrow \\ (\alpha E)0\colon A' & \longrightarrow & \ker\varphi & \longrightarrow & 0 \end{array} \qquad \begin{array}{rcccc} E\colon A & \longrightarrow & B & \longrightarrow & C \\ \big\| & & \big\uparrow & & \big\uparrow \\ E0\colon A & = & A & \longrightarrow & 0 \\ {\scriptstyle\alpha}\big\downarrow & & \big\downarrow & & \big\| \\ \alpha(E0)\colon A' & = & A' & \longrightarrow & 0\,. \end{array}$$

So $(\alpha E)0 = \alpha(E0)$ implies A' maps isomorphically onto $\ker\varphi$. So αE is exact. Similarly, $E\beta$ is exact.

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