SUBBASES, CONVEX SETS, AND HYPERSPACES

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We introduce a notion of topological convexity. Main topics are: compactness of convexity structures, continuity of convex closure operators, and characterization of convex sets.

1. **Closed subbases.** It will be assumed that all spaces are T_1. We agree to use the word "subbase" for "closed subbase".

A subbase $\mathscr{S}$ of a space X is called a T_1 *subbase* if for each $S \in \mathscr{S}$ and for each $x \in X - S$ there is an $S' \in \mathscr{S}$ with $x \in S' \subset X - S$. $\mathscr{S}$ is called a *normal subbase* if for each pair of sets $S_1, S_2 \in \mathscr{S}$ with $S_1 \cap S_2 = \varnothing$ there exist $S'_1, S'_2 \in \mathscr{S}$ such that

$$S_1 \subset S'_1 - S'_2; \ S_2 \subset S'_2 - S'_1; \ S'_1 \cup S'_2 = X .$$

S'_1 and S'_2 are then said to *separate* (or: to *screen*) S_1 and S_2. Finally, $\mathscr{S}$ is called a *binary subbase* if each linked system $\mathscr{S}' \subset \mathscr{S}$ (i.e., a subcollection $\mathscr{S}'$ of $\mathscr{S}$ of which any two members meet) satisfies $\cap \mathscr{S}' \neq \varnothing$.

It is well-known that a binary subbase is T_1 (cf. van Mill [9, Lemma 1]), that a space carrying a binary subbase is compact (use Alexander's lemma), and that a space is completely regular iff it admits a normal T_1 (sub)base (cf. Frink [6, Thm. 1]; de Groot and Aarts [1, Thm. 2]).

2. **Topological convexity structures.** The analysis of convex sets in Euclidean space has led to the introduction of axiomatic convexity theory. One of the main purposes of this theory is to investigate in an abstract setting the relationship between various convexity invariants, modelled after famous theorems of Caratheodory, Helly and Radon on convex sets of R^n.

In this paper we will study convexity structures compatible with a topological structure. Our purpose is not to study the above mentioned invariants, as we do not expect the introduction of a topology to give rise to new relationships in general. Instead, we are mainly concerned with the interaction between the two structures. Some of the main results are summed up at the end of this section. More results and applications can be found in [13], [14], [15], and [18].

In [7, p. 471], Kay and Womble define a *convexity (structure)* as a pair $(X, \mathscr{C})$, where $\mathscr{C}$ is a collection of subsets of X, such that $\varnothing, X \in \mathscr{C}$ and $\cap \mathscr{C}' \in \mathscr{C}$ for each nonempty family $\mathscr{C}' \subset \mathscr{C}$. $\mathscr{C}$ is also called a *convexity structure for (on)* X, and the members of $\mathscr{C}$

are said to be the *convex sets of the structure*, or $\mathscr{C}$-*convex sets*. For our purposes, it is more convenient to omit the empty set, and consequently, to require $\cap \mathscr{C}' \in \mathscr{C}$ only when $\cap \mathscr{C}' \neq \varnothing$.

If a convexity $\mathscr{C}$ is to be compatible with the topological structure of X then it is natural to require that the members of $\mathscr{C}$ be closed, and in fact that $\mathscr{C}$ be a subbase, so that there are sufficiently many convex sets. This leads to the following definition:

DEFINITION 2.1. *A topological convexity structure* (briefly: a *convexity*) is a pair $(X, \mathscr{C})$, where X is a topological space, and $\mathscr{C}$ is both a convexity structure on the underlying set of X and a subbase for the topology of X. As most spaces considered below will be compact, the latter condition is not very restrictive.

This notion of "subbase convexity" arose from investigations on the Lefschetz fixed point property of superextensions (van de Vel [17]), and it yields an appropriate background for so-called interval structures (van Mill and Schrijver [12]) and for subbasic closure operations on supercompact spaces (van Douwen and van Mill [3]):

DEFINITION 2.2. Let $(X, \mathscr{C})$ be a topological convexity structure, and let $A \subset X$ be a nonempty (closed) set. *The convex closure* of A *relative to* $\mathscr{C}$ is defined to be the set

$$I_{\mathscr{C}}(A) = \cap \{C \,|\, A \subset C \in \mathscr{C}\} \,.$$

See [3, 2.4]. If no confusion can arise, we write I for $I_{\mathscr{C}}$. In case $A = \{x_1, x_2\}$, a two point set, we also write $I(x_1, x_2)$ instead of $I(\{x_1, x_2\})$. This set is called *the interval between x_1 and x_2*, see [12, §2].

EXAMPLES 2.3. The above definitions cover two well-known concepts of convexity in topology.

(i) Let X be a totally ordered space. A nonempty closed set of X is *order-convex* iff it is convex relative to the subbase of all closed segments. An interval in the above sense is an *order-theoretic interval*.

(ii) Let X be a compact convex subset of a locally convex linear space. Then the collection of all closed linearly convex sets in X forms a subbase. Intervals are simply *line-segments*. This convexity is normal, as can be seen from a Hahn-Banach theorem (Rudin [16, p. 58]).

In the sequel we shall concentrate on compact spaces and on convexities with an abundancy of convex sets. We make "abundancy" precise by requiring the convexity to be normal. The normality

condition can be compared with the separation of disjoint convex closed sets by a hyperplane.

For normal binary convexities we shall prove a.o. the following results.

THEOREM 2.4. *Let $\mathscr{C}$ be a normal binary convexity structure on the space X. Then a nonempty closed set $C \subset X$ is $\mathscr{C}$-convex iff $I_\centerdot(x, x') \subset C$ for each $x, x' \in X$.*

THEOREM 2.5. *Let $\mathscr{C}$ be a normal binary convexity structure on X. Then each $\mathscr{C}$-convex set is a retract of X.*

Many other convexity structures satisfy the characterization of convex sets as in 2.4, e.g., linear convexity structures. Theorem 2.5 parallels a well-known retraction property of convex sets in locally convex linear spaces (cf. Dugundji [4, Cor. 4.2]). Proofs are given in §3, together with some other results.

If $\mathscr{C}$ is a topological convexity on X, then the collection $\mathscr{C}$ can be topologized by considering $\mathscr{C}$ as a subspace of $H(X)$, the hyperspace of X. It is then natural to ask when $\mathscr{C}$ is closed in $H(X)$, and when the convex closure operator $I_\mathscr{C} : H(X) \to \mathscr{C}$ is continuous. Our main result is that these questions are equivalent for compact X:

THEOREM 2.6. *If X is a compact space, and if $\mathscr{C}$ is a normal convexity on X, then the following assertions are equivalent:*
(i) *$\mathscr{C}$ is compact;*
(ii) *$I_\mathscr{C} : H(X) \to \mathscr{C}$ is continuous;*
(iii) *$\mathscr{C}$ is a retract of $H(X)$.*

For other equivalent conditions, and for a proof of 2.6, see §4.

Each subbase $\mathscr{S}$ of a space X generates a topological convexity, namely the family

$$\mathscr{S}^{\wedge} = \{ \cap \cdot \mathscr{A} \mid \varnothing \neq \mathscr{A} \subset \mathscr{S}, \ \cap \mathscr{A} \neq \varnothing \} \cup \{X\} \, .$$

This can be used to produce binary convexities, as follows from our next observation, the simple proof of which is omitted:

FACT 2.7. *If $\mathscr{S}$ is a normal binary subbase of X, then so is $\mathscr{S}^{\wedge}$.* Unfortunately "binary" is essential in the following sense: it need not be true that $\mathscr{S}^{\wedge}$ is normal if $\mathscr{S}$ is a normal subbase for X, see 5.3 below.

3. Binary convexity structures. Binary subbases were intro-

duced by De Groot as a part of (super)compactification theory. They can be interpreted as *topological convexity structures of Helly number* 2 on compact spaces (the Helly number of a set-theoretic convexity $(X, \mathscr{C})$ is the least natural number n with the property that for each finite $\mathscr{C}' \subset \mathscr{C}$, $\cap \mathscr{C}' \neq \varnothing$ whenever each n-tuple of members of $\mathscr{C}'$ has a nonempty intersection).

As we shall show in §4, every topological convexity can be "modelled" into a binary one, and this modelling will be used to prove our Theorem 2.6.

We begin with two auxiliary results.

LEMMA 3.1. *Let* $\mathscr{C}$ *be a normal binary convexity on* X, *let* $A \subset X$ *be nonempty and closed, and let* $x \in X$. *Then*
  (i) *the set* $\bigcap_{a \in A} I_{\mathscr{C}}(x, a) \cap I_{\mathscr{C}}(A)$ *contains exactly one point;*
  (ii) *if* $x \in I_{\mathscr{C}}(A)$, *then* $\{x\} = \bigcap_{a \in A} I_{\mathscr{C}}(x, a)$.

Proof of (i). The collection

$$\mathscr{C}' = \{I_{\mathscr{C}}(x, a) \mid a \in A\} \cup \{I_{\mathscr{C}}(A)\} \subset \mathscr{C}$$

is obviously linked. Hence, as $\mathscr{C}$ is binary, $\cap \mathscr{C}' \neq \varnothing$. Assume that $p \neq q$ are in $\cap \mathscr{C}'$. Using the fact that $\mathscr{C}$ is a normal T_1 subbase and that X is a T_1 space, we find C_0, C_1 in $\mathscr{C}$ such that

$$p \in C_0 - C_1 \ ; \quad q \in C_1 - C_0 \ ; \quad C_0 \cup C_1 = X \ .$$

We may assume that $x \in C_0$. If $A \cap C_0 \neq \varnothing$, then $I_{\mathscr{C}}(x, a) \subset C_0$ for some $a \in A$, contradicting that $q \in C_0$. Hence, $A \subset C_1$, and consequently $I_{\mathscr{C}}(A) \subset C_1$, contradicting that $p \notin C_1$.

Proof of (ii). If $x \in I_{\mathscr{C}}(A)$, then $I_{\mathscr{C}}(x, a) \subset I_{\mathscr{C}}(A)$ for each $a \in A$, whence

$$x \in \bigcap_{a \in A} I_{\mathscr{C}}(x, a) = \bigcap_{a \in A} I_{\mathscr{C}}(x, a) \cap I_{\mathscr{C}}(A) \ . \qquad \square$$

The next result is a reformation of [12, Thm. 2.11]:

LEMMA 3.2. *Let* $\mathscr{S}$ *be a subbase for the compact space* X. *Then* $\mathscr{S}$ *is binary iff there is a function*

$$I: X \times X \longrightarrow \mathscr{P}(X) \quad (the \ power \ set \ of \ X)$$

with the following properties:
  (i) *for each* $u, u', x, x' \in X$: $u, u' \in I(x, x')$ *iff* $I(u, u') \subset I(x, x')$;
  (ii) *for each* $x_1, x_2, x_3 \in X$: $I(x_1, x_2) \cap I(x_2, x_3) \cap I(x_3, x_1) \neq \varnothing$;
  (iii) *for each* $S \in \mathscr{S}$ *and* $x, x' \in S$, $I(x, x') \subset S$. $\qquad \square$

3.3. Proof of Theorem 2.4. Necessity is obvious. To prove sufficiency, let $C_0 \subset X$ be a nonempty closed set such that $I_{\mathscr{C}}(x, x') \subset C_0$ for each $x, x' \in C_0$. Define

$$I \colon C_0 \times C_0 \longrightarrow \mathscr{P}(C_0)$$

by $I(x, x') = I_{\mathscr{C}}(x, x')$, and let

$$\mathscr{C} \mid C_0 = \{C \cap C_0 \mid C \in \mathscr{C}\} \,.$$

Then the subbase $\mathscr{C} \mid C_0$ of C_0 and the map I satisfy the conditions (i), (ii) and (iii) of Lemma 3.2, proving that $\mathscr{C} \mid C_0$ is binary.

Let $p \in I_{\mathscr{C}}(C_0)$, and let

$$\mathscr{C}' = \{I_{\mathscr{C}}(p, c) \cap C_0 \mid c \in C_0\} \,.$$

Then $\mathscr{C}' \subset \mathscr{C} \mid C_0$, and for each pair of points $c_1, c_2 \in C_0$,

$$\varnothing \neq I_{\mathscr{C}}(p, c_1) \cap I_{\mathscr{C}}(p, c_2) \cap I_{\mathscr{C}}(c_1, c_2) \subset I_{\mathscr{C}}(p, c_1) \cap I_{\mathscr{C}}(p, c_2) \cap C_0$$

by the binarity of $\mathscr{C}$ and by our assumption on C_0. This shows that $\mathscr{C}'$ is a linked system, whence by the binarity of $\mathscr{C} \mid C_0$, $\cap \mathscr{C}' \neq \varnothing$. However,

$$\cap \mathscr{C}' = \bigcap_{c \in C} I_{\mathscr{C}}(p, c) \cap C_0 \subset \bigcap_{c \in C} I(p, c) = \{p\}$$

by Lemma 3.1(ii), proving that $p \in C_0$. It follows that $C_0 = I_{\mathscr{C}}(C_0) \in \mathscr{C}$. $\qquad\square$

Another characterization of (normal binary) convexity on super-extensions, using partial orderings, can be found in [17, Lemma 2.2].

Our next result is a generalization of [19, III, 2.8], of [11, Thm. 2.2], and of [17, Thm. 2.3]. We need the following notation: Let $H(X)$ be the hyperspace of X. If $A_1, \cdots, A_n$ are nonempty subsets of X, then we write

$$\langle A_1, \cdots, A_n \rangle = \left\{C \mid C \in H(X), C \subset \bigcup_{i=i}^{n} A_i, C \cap A_i \neq \varnothing \text{ for } 1 \leqq i \leqq n\right\} \,.$$

The topology of $H(X)$ is generated by the open base

$$\{\langle 0_1, \cdots, 0_n \rangle \mid n \in N, 0_1, 0_2, \cdots, 0_n \text{ nonempty and open}\} \,,$$

or, equivalently, by the open subbase

$$\{\langle 0 \rangle \mid 0 \text{ nonempty open}\} \cup \{\langle 0, X \rangle \mid 0 \text{ nonempty open}\} \,.$$

3.4. Nearest point mapping theorem. *Let $\mathscr{C}$ be a normal binary convexity on X. Then for each $x \in X$ and $C \in \mathscr{C}$, the set*

$$P(x,\, C) = \bigcap_{c \in C} I_{\mathscr{C}}\{x,\, c\} \cap C$$

contains exactly one point, and the function

$$p\colon X \times \mathscr{C} \longrightarrow X\,,$$

defined by $\{p(x,\, C)\} = P(x,\, C)$, *is continuous.*

Proof. That $P(x,\, C)$ is a singleton follows from Lemma 3.1(i) and from $C = I_{\mathscr{C}}(C)$. Let $C \in \mathscr{C}$, and assume $(x,\, D) \in (X \times \mathscr{C}) - p^{-1}(C)$. Then $p(x,\, D) \notin C$, and since $\mathscr{C}$ is normal T_1, there exist $C_0,\, C_1$, in $\mathscr{C}$ such that

$$p(x,\, D) \in C_0 - C_1\,;\quad C \subset C_1 - C_0\,;\quad C_0 \cup C_1 = X\,.$$

In particular, $D \cap (X - C_1) \neq \varnothing$ since $p(x,\, D) \in D - C_1$. If $D \not\subset X - C_1$, then $x \notin C_1$. Indeed, there is a point $d \in D \cap C_1$, and $x \in C_1$ then implies that

$$p(x,\, D) \in I_{\mathscr{C}}(x,\, d) \subset C_1\,,$$

contradicting our assumption. Define

$$
\begin{aligned}
V &= X \times (\langle X - C_1 \rangle \cap \mathscr{C})\,, &\quad \text{if} \quad D \subset X - C_1 \quad (1)\\
&= (X - C_1) \times (\langle X - C_1,\, X \rangle \cap \mathscr{C})\,, &\quad \text{if} \quad D \not\subset X - C_1 \quad (2)\,.
\end{aligned}
$$

Then V is a neighborhood of $(x,\, D)$ in $X \times \mathscr{C}$ which does not meet $p^{-1}(C)$. For take $(x',\, D') \in V$. If V is as in (1), then $D' \subset X - C_1$, whence

$$p(x',\, D') \in X - C_1 \subset X - C\,.$$

If V is as in (2), then fix $d' \in D' \cap (X - C_1)$. We find

$$\{d',\, x'\} \subset X - C_1 \subset C_0\,,$$

whence $p(x',\, D') \in I_{\mathscr{C}}(d',\, x') \subset C_0 \subset X - C$. This shows that $p^{-1}(C)$ is closed. As $\mathscr{C}$ is a subbase for X, it follows that p is continuous. $\square$

The mapping p is truly a "nearest point" map: see [17, 2.2, 2.3] for the behavior of p with respect to certain orderings on superextensions, and [13, Thm. 3.3] for the metric behavior of p.

This map has been discovered in various steps, each leading to rather deep consequences. Van Mill used it to prove that a certain subspace of the superextension of [0, 1] is homeomorphic to the pseudo-boundary of the Hilbert cube, [11, Thm. 3.4], and Van de Vel used it to prove the Lefschetz fixed point property for spaces with binary normal subbases, [17, Cor. 3.7].

3.5.   Proof of Theorem 2.5.   Let $\mathscr{C}$ be a normal binary convexity on X, and let $C \in \mathscr{C}$. Then then the restriction

$$p(-, C): X \longrightarrow X$$

of the nearest map p obviously retracts X onto C. $\square$

Let $\mathscr{C}$ be a topological convexity structure on the space X. We let $L(X, \mathscr{C})$ denote the subspace of the hyperspace $H(\mathscr{C})$, consisting of all linked systems $\mathscr{L} \subset \mathscr{C}$ which are closed and nonempty.

THEOREM 3.6.   *Let $\mathscr{C}$ be a normal binary convexity on X. Then the intersection operator*

$$L(X, \mathscr{C}) \longrightarrow \mathscr{C} ,$$

sending $\mathscr{L} \in L(X, \mathscr{C})$ to $\cap \mathscr{L}$, is continuous.

Proof. X is compact and completely regular (see §1), and hence normal. Notice that $\cap \mathscr{L} \neq \varnothing$ for each $\mathscr{L} \in L(X, C)$ by binarity. Fix a nonempty open set $0 \subset X$, and let $\mathscr{L} \in L(X, \mathscr{S})$. We have to consider two cases for a continuity proof:

(i) $\cap \mathscr{L} \subset 0$. X being compact, there exist $L_1, \cdots, L_n \in \mathscr{L}$ such that $\bigcap_{i=1}^{n} L_i \subset 0$. By the normality of X, there exist open sets $0_1, \cdots, 0_n$ such that $L_i \subset 0_i (1 \leq i \leq n)$ and $\bigcap_{i=1}^{n} 0_i \subset 0$.   Then

$$V = \langle \langle 0_1 \rangle \cap \mathscr{C}, \cdots, \langle 0_n \rangle \cap C \rangle \cap L(X, \mathscr{C})$$

is a neighborhood of $\mathscr{L}$ which is easily seen to be mapped into $\langle 0 \rangle \cap \mathscr{C}$.

(ii) $\cap \mathscr{L} \cap 0 \neq \varnothing$.   Fix a point $x \in \cap \mathscr{L} \cap 0$. As $\mathscr{C}$ is a normal T_1 subbase, closed under (nonempty) intersections, there exists a $C \in \mathscr{C}$ such that

$$x \in \mathrm{int}\, C \subset C \subset 0 .$$

Then the set

$$V = \langle \langle C, X \rangle \cap \mathscr{C} \rangle \cap L(X, \mathscr{C})$$

is a neighborhood of $\mathscr{L}$, and V is mapped into $\langle 0, X \rangle \cap \mathscr{C}$. Indeed, if $\mathscr{L}' \in V$ then $L' \cap C \neq \varnothing$ for each $L' \in \mathscr{L}'$, whence $\mathscr{L}' \cup \{C\}$ is a linked system. $\mathscr{C}$ being binary,

$$\varnothing \neq \cap \mathscr{L}' \cap C \subset 0 ,$$

proving that $\cap \mathscr{L}' \in \langle 0, X \rangle \cap \mathscr{C}$. $\square$

Continuity of $A \cap B$ in the variables A and B (or even in one

variable at the time) seems to be fairly exceptional, as can be seen from the next example using only "nice" sets:

Let X be the unit 2-cell, regarded as a subset of the complex numbers. For each $t \in [0, \pi]$ we let $F_t \in H(X)$ denote the line-segment joining 0 with e^{it}. The resulting mapping $F: [0, \pi] \to H(X)$, sending t to F_t, is obviously continuous. The map

$$G: F([0, \pi]) \longrightarrow H(X) ,$$

assigning to F_t the set $F_\pi \cap F_t$, is not continuous, since the image of $G \circ F$ consists of the two points $\{0\}$ and F_π of $H(X)$.

We shall now work towards the continuity of the convex closure operator for normal binary convexities. A lemma is needed:

LEMMA 3.7. *Let $\mathscr{C}$ be a normal binary convexity for X, and let $n \geq 2$. For each $(n + 1)$-tuple $(x, x_1, \cdots, x_n) \in X^{n+1}$ the set*

$$F(x, x_1, \cdots, x_n) = \bigcap_{i=1}^{n} I_{\mathscr{C}}(x, x_i) \cap I_{\mathscr{C}}(x_1, \cdots, x_n)$$

contains exactly one point, and the function

$$f: X^{n+1} \longrightarrow X ,$$

defined by $\{f(x, x_1, \cdots, x_n)\} = F(x, x_1, \cdots, x_n)$, is continuous.

Proof. The first part of the statement follows from Lemma 3.1 (i). To prove continuity, let $C \in \mathscr{C}$ and let $(x, x_1, \cdots, x_n) \in X^{n+1} - f^{-1}(C)$. Then

$$\bigcap_{i=1}^{n} I_{\mathscr{C}}(x, x_i) \cap I_{\mathscr{C}}(x_1, \cdots, x_n) \cap C = \varnothing ,$$

and by the binarity of $\mathscr{C}$, we have either that $I_{\mathscr{C}}(x, x_i) \cap C = \varnothing$ for some $i \in \{1, \cdots, n\}$, or that $I_{\mathscr{C}}(x_1, \cdots, x_n) \cap C = \varnothing$. In the first case, it follows from the normality of $\mathscr{C}$ that there is a $C_0 \in \mathscr{C}$ with

$$I (x, x_i) \subset \text{int } C_0 \subset C_0 \subset X - C .$$

Let $\pi_j: X^{n+1} \to X$ denote the projection onto the jth-coordinate ($j = 0, \cdots, n$). Then

$$0 = \pi_0^{-1}(\text{int } C_0) \cap \pi_i^{-1}(\text{int } C_0)$$

is a neighborhood of $(x, x_1, \cdots, x_n)$ which does not meet $f^{-1}(C)$. For if $(y, y_1, \cdots, y_n) \in 0$, then $\{y, y_i\} \subset \text{int } C_0 \subset C_0$, whence

$$f(y, y_1, \cdots, y_n) \in I_{\mathscr{C}}(y, y_i) \subset C_0 \subset X - C .$$

In the second case, one can proceed in the same way. First, choose $C_0 \in \mathscr{C}$ such that

$$I_{\mathscr{C}}(x_1, \cdots, x_n) \subset \text{int } C_0 \subset C_0 \subset X - C .$$

Then put

$$0 = \bigcap_{i=1}^{n} \pi_i^{-1}(\text{int } C_0) .$$

This set is a neighborhood of $(x, x_1, \cdots, x_n)$ not meeting $f^{-1}(C)$. $\square$

THEOREM 3.8. *Let $\mathscr{C}$ be a normal binary convexity on X. Then the convex closure map*

$$I_{\mathscr{C}}\colon H(X) \longrightarrow \mathscr{C}$$

is a retraction of $H(X)$ onto $\mathscr{C}$.

Proof. We use the open subbase of $\mathscr{C} \subset H(X)$, induced by the sets of type $\langle 0 \rangle$ or $\langle 0, X \rangle$, where 0 is open in X. Fix an open set $0 \subset X$, and assume that $I_{\mathscr{C}}(A) \subset 0$ for a given $A \in H(X)$. As X is compact, and as $\mathscr{C}$ is a normal subbase which is closed under intersection, there exist $C_1, \cdots, C_n \in \mathscr{C}$ such that

$$X - 0 \subset \bigcup_{i=1}^{n} C_i \subset X - I_{\mathscr{C}}(A) .$$

$\mathscr{C}$ being normal, there exist $D_i, D_1' \in \mathscr{C}$ with

$$C_i \subset D_1' - D_i ; \quad I_{\mathscr{C}}(A) \subset D_i - D_1' ; \quad D_i \cup D_1' = X(1 \leq i \leq n)$$

Let $D = \bigcap_{i=1}^{n} D_i$. Then $D \in \mathscr{C}$ and

$$I_{\mathscr{C}}(A) \subset \text{int } D \subset D \subset \bigcap_{i=1}^{n} (X - C_i) \subset 0 .$$

For each $A' \in \langle \text{int } D \rangle$, we then have $I_{\mathscr{C}}(A') \subset D \subset 0$.

Assume next that $I_{\mathscr{C}}(A) \cap 0 \neq \varnothing$, and fix a point $p \in I_{\mathscr{C}}(A) \cap 0$. By Lemma 3.1 (ii)

$$\{p\} = \bigcap_{a \in A} I_{\mathscr{C}}(a, p) \subset 0 .$$

As X is compact, and as 0 is open, there exist already finitely many $a_i \in A(1 \leq i \leq n)$ such that

$$\bigcap_{i=1}^{n} I_{\mathscr{C}}(a_i, p) \subset 0 .$$

Consequently, with the notation of Lemma 3.7,

$$\{f(p, a_1, \cdots, a_n)\} = \bigcap_{i=1}^{n} I_{\mathscr{C}}(a_i, p) \cap I_{\mathscr{C}}(a_1, \cdots, a_n) \subset 0 .$$

By the continuity of f there exist open neighborhoods V_i of a_i, $1 \leqq i \leqq n$, such that for each n-tuple $(a_1', \cdots, a_n') \in \prod_{i=1}^{n} V_i$,

$$f(p, a_1', \cdots, a_n') \in 0 .$$

The set $\langle V_1, \cdots, V_n, X \rangle$ is a neighborhood of $A \in H(X)$, which we now show to be mapped into $\langle 0, X \rangle$. Let $A' \in \langle V_1, \cdots, V_n, X \rangle$ and choose $a_i' \in A' \cap V_i, 1 \leqq i \leqq n$.

Then

$$\varnothing \neq \bigcap_{i=1}^{n} I(a_i', p) \cap I_{\mathscr{C}}(a_1', \cdots, a_n') \subset 0 \cap I_{\mathscr{C}}(A') ,$$

since $\{a_1', \cdots, a_n'\} \subset A'$, and $(a_1', \cdots, a_n') \in \prod_{i=1}^{n} V_i$. In particular, $0 \cap I_{\mathscr{C}}(A') \neq \varnothing$. This proves the continuity of $I_{\mathscr{C}}$. $I_{\mathscr{C}}$ is then a retraction, since $I_{\mathscr{C}}(C) = C$ for each $C \in \mathscr{C}$. □

Theorem 3.6 may suggest another way to prove the continuity of convex closure: if $\mathscr{C}$ is a normal binary convexity on X, then $I_{\mathscr{C}}$ equals the composition

$$H(X) \xrightarrow{F} L(X, \mathscr{C}) \xrightarrow{G} \mathscr{C} ,$$

where

$$F(A) = \{C \mid A \subset C \in \mathscr{C}\} \quad (A \in H(X)) ,$$
$$G(\mathscr{L}) = \cap \mathscr{L} \quad (\mathscr{L} \in L(X, \mathscr{C})) .$$

Difficulties arise when proving the continuity of F: there is no nice open or closed subbase available for $L(X, \mathscr{C})$.

4. Compact convexity structures. As a consequence of Theorem 3.8, a normal binary convexity $\mathscr{C}$ on a space X is a compact subspace of $H(X)$. The class of such "compact" convexities appears to be considerably larger than the rather restrictive class of normal binary convexities: it contains for instance the linear convexities on compact convex sets in locally convex linear spaces (cf. §5).

We begin with modelling subbases (or convexities) into binary ones.

4.1. Superextensions. Let $\mathscr{S}$ be a closed subbase of X. Then $\lambda(X, \mathscr{S})$ denotes the set of all maximal linked systems in $\mathscr{S}$ (see §1), equipped with the topology generated by the closed subbase

$$\mathscr{S}^{+} = \{S^{+} \mid S \in \mathscr{S}\}$$

where $S^{+} = \{\mathscr{L} \mid \mathscr{L} \in \lambda(X, \mathscr{S}), S \in \mathscr{L}\}$. The resulting space is called *the superextension of X relative to $\mathscr{S}$.* The subbase $\mathscr{S}^{+}$ is automatically binary. It is easy to see that for each $S_1, S_2 \in \mathscr{S}$:

$$S_1 \cup S_2 = X \quad \text{iff} \quad S_1^+ \cup S_2^+ = \lambda(X, \mathscr{S}) \; ;$$
$$S_1 \cap S_2 = \varnothing \quad \text{iff} \quad S_1^+ \cap S_2^+ = \varnothing \; ;$$
$$S_1 \subset S_2 \qquad \text{iff} \quad S_1^+ \subset S_2^+ \; .$$

In this way $\mathscr{S}^+$ can be regarded as a kind of a model of $\mathscr{S}$. The above equivalences imply that $\mathscr{S}$ is normal iff $\mathscr{S}^+$ is normal. Being a Wallman-type extension, the space $\lambda(X, \mathscr{S})$ is automatically T_1. Normality of $\mathscr{S}(\mathscr{S}^+)$ implies that $\lambda(X, \mathscr{S})$ is Hausdorff.

If X is a (T_1) space with a T_1 subbase $\mathscr{S}$, then there is an embedding $i \colon X \to \lambda(X, \mathscr{S})$ defined by

$$i(x) = \{S \mid x \in S \in \mathscr{S}\} \, , \qquad x \in X \, .$$

In the sequel, we shall identify X with the subspace $i(X)$ of $\lambda(X, \mathscr{S})$.

For detailed information on superextension theory, the reader can consult Verbeek [19]. A simple example that $\lambda(X, \mathscr{S}) \neq \lambda(X, \mathscr{S}^\frown)$ can be found in [9, p. 13]. It is very obvious that $\mathscr{S}^+ \neq (\mathscr{S}^+)^\frown$ in general.

We now come to a proof of Theorem 2.6. For proof-technical reasons, we give an extended formulation of 2.6, including three more equivalent conditions.

THEOREM 4.2. *Let X be a compact space, and let $\mathscr{C}$ be a normal convexity on X. Then the following conditions are equivalent.*

(i) $\mathscr{C}$ *is compact*;

(ii) $I_{\mathscr{C}} \colon H(X) \to \mathscr{C}$ *is continuous*;

(iii) $\mathscr{C}$ *is a retract of $H(X)$*;

(iv) *the mapping $\mathscr{C} \to \mathscr{C}^+$, sending $C \in \mathscr{C}$ to $C^+ \in \mathscr{C}^+$, is a homeomorphism*;

(v) *the collection*

$$\{\langle C \rangle \cap \mathscr{C} \mid C \in \mathscr{C}\} \cup \{\langle C, X \rangle \cap \mathscr{C} \mid C \in \mathscr{C}\}$$

is a normal T_1 subbase for the space $\mathscr{C}$;

(vi) *the restriction of $I_{\mathscr{C}}$ to the space of all finite subsets of X is continuous, and a nonempty closed set $A \subset X$ is $\mathscr{C}$-convex iff for each finite $F \subset A$, $I_{\mathscr{C}}(F) \subset A$.*

The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) are obvious, since $H(X)$ is compact (cf. Michael [8, Thm. 4.2]). We shall prove the following statements: (i) $\Rightarrow$ (ii) and (iv); (iv) $\Rightarrow$ (i) $\Rightarrow$ (v) $\Rightarrow$ (iv); (i) $\Leftrightarrow$ (vi).

Proof of (i) $\Rightarrow$ (ii) *and* (iv). Recall that X is regarded as a subspace of $\lambda(X, \mathscr{C})$, which at present is closed by the compactness of X. Let g denote the restriction of $I_{\mathscr{C}^+}$ to $H(X) \subset H(\lambda(X, \mathscr{C}))$,

and let h denote the restriction of g to $\mathscr{C} \subset H(X)$. The functions g and h are continuous by Theorem 3.8. For each $A \in H(X)$ we have

$$I_{\mathscr{C}} + (A) = \cap \{C^+ \,|\, C \in \mathscr{C}, \, A \subset C^+\} = \cap \{C^+ \,|\, A \subset C \in \mathscr{C}\} \,,$$

and it easily follows that

$$I_{\mathscr{C}+}(A) = (I_{\mathscr{C}}(A))^+ \,.$$

This shows that $g(H(X)) = h(\mathscr{C}) \subset \mathscr{C}^+$.

Regarding h as a map $\mathscr{C} \to \mathscr{C}^+$, one can easily see that this map has a two-sided inverse, which is the mapping $\mathscr{C}^+ \to \mathscr{C}$ sending C^+ to $C = C^+ \cap X$. As $\mathscr{C}$ is compact and Hausdorff, it follows that $h \colon \mathscr{C} \to \mathscr{C}^+$ is a homeomorphism. We then obtain a well-defined continuous mapping

$$h^{-1}g \colon H(X) \longrightarrow \mathscr{C}^+ \longrightarrow \mathscr{C} \,,$$

which equals $I_{\mathscr{C}}$.

Proof of (iv) $\Rightarrow$ (i). Assume that $\mathscr{C}$ and $\mathscr{C}^+$ are homeomorphic spaces. We show that $\mathscr{C}^+$ is a closed subset of $(\mathscr{C}^+)\hat{\;}$. The latter being compact, we then find that $\mathscr{C}^+$, and hence also $\mathscr{C}$, is compact. Let $A \in (\mathscr{C}^+)\hat{\;} - \mathscr{C}^+$. If $A \cap X = \varnothing$, then $\langle \lambda(X, \mathscr{C}) - X \rangle$ is a neighborhood of A not meeting $\mathscr{C}^+$ (since each $C^+ \in \mathscr{C}^+$ satisfies $C^+ \cap X = C \neq \varnothing$). So assume that $A \cap X \neq \varnothing$, and let $\mathscr{A} \subset \mathscr{C}^+$ be such that $A = \cap \mathscr{A}$. Then

$$A \cap X = \cap \{C^+ \,|\, C^+ \in \mathscr{A}\} \cap X = \cap \{C \,|\, C^+ \in \mathscr{A}\} \in \mathscr{C} \,.$$

Hence we can consider $(A \cap X)^+$. As we noticed in the proof of (i) $\Rightarrow$ (ii) and (iv), $(A \cap X)^+ = I_{\mathscr{C}+}(A \cap X)$, whence by the convexity of $A \subset \lambda(X, \mathscr{C})$, $(A \cap X)^+ \subset A$. As $A \notin \mathscr{C}^+$, we find that $(A \cap X)^+ \neq A$, and hence there must be a maximal linked system

$$\mathscr{L} \in A - (A \cap X)^+ \,.$$

Fix $L \in \mathscr{L} \subset \mathscr{C}$, such that $L \cap (A \cap X) = \varnothing$. $\mathscr{C}$ being normal, we can find $C_0, C_1 \in \mathscr{C}$ such that

$$L \subset X - C_0 \,; \quad A \cap X \subset X - C_1 \,; \quad (X - C_0) \cap (X - C_1) = \varnothing \,.$$

Observe that $\mathscr{L} \in L^+ \cap A$, whence $L^+ \cap A \neq \varnothing$, and that

$$L^+ \subset \lambda(X, \mathscr{C}) - C_0^+$$

as one can easily see from $L \subset X - C_0$. Hence, the collection

$$V = \langle \lambda(X, \mathscr{C}) - C_0^+, \lambda(X, \mathscr{C}) \rangle \cap \langle \lambda(X, \mathscr{C}) - C_1^+, \lambda(X, \mathscr{C}) - X \rangle$$

is a neighborhood of A. We show that $V \cap \mathscr{C}^+ = \varnothing$. Indeed, assume that $D^+ \in V \cap \mathscr{C}^+$. We find that

$$D = D^+ \cap X \subset ((\lambda(X, \mathscr{C}) - C_1^+) \cup (\lambda(X, \mathscr{C}) - X)) \cap X = X - C_1 ,$$

$$\varnothing \neq D^+ \cap (\lambda(X, \mathscr{C}) - C_0^+), \quad \text{whence} \quad \varnothing \neq D \cap (X - C_0) .$$

This contradicts with the fact that $X - C_0$ and $X - C_1$ are disjoint.

Proof of (i) $\Rightarrow$ (v). First notice that for each $C \in \mathscr{C}$,

$$H(X) - \langle C \rangle = \langle X - C, X \rangle ;$$
$$H(X) - \langle C, X \rangle = \langle X - C \rangle ,$$

and hence that the sets $\langle C \rangle \cap \mathscr{C}$, $\langle C, X \rangle \cap \mathscr{C}$ are closed in $\mathscr{C}$. Assume that $\mathscr{C}$ is compact, let $\mathscr{B} \subset \mathscr{C}$ be closed, and let $D \in \mathscr{C} - \mathscr{B}$. For each $B \in \mathscr{B}$ we have either that $B \not\subset D$, or that $D \not\subset B$.

If $B \not\subset D$, then choose $x \in B - D$, and fix $C_B, C_D \in \mathscr{C}$ such that

$$x \in C_B - C_D ; \quad B \subset C_D - C_B ; \quad C_D \cup C_B = X .$$

In particular, $x \in B \cap \operatorname{int} C_B$, and it follows that $\langle C_B, X \rangle$ is a neighborhood of B which does not contain D.

If $D \not\subset B$, then choose $y \in D - B$, and fix $C_B, C_D \in \mathscr{C}$ such that

$$y \in C_D - C_B ; \quad B \subset C_B - C_D ; \quad C_D \cup C_B = X .$$

We then find that $\langle C_B \rangle$ is a neighborhood of B, and that $D \notin \langle C_B \rangle$. By the compactness of $\mathscr{B}$, a finite number of the selected neighborhoods of type $\langle C_B, X \rangle$ or $\langle C_B \rangle$ suffices to cover $\mathscr{B}$. It easily follows that the sets of type $\langle C \rangle \cap \mathscr{C}$ or $\langle C, X \rangle \cap \mathscr{C}$, $C \in \mathscr{C}$, form a subbase for the space $\mathscr{C}$.

The proof that this subbase is normal and T_1 is a routine argument, involving the corresponding assumptions on $\mathscr{C}$. We leave this to the interested reader.

Proof of (i) $\Rightarrow$ (vi). We only have to show that if $A \in H(X)$ is such that for each finite $F \subset A$, $I_{\mathscr{C}}(F) \subset A$, then $A \in \mathscr{C}$. Let $\langle 0_1, \cdots, 0_n \rangle$ be a basic open neighborhood of A. Fix $a_i \in A \cap 0_i$, $1 \leq i \leq n$, and let $F = \{a_1, \cdots, a_n\}$. By assumption,

$$I_{\mathscr{C}}(F) \subset A \subset \bigcup_{i=1}^{n} 0_i ; \quad I_{\mathscr{C}}(F) \cap 0_i \neq \varnothing \quad \text{for} \quad 1 \leq i \leq n ,$$

and hence $\langle 0_1, \cdots, 0_n \rangle$ meets $\mathscr{C}$. It follows that

$$A \in \text{closure of} \quad \mathscr{C} = \mathscr{C} .$$

Proof of (vi) $\Rightarrow$ (i). Let $A \in H(X) - \mathscr{C}$. By assumption, there

is a finite set $F = \{a_1, \cdots, a_n\} \subset A$ and a point $x \in I_{\mathscr{C}}(F) - A$. By the regularity of X, there exist open sets $0, P \subset X$ such that $x \in P$, $A \subset 0$, and $0 \cap P = \varnothing$. Since $I_{\mathscr{C}}$ is continuous on the space of finite sets in X, there exist open sets $0_i \subset 0$ such that $a_i \in 0_i (1 \leq i \leq n)$ and such that for each n-tuple $(a'_1, \cdots, a'_n) \in \prod_{i=1}^n 0_i$, $I_{\mathscr{C}}(a'_1, \cdots, a'_n) \cap P \neq \varnothing$. The set $V = \langle 0, 0_1, \cdots, 0_n \rangle$ is a neighborhood of A. We show that it does not meet $\mathscr{C}$: let $B \in V$, and fix $b_i \in B \cap 0_i$, $1 \leq i \leq n$. We find that

$$I_{\mathscr{C}}(b_1, \cdots, b_n) \cap P \neq \varnothing \; ; \quad B \subset \bigcup_{i=1}^n 0_i \cup 0 = 0 \; .$$

It follows that $I_{\mathscr{C}}(b_1, \cdots, b_n) \not\subset B$, whence $B \notin \mathscr{C}$. $\qquad \square$

4.3. Comments. Except for the rather surprising equivalence between compactness of $\mathscr{C}$ and continuity of $I_{\mathscr{C}}$, two other equivalent conditions seem to be interesting: (iv) and (vi). As for (iv), we noticed already that the modelling of $\mathscr{C}$ into a binary $\mathscr{C}^+$ is combinatorically faithful. Compact convexities are characterized by the fact that this modelling is also *topologically faithful.*

On the other hand, condition (vi) states that compact convexities satisfy a slightly weaker form of a characterization of convex sets when compared with binary convexities (cf. Theorem 2.6).

The condition (v) appears to be rather technical, though it has proved useful already in various applications ([14, Thm. 2.1.] and [18, Thms. 8, 9 and 10]).

5. Some examples and corollaries. Theorem 4.2 shows that a subbase which

    (a)   is normal and T_1;

    (b)   is closed under (nonempty) intersection;

    (c)   is compact as a hyperspace of convex sets;

must satisfy quite stringent properties. The most interesting types of examples are the normal binary subbases (including Example 2.3(i)), and the ones described in 5.1 below.

EXAMPLE 5.1. Let X be a compact convex subspace of a locally convex linear space, equipped with the subbase (convexity) $\mathscr{C}$ of all closed and linearly convex subsets of X (cf. 2.3(ii)). By Hahn-Banach's theorem ([16, p. 58]) $\mathscr{C}$ is also normal. This convexity is compact, as one can see from an obvious argument on line segments, using the continuity of the algebraic operations.

It follows from Theorem 4.2 that the hyperspace of all closed and linearly convex sets of X is a retract of $H(X)$, with a retraction equal to the convex closure operator.

EXAMPLE 5.2. Let X be a locally connected continuum, and let $C(X)$ denote the hyperspace of all subcontinua of X. Then $C(X)$ is easily seen to be a normal T_1 subbase for X, and $C(X)$ is compact. $C(X)$ is a convexity if and only if X is hereditarily unicoherent. A locally connected hereditarily unicoherent continuum is a *tree* (cf. Whyburn [21, Thm. 9.1]; a tree is a continuum in which every two distinct points can be separated by a third point, cf. Ward [20, p. 992]). In this case, $C(X)$ is even a binary subbase, as can be derived from the proof of [12, Thm. 4.3]. A direct proof of this fact can be obtained from Lemma 3.2 (which was also taken from [12]): using unicoherence, one can find for each $a, b \in X$ a smallest continuum $I(a, b)$ with $a, b \in I(a, b)$. It is now easy to see that the resulting map $I: X^2 \to \mathscr{P}(X)$ and the subbase $C(X)$ satisfy the assumptions (i) (ii) and (iii) of Lemma 3.2, whence $C(X)$ is binary.

In [2, Thm. 2], Curtis and Schori have shown that $C(X)$ is a Hilbert Cube factor (that is, a space whose product with the Hilbert Cube is homeomorphic to the Hilbert Cube), iff X is a Peano continuum. In particular, $C(X)$ is then a retract of $H(X)$. Theorem 4.2 (or 3.8) implies that for the class of dendra (metrizable trees) such a retraction can be well described as the convex closure operator relative to $C(X)$.

EXAMPLES 5.3. Let S^1 denote the unit circle. The following collections are easily seen to be closed subbases for S^1 ($0 < r \leqq 2\pi$):

$$\mathscr{S}_r = \{C \in C(S^1) \mid \text{arc length of } C \leqq r\},$$
$$\mathscr{S}_r' = \{C \in C(S^1) \mid \text{arc length of } C < r\}.$$

There is a well-known homeomorphism $h: C(S^1) \to E^2$ (the unit 2-cell) constructed as follows: $h(S^1) = 0$, and for each $C \in C(S^1) - \{S^1\}$, $h(C)$ is the point of E^2 on the line segment joining 0 with the middle point of the arc C on a distance

$$1 - \frac{1}{2\pi}(\text{arc lenght of } C)$$

of the origin. Applying this map to the subspaces $\mathscr{S}_r$, $\mathscr{S}_r'$ of $C(S^1)$, it is easy to see that each $\mathscr{S}_r$ is compact and each $\mathscr{S}_r'$ is noncompact. $\mathscr{S}_r$ (resp. $\mathscr{S}_r'$) is closed under intersection iff $r < \pi$ (resp. iff $r \leqq \pi$). $\mathscr{S}_r$ is nonnormal for each $r < 2\pi$, and $\mathscr{S}_r'$ is normal iff $r > \pi$.

None of the above subbases therefore satisfies (a), (b), and (c) simultaneously. Notice that $\mathscr{S}_r$ and $\mathscr{S}_r'$ are even binary if $r < 2\pi/3$. Let $\mathscr{S} = \mathscr{S}_r'$ with $r = 2\pi/3$. Then $\mathscr{S}$ is a normal nonbinary subbase for the circle. The members

$$A = \{(0, 1), (0, -1)\}, \quad B = \{(1, 0), (-1, 0)\}$$

of the generated convexity $\mathscr{S}^\frown$ can only be screened by *disconnected* closed sets. One easily sees that disconnected members of $\mathscr{S}^\frown$ have measure less than π. Hence $\mathscr{S}^\frown$ is *not normal*.

We now present some corollaries of the results obtained in §§3 and 4.

CoROLLARY 5.4. *Let $\mathscr{C}$ be a normal binary convexity on X. Then the space $\mathscr{C}$ has a normal binary convexity.*

Proof. By Theorems 3.8 and 4.2, $\mathscr{C}$ admits a normal T_1 subbase consisting of all sets of type $\langle C \rangle \cap \mathscr{C}$ or $\langle C, X \rangle \cap \mathscr{C}$, where $C \in \mathscr{C}$. This subbase is binary: let the collection

$$\mathscr{L} = \{\langle C_i \rangle \cap \mathscr{C} \mid i \in I\} \cup \{\langle D_j, X \rangle \cap \mathscr{C} \mid j \in J\}$$

be linked, where $C_i, D_j \in \mathscr{C}$ for each $i \in I$ and $j \in J$. Choose

$$C_{ii'} \in \langle C_i \rangle \cap \langle C_{i'} \rangle \cap \mathscr{C} \qquad i, i' \in I,$$
$$D_{ij} \in \langle C_i \rangle \cap \langle D_j, X \rangle \cap \mathscr{C} \qquad i \in I, j \in J.$$

Hence

$$\varnothing \neq C_{ii'} \subset C_i \cap C_i,$$
$$D_{ij} \subset C_i; \quad D_{ij} \cap D_j \neq \varnothing,$$

proving that for each $j \in J$ the collection

$$\{C_i \mid i \in I\} \cup \{D_j\} \subset \mathscr{C}$$

is linked. Choose

$$x_j \in \bigcap_{i \in I} C_i \cap D_j, \quad j \in J,$$

and let $A = I_\mathscr{C}(\{x_j \mid j \in J\})$. Then $A \subset \bigcap_{i \in I} C_i$, and $A \cap D_j \neq \varnothing$ for each $j \in J$. Consequently, $\cap \mathscr{L} \neq \varnothing$. $\quad\square$

CoROLLARY 5.5. *Let X be a continuum with a normal binary convexity $\mathscr{C}$. Then*
  (i) *the space $\mathscr{C}$ has the fixed point property for continuous maps;*
  (ii) *if X is metrizable, then $\mathscr{C}$ is a metric* AR.

Proof. The space $H(X)$ is connected, and hence its retract $\mathscr{C}$ is connected. A connected space carrying a normal binary subbase is an acyclic Lefschetz space (cf. van de Vel [17, Cor. 3.7]). If X is in addition metrizable, then $\mathscr{C} \subset H(X)$ is metrizable. It then follows from a result of van Mill ([10, Thm. 4]) that $\mathscr{C}$ is a metric AR. $\quad\square$

Statement (ii) can also be derived as follows. A continuum X carrying a normal binary subbase is locally connected (cf. Verbeek [19, Thm. III, 4.1]). If X is metrizable, then $H(X)$ is an AR (cf. Wojdyslawski [22, p. 247]), and hence $\mathscr{C}$ is an AR, being a retract of an AR.

5.6. Construction of hyperspaces which are AR's. By the above cited result of Wojdyslawski, the hyperspace of a Peano continuum is an AR. In case a metric continuum is not locally connected, the techniques discussed in the present paper provide a way to construct hyperspaces which are AR's and which are close to the original hyperspace.

Let $\mathscr{S}$ be a normal T_1 subbase for the metric continuum X. Then $\lambda(X, \mathscr{S})$ is Peanian ([19, Cor. IV 2.4, Thm. III 4.1]), and consequently

$(\mathscr{S}^+)^\wedge$ is an AR, being a retract of an AR (Theorem 3.8) .

By a recent result of Edwards (cf. [5]) every (compact metric) AR is a Hilbert cube factor. It would be desirable to find conditions on $(X, \mathscr{S})$ such that $(\mathscr{S}^+)^\wedge$ is not only a Hilbert cube factor but is homeomorphic to the Hilbert cube itself.

Added in proof. (1) The equivalence of (i) and (ii) in Theorem 4.2 has been obtained more economically by R. E. Jamison in his 1974 dissertation (University of Washington, Seattle). Rather unfortunately, no part of this dissertation has been published, so that the authors became aware of it only recently.

(2) In [18] a notion of "convexity" has been developed for nonclosed sets, and which is based on the above studied type of topological convexity structure. If one adds the nonclosed "convex" sets to the given convexity structure, one obtains the "topological alignments" which were studied by R. E. Jamison. In this way, both approaches are basically the same, and a unifying account is given in a forthcoming paper of the second author, entitled: "Pseudo-boundaries and pseudo-interiors for topological convexities".

(3) In another forthcoming paper of the second author, entitled: "Finite dimensional convexity structures II: the invariants", it will be shown that the "classical" invariants of a convexity are rather deeply affected by certain topological conditions, e.g., concerning dimension. This seemingly contradicts with the expectation raised in §2 above. We emphasize, however, that these results only work under rather stringent (though natural) restrictions such as connectedness of convex sets.

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