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BASE CHANGE FOR TEMPERED IRREDUCIBLE REPRESENTATIONS OF GL $(n, \boldsymbol{R})$

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Let π be a tempered irreducible representation of GL $(n, \boldsymbol{R})$. We prove the expected relation between the characters of π and its base change lifting.

0. Introduction. To each irreducible representation π of GL $(n, \boldsymbol{R})$ is associated its "base change lifting", an irreducible representation Π of GL $(n, \boldsymbol{C})$. It is expected that the characters of these two representations are related in a certain way, at least if π is tempered, and this relation has in fact been proved for GL $(2, \boldsymbol{R})$ by Shintani [4], and for representations of GL $(n, \boldsymbol{R})$ induced from unramified quasicharacters of a minimal parabolic subgroup by Clozel [1]. The purpose of this paper is to prove the relation for arbitrary tempered irreducible representations of GL $(n, \boldsymbol{R})$.

The proof involves computations not unlike those used to calculate the character of an induced representation. The representations in question are all induced from parabolic subgroups whose Levi components are products of copies of GL (2) and GL (1), so we are able to use Shintani's results for GL (2) as a starting point. It is to be expected that a similar "inductive step" can be proved for the general quasi-split connected real reductive group, but technical problems make that more difficult.

1. Notation and preliminaries. Let $G = \mathrm{GL}(n)$, $n \geqq 3$. Every irreducible tempered representation π of G_R is induced from a cuspidal parabolic subgroup P_R. After conjugation, we may assume $P = MN$, where the Levi component M consists of 2×2 and/or 1×1 blocks along the diagonal and N, the unipotent radical of P, consists of upper triangular matrices with diagonal entries all equal to 1 and with zero for those entries which lie inside the blocks of M. Thus $M \cong \mathrm{GL}(2)^k \times \mathrm{GL}(1)^{n-2k}$. Also let $K = U(n)$.

We recall some remarks about σ-conjugacy (see, e.g., [1], § 2). Write g^σ for the complex conjugate of an element $g \in G_C$. Two elements $g, g' \in G_C$ are σ-conjugate if $g = h^\sigma g' h^{-1}$, for some $h \in G_C$. If $g \in G_C$, its norm is defined by $Ng = g^\sigma g$. If g and g' are σ-conjugate, then Ng and Ng' are conjugate in G_C. As usual, we write G'_C for the regular elements of G_C; we shall say g is σ-regular if $Ng \in G'_C$, and write G''_C for the σ-regular elements of G_C. The complement of G''_C is a real analytic subvariety of measure zero.

With $M \cong \mathrm{GL}(2)^k \times \mathrm{GL}(1)^{n-2k}$ as above (blocks along the diagonal), we wish to find representatives for the conjugacy classes of Cartan subgroups of M_R. Inside each 2×2 block, we may take either the split Cartan subgroup, consisting of the diagonal matrices, or the nonsplit Cartan subgroup $Z_R \cdot \mathrm{SO}(2)$, where Z is the scalar matrices. Thus for each i, $0 \leqq i \leqq k$, we have $\binom{k}{i}$ subgroups which have i nonsplit factors. We label them T^i_j, $1 \leqq j \leqq \binom{k}{i}$, in any order. For fixed i, all the T^i_j are conjugate in G_R, though not in M_R. Fix i, j; for each l, $1 \leqq l \leqq \binom{k}{i}$, let $s_l \in G_R$ be such that $s_l T^i_j s_l^{-1} = T^i_l$; let $S^i_j = \{s_1, s_2, \cdots\}$.

By [1], Corollaire, p. 28, every element $g \in G''_C$ is σ-conjugate to an element of G_R, and Ng is conjugate to an element of G_R. Likewise every element $m \in M''_C$ is σ-conjugate to an element of M_R, by an element of M_C, and Nm is conjugate, in M_C, to an element of M_R.

2. Representations. The irreducible representation π of G_R is induced from a representation of P_R. Specifically, let ω be a discrete series representation of M_R, and extend it to the representation $\omega \otimes 1$ of $P_R = M_R N_R$, trivial on N_R. Then $\pi = \mathrm{Ind}_{P_R}^{G_R} \omega \otimes 1$, and all tempered irreducible representations of G_R arise in this way, for some P and ω of this type.

The representation ω is associated in the usual way to a (Weyl group orbit of) character(s) λ of $(T^k_1)_R$, the compact Cartan subgroup of M_R. The restriction of the norm gives a homomorphism N: $(T^k_1)_C \to (T^k_1)_R$, so $\Lambda = \lambda \circ N$ is a character of $(T^k_1)_C$. The base change lifting Π of π is the representation of G_C induced from the extension of Λ to any minimal parabolic subgroup of G_C containing $(T^k_1)_C$. We may choose the minimal parabolic subgroup to be contained in P_C. The lifting Ω of ω to M_C is induced from the restriction of this character to the intersection of the minimal parabolic subgroup with M_C.

To obtain Π, we may first induce to P_C and then to G_C, so we see that $\Pi = \mathrm{Ind}_{P_C}^{G_C} \Omega \otimes 1$.

The liftings Π and Ω are equivalent to their conjugates Π^σ and Ω^σ ($\Pi^\sigma(g) = \Pi(g^\sigma)$, etc.); i.e., there are involutions A_Π and A_Ω so that $A_\Pi \circ \Pi(g) \circ A_\Pi = \Pi^\sigma(g)$, $A_\Omega \circ \Omega(m) \circ A_\Omega = \Omega^\sigma(m)$. If $f \in C_c^\infty(G_C)$, then $\Pi(f) = \int f(g)\Pi(g)dg$ is an operator of trace class, and moreover $f \mapsto \mathrm{trace}\,(\Pi(f) \circ A_\Pi)$ is a distribution. We wish to show that this distribution is in fact given by a function θ^σ_Π, and that $\theta^\sigma_\Pi(g) = \theta_\pi(Ng)$, where on the right side θ_π is the character of π, extended to a conjugate-invariant function on $G_R^{G_C}$.

Shintani [4] has proved this relation for GL(2), so it follows immediately for $M \cong \mathrm{GL}(2)^k \times \mathrm{GL}(1)^{n-2k}$; in particular, $f \mapsto$ trace $(\Omega(f)\circ A_\Omega)$ is given by a function θ_Ω^σ, and

$$\theta_\Omega^\sigma(m) = \theta_\omega(Nm) . \tag{2.1}$$

(Actually, there is an ambiguous sign in the definition of the involution A_Ω, but we fix it so as to make (2.1) hold.)

Suppose Ω acts on the Hilbert space $\mathscr{H}_\Omega$. Then Π acts by right translation on the space $\mathscr{H}_\Pi$ of functions $\phi: G_C \to \mathscr{H}_\Omega$ such that $\phi(pg) = \delta_C^{1/2}(p)(\Omega \otimes 1)(p)\phi(g)$, $(p \in P_C)$, and $\phi|_K \in L^2(K, \mathscr{H}_\Omega)$—here δ_C is the modular function of P_C. Define the operator A_Π on $\mathscr{H}_\Pi$ by $A_\Pi\phi(g) = A_\Omega\phi(g^\sigma)$.

LEMMA 2.1. (i) *If* $\phi \in \mathscr{H}_\Pi$, *then* $A_\Pi\phi \in \mathscr{H}_\Pi$.
(ii) *For* $g \in G_C$, $A_\Pi \circ \Pi(g) = \Pi(g^\sigma)\circ A_\Pi$.

Proof. (i) The square-integrability is easy. If $g \in G_C$, $p \in P_C$, then $A_\Pi\phi(pg) = A_\Omega\phi(p^\sigma g^\sigma) = \delta_C^{1/2}(p^\sigma)A_\Omega\circ(\Omega \otimes 1)(p^\sigma)\phi(g^\sigma) = \delta_C^{1/2}(p)(\Omega\otimes 1)(p)\circ A_\Omega\phi(g^\sigma) = \delta_C^{1/2}(p)(\Omega \otimes 1)(p)A_\Pi\phi(g)$.

(ii) $A_\Pi\circ\Pi(g)\phi(g') = A_\Omega(\Pi(g)\phi)(g'^\sigma) = A_\Omega\phi(g'^\sigma g) = A_\Pi\phi(g'g^\sigma)$. □

3. Jacobians. Given the parabolic subgroup $P = MN$, as above, we let $\mathfrak{n}$ be the Lie algebra of N. If $m \in M_R$ (resp. M_C), then $\mathfrak{n}_R$ (resp. $\mathfrak{n}_C$) is $Ad(m)$-invariant. We denote by σ the complex conjugation on $\mathfrak{n}_C$, and by δ_R (resp. δ_C) the modular function of P_R (resp. P_C); i.e., $\delta_R(m) = \det(Ad(m)|_{\mathfrak{n}_R})$; $\delta_C = \det(Ad(m)|_{\mathfrak{n}_C})$.

DEFINITION. If $m \in M_R$, define $\Delta(m) = \delta_R(m)^{-1/2}\det(I - Ad(m))_{\mathfrak{n}_R}$. If $m \in M_C$, define $\Delta^\sigma(m) = \delta_C(m)^{-1/2}\det(I - Ad(m)\circ\sigma)_{\mathfrak{n}_C}$.

We remark that $\Delta(m)$ is invariant under conjugation by M_R, so we may extend it to an M_C-conjugate-invariant function on $M_R^{M_C}$, the elements of M_C which are conjugate to elements of M_R. We remark too that Δ^σ is σ-conjugate invariant—in fact both factors are σ-conjugate invariant.

PROPOSITION 3.1. *If* $m \in M_C$ *is* σ-*regular, then* $\Delta^\sigma(m) = \Delta(Nm)$.

Proof. Note that the right side makes sense, since $Nm \in M_R^{M_C}$. By the σ-conjugate invariance of Δ^σ, we may assume $m \in T_R$, where T is a Cartan subgroup of M defined over $\boldsymbol{R}$. Thus $Nm = m^2$, and $\Delta(Nm) = \delta_R(m^2)^{-1/2}\prod(1 - \alpha(m^2))$, where the product is over those roots α which appear in the decomposition of the action of T_R on $\mathfrak{n}_R$.

On the other hand, in the action of T_C on $\mathfrak{n}_C$, the roots occur in conjugate pairs β, β^σ, where $\beta^\sigma(t) = \beta(t^\sigma)$, and where $\beta|_{T_R} = \beta^\sigma|_{T_R}$ is one of the roots α of T_R in $\mathfrak{n}_R$. Moreover, the conjugation σ on $\mathfrak{n}_C$ interchanges the root spaces corresponding to β and β^σ. Thus on the span of these two root spaces, relative to a basis of root vectors, $Ad(m)\circ\sigma$ is given by the matrix

$$\begin{pmatrix}\beta(m) & 0\\ 0 & \beta^\sigma(m)\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}0 & \beta(m)\\ \beta^\sigma(m) & 0\end{pmatrix}.$$

The matrix of $Ad(m)\circ\sigma$ on all of $\mathfrak{n}_C$ is a sum of blocks of this type, so $\det(I - Ad(m)\circ\sigma)_{\mathfrak{n}_C} = \prod \det\begin{pmatrix}1 & -\beta(m)\\ -\beta^\sigma(m) & 1\end{pmatrix} = \prod(1-\beta(m)\beta^\sigma(m))$. And for our real m, $\beta^\sigma(m) = \beta(m) = \alpha(m)$, so

$$\Delta^\sigma(m) = \delta_C(m)^{-1/2}\prod(1-\alpha(m)\alpha(m)) = \delta_R(m)^{-1}\prod(1-\alpha(m^2)) = \Delta(m^2) = \Delta(Nm)\ ,$$

as desired. □

4. **Integration formulas.** We need to develop integral formulas that are adapted to integration over σ-conjugacy classes, analogous to the familiar formulas for ordinary conjugacy.

Let $T = T^i_j$ and consider the mapping $G_R/T_R \times T'_R \to G_R$ given by $(\dot g, t) \mapsto \dot g t \dot g^{-1}$. It has order equal to $w^i_G = |w(G_R, (T^i_j)_R)|$. The restriction of this map to $M_R/T_R \times T'_R$ has order $w^i_M = |w(M_R, (T^i_j)_R)|$.

The σ-twisted analogues of these mappings are the map $G_C/T_R \times T''_R \to G_C$ given by $(\dot g, t) \mapsto \dot g^\sigma t \dot g^{-1}$, and its restriction $M_C/T_R \times T''_R \to M_C$.

We calculate their orders: suppose $g^\sigma t g^{-1} = h^\sigma s h^{-1}$, $g, h \in G_C$; $s, t \in T''_R$. Taking norms, we find $h^{-1}gt^2g^{-1}h = s^2$. Letting S be a set of representatives for $W(G_R, T_R)$, we see that $t^2 = ws^2w^{-1}$, for some $w \in S$. Thus $wh^{-1}g \in T_C$, i.e., $h^{-1}g = w^{-1}t'$, some $t' \in T_C$. So $s = h^{-\sigma}g^\sigma t g^{-1}h = w^{-\sigma}t'^\sigma t t'^{-1}w = w^{-1}t'^\sigma t'^{-1}tw$, or $t'^\sigma t'^{-1}t = wsw^{-1} \in T_R$, so $t'^\sigma t'^{-1} \in T_R$. Modulo T_R, this allows only finitely many possibilities for t'. It is in fact easy to see that there are 2^{n-i} possibilities for t', modulo T_R. The same analysis applies to M_C, with the difference that w must be in M_R, though the possibilities for t' remain the same. We record the result as

LEMMA 4.1. *Let $T = T^i_j$, for some i, j. The maps $(\dot g, t) \mapsto \dot g^\sigma t \dot g^{-1}$, $G_C/T_R \times T''_R \to G_C$ and its restriction to $M_C/T_R \times T''_R$ have orders $2^{n-i}w^i_G$ and $2^{n-i}w^i_M$ respectively.*

Next we prove the σ-twisted analogue of a familiar result ([3], Lemma 5.2; cf. [5], Theorem 1.1.4.4).

PROPOSITION 4.2. *If $m \in M''_C$ is such that $\Delta^\sigma(m) \neq 0$, then the* *map* $n \mapsto m^{-1}n^{\sigma}mn^{-1}$ *is an analytic diffeomorphism* $N_C \to N_C$, *with Jacobian equal to* $\det(Ad(m^{-1})\circ\sigma - I)_{\mathfrak{n}_C}$.

Proof. Let $\mathfrak{n}_{n-1} = \{0\}$, $\mathfrak{n}_r = \{X \in \mathfrak{n}_C: [X, \mathfrak{n}_C] \subseteq \mathfrak{n}_{r+1}\}$. Then $\mathfrak{n}_C = \mathfrak{n}_0 \supseteq \mathfrak{n}_1 \supsetneqq \mathfrak{n}_2 \supsetneqq \cdots \supsetneqq \mathfrak{n}_{n-1} = \{0\}$. Letting $A_1 = Ad(m^{-1})\circ\sigma$, $A_2 = -I$, we can apply [5], Lemma 1.1.4.2. □

Fix Haar measures on M_R, N_R, M_C, N_C, K, $K \cap G_R$, and use them to define Haar measure on G_R by

$$\int f(g)dg = \int_{K\cap G_R}\int_{N_R}\int_{M_R} f(mnk)dmdndk\ ,$$

and similarly for G_C.

We apply [5], formula II in § 8.1.2, to M_R and G_R and find that for $\phi \in C_c^\infty(M_R)$, $f \in C_c^\infty(G_R)$,

$$\int_{M_R}\phi(m)dm = \sum_{0\le i\le k}\sum_{1\le j\le\binom{k}{i}}(w_M^i)^{-1} \cdot\int_{M/T_j^i}\int_{T_j^i}\phi(\dot{m}t\dot{m}^{-1})\,|\det(Ad(t^{-1}) - I)_{\mathfrak{m}/\mathfrak{t}_j^i}|\,dtd\dot{m}$$

$$\int_{G_R} f(g)dg = \sum_{0\le i\le k}(w_G^i)^{-1} \cdot\int_{G/T_1^i}\int_{T_1^i} f(\dot{g}t\dot{g}^{-1})\,|\det(Ad(t^{-1}) - I)_{\mathfrak{g}/\mathfrak{t}_1^i}|\,dtd\dot{g}\ .$$

Here $\mathfrak{m}$, $\mathfrak{g}$, $\mathfrak{t}_j^i$ are the Lie algebras of M, G, T_j^i, and we have suppressed the subscript $\boldsymbol{R}$ on T_j^i, M, G, $\mathfrak{t}_j^i$, $\mathfrak{m}$, and $\mathfrak{g}$. For fixed i, all the T_j^i's are conjugate in G_R, so T_1^i in the second formula could be replaced by any T_j^i, or better yet by their average:

$$\int_{G_R} f(g)dg = \sum_{0\le i\le k}\sum_{1\le j\le\binom{k}{i}}\binom{k}{i}^{-1}(w_G^i)^{-1} \cdot\int_{G/T_j^i}\int_{T_j^i} f(\dot{g}t\dot{g}^{-1})\,|\det(Ad(t^{-1}) - I)_{\mathfrak{g}/\mathfrak{t}_j^i}|\,dtd\dot{g}\ .$$

Replacing $\dot{g}$ by $kn\dot{m}$, with $k \in K \cap G_R$, $n \in N_R$, $\dot{m} \in M_R/(T_j^i)_R$, we see the above integral equals

$$\int_{K\cap G_R}\int_{N_R}\int_{M_R/(T_j^i)_R}\int_{(T_j^i)_R} f(kn\dot{m}t\dot{m}^{-1}n^{-1}k^{-1})\,|\det(Ad(t^{-1}) - I)_{\mathfrak{m}/\mathfrak{t}_j^i}|\,dtd\dot{m} \cdot|\det(Ad(t^{-1}) - I)_{\mathfrak{g}/\mathfrak{t}_j^i}|\cdot|\det(Ad(t^{-1}) - I)_{\mathfrak{m}/\mathfrak{t}_j^i}|^{-1}dndk\ .$$

Now

$$|\det Ad(t^{-1}) - I)_{\mathfrak{g}/\mathfrak{t}_j^i}|\cdot|\det(Ad(t^{-1}) - I)_{\mathfrak{m}/\mathfrak{t}_j^i}|^{-1} = |\det(Ad(t^{-1}) - I)_{\mathfrak{g}/\mathfrak{m}}| = \Delta(t)^2\ .$$

Collecting constants, if $m \in M'_R$ is conjugate to an element of T^i_j, we define $r(m) = \binom{k}{i}^{-1} w^i_M / w^i_G$. Then

$$\text{(4.1)} \qquad \int f(g)dg = \int_{K \cap G_R} \int_{N_R} \int_{M_R} r(m) f(knmn^{-1}k^{-1}) \Delta(m)^2 dm dn dk \ .$$

Next we turn to the σ-conjugate situation. Let $M^i_j = \{m^\sigma t m^{-1}: m \in M_C,\ t \in (T^i_j)_R\}$, $M^i = \bigcup_j M^i_j$, $G^i = \{g^\sigma m g^{-1}: g \in G_C,\ m \in M^i\}$. Using Proposition 4.2 and an argument analogous to the one used to prove the corresponding untwisted formulas ([2], Corollary 2, p. 94; [5], § 8.1.2), we find that for $f \in C_c^\infty(G^i)$, the integral

$$\int_K \int_{N_C} \int_{M^i_j} f(k^\sigma n^\sigma m n^{-1} k^{-1}) \Delta^\sigma(m)^2 dm dn dk$$

is a constant multiple of $\int f(g)dg$. Moreover, from Lemma 4.1 we see that the constant is $2^{n-i} w^i_G / (2^{n-i} w^i_M) = w^i_G / w^i_M$. It works equally well for any j, so, averaging, we find

$$\int_{G^i} f(g)dg = \binom{k}{i}^{-1} w^i_M / w^i_G \int_K \int_{N_C} \int_{M^i} f(k^\sigma n^\sigma m n^{-1} k^{-1}) \Delta^\sigma(m)^2 dm dn dk \ .$$

Combining all the G^i's, we can write, for $f \in C_c^\infty(\bigcup_i G^i)$,

$$\text{(4.2)} \qquad \int f(g)dg = \int_K \int_{N_C} \int_{M_C} r(m) f(k^\sigma n^\sigma m n^{-1} k^{-1}) \Delta^\sigma(m)^2 dm dn dk$$

where for $m \in M^i$, $r(m) = r(Nm) = \binom{k}{i}^{-1} w^i_M / w^i_G$.

5. Integral operators. For $f \in C_c^\infty(G_C)$, we express $\Pi(f) \circ A_\Pi$ as an integral operator. If $\phi \in \mathscr{H}_\Pi$, $k_0 \in K$,

$$\begin{aligned}
\Pi(f) \circ A_\Pi \phi(k_0) &= \int_{G_C} f(g) A_\Pi \phi(k_0 g) dg = \int f(k_0^{-1} g) A_\Pi \phi(g) dg \\
&= \int_K \int_{N_C} \int_{M_C} f(k_0^{-1} mnk) A_\Pi \phi(mnk) dm dn dk \\
&= \iiint f(k_0^{-1} mnk) \delta_C(m)^{1/2} \Omega(m) A_\Omega \phi(k^\sigma) dm dn dk \\
&= \iiint f(k_0^{-1} n^\sigma m n^{-1} k^\sigma) \Delta^\sigma(m) \Omega(m) A_\Omega \phi(k) dm dn dk \\
&= \int_K K_f(k_0, k) \phi(k) dk \ ,
\end{aligned}$$

where $K_f(k_0, k)$ is the operator-valued kernel

$$\int_{N_C} \int_{M_C} f(k_0^{-1} n^\sigma m n^{-1} k^\sigma) \Delta^\sigma(m) \Omega(m) A_\Omega dm dn \ .$$

To find the trace of this operator we use Hirai's generalization ([3], § 4) of the usual procedure and integrate the kernel along the diagonal and find the trace of the resulting operator, i.e.,

$$\begin{aligned}\operatorname{trace}(\Pi(f)\circ A_\Pi) &= \operatorname{trace}\int_K K_f(k, k)dk \\ &= \operatorname{trace}\left[\int_K\int_{N_C}\int_{M_C} f(k^{-1}n^\sigma mn^{-1}k^\sigma)\Delta^\sigma(m)\Omega(m)dmdndk\circ A_\Omega\right] \\ &= \int_{M_C}\int_{N_C}\int_K f(k^\sigma n^\sigma mn^{-1}k^{-1})\Delta^\sigma(m)\theta_\Omega^\sigma(m)dkdndm \\ &= \int_{G_C} f(g)\theta_\Pi^\sigma(g)dg\ ,\end{aligned}$$

where, using (4.2) and symmetrizing $\Delta^{\sigma-1}\theta_\Omega^\sigma$, we have $\theta_\Pi^\sigma(h^\sigma gh^{-1}) = \theta_\Pi^\sigma(g)$, for $g, h \in G_C$, and if $t \in (T_j^i)_R$

$$\begin{aligned}\theta_\Pi^\sigma(t) &= r(t)^{-1}\binom{k}{i}^{-1} w_M^i/w_G^i \sum_{s\in S_j^i}\sum_w \Delta^{\sigma-1}\theta_\Omega^\sigma(wsts^{-1}w^{-1}) \\ &= \sum_s\sum_w \Delta^{\sigma-1}\theta_\Omega^\sigma(wsts^{-1}w^{-1})\ .\end{aligned} \tag{5.1}$$

The inner sum is over $w \in W(M_R, T_j^i)\backslash W(G_R, T_j^i)^s$, and the outer sum over $s \in S_j^i$ averages over the various T_l^i's. Also $\theta_\Pi^\sigma(g) = 0$ unless $g \in \bigcup_{i=0}^k G^i$.

6. The character relation.

We are now able to state:

THEOREM. *Let π be an irreducible tempered representation of* GL(n, $\boldsymbol{R}$), *Π its base change lifting. Let A_Π be an involution with $A_\Pi\circ\Pi(g) = \Pi^\sigma(g)\circ A_\Pi$. The distribution*

$$f \longmapsto \operatorname{trace}(\Pi(f)\circ A_\Pi) \qquad (f \in C_c^\infty(\mathrm{GL}(n, \boldsymbol{C})))$$

is given by a function θ_Π^σ on GL(n, $\boldsymbol{C}$), *and the sign of A_Π may be chosen so that $\theta_\Pi^\sigma(g) = \theta_\pi(Ng)$.*

Proof. The result is trivial for GL(1), and for GL(2) has been done by Shintani [4]. For $n \geqq 3$, all that remains to be shown is the last identity, and, ignoring a null set, it suffices to consider $g \in G_C''$. We fix A_Π as in § 2.

By the familiar untwisted analogue of the computation in § 5, we can use (4.1) to calculate θ_π (cf. [3]). For $t \in (T_j^i)_R$, we find $\theta_\pi(t) = \sum_{s\in S_j^i}\sum_w \Delta^{-1}\theta_\omega(wsts^{-1}w^{-1})$. The inner sum is over $w \in W(M_R, (T_j^i)^s)\backslash W(G_R, T_j^i)^s$. Also $\theta_\pi(g) = 0$ unless $g \in M_R^{G_R}$.

We know from § 5 that $\theta_\Pi^\sigma(g) = 0$ unless $g \in \cup G^i$. Thus the desired relation holds for $g \notin \cup G^i$, so we may suppose $g \in G^i$. By the invariance of θ_π and the σ-conjugate invariance of θ^σ_Π, we can assume $g = t \in (T^i_j)_R$, so $Ng = t^2$.

The result follows by comparing the above formula for $\theta_\pi(Nt) = \theta_\pi(t^2)$ with formula (5.1) for $\theta^\sigma_\Pi(t)$, and applying Proposition 3.1 and formula (2.1). □

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