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EMBEDDINGS OF THE PSEUDO-ARC IN E^2

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In this paper, we show that there exists an embedding, P_s, of the pseudo-arc in the plane such that any two accessible points lie in distinct composants of P_s. We also show that there are $c=2^{\omega_0}$ distinct embeddings of the pseudo-arc in the plane, including for each positive integer n, one with exactly n composants accessible. This answers some questions and a conjecture of Brechner.

For definitions and notation of chain (from p to q), link, crooked, etc., see [1] and [7]. The links of our chains will always be the interiors of disks, and if two links of a chain intersect their intersection is the interior of a disk. When a chain D refines a chain C, we shall always require that the closure of each link of D be contained in a link of C.

First we describe the special embedding P_s, then prove Brechner's conjecture that any two distinct accessible points of P_s lie in distinct composants. Let C_0 be a chain in E^2 from point p to point q which runs straight across from left to right horizontally. Let C_1 be a chain also running from p to q which is crooked in C_0 and descending, as in Figure 1. If we think of C_1 as straightened out

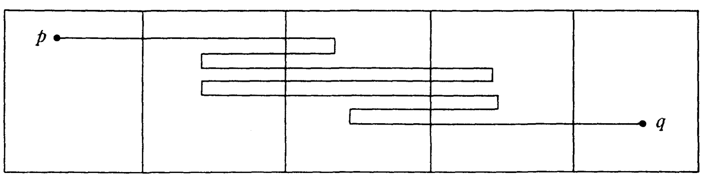

FIGURE 1
Only the nerve of C_1 in C_0 is shown.

with p on the left and q on the right, then C_2 is a chain from p to q which is crooked in C_1 and ascending. We continue in this manner, alternating descending and ascending chains, so that C_i runs from p to q, mesh $(C_i) < 1/2^i$, C_{i+1} refines and is crooked in C_i, and C_{i+1} is descending (ascending) in C_i if i is even (odd). The pseudo-arc P_s is $\bigcap_{i\in\omega_0} C_i^*$. (If A is a collection of sets, A^* is the union of A.)

THEOREM. *Any two distinct accessible points of P_s are in distinct components.*

Proof. We can draw horizontal rays to the left from p and to the right from q. A top accessible point will be a point of P_s which is accessible by an arc lying in the upper complementary domain of P_s plus the two rays (except for the endpoint of the arc in P_s). A bottom accessible point is defined similarly. We will show that any two distinct top accessible points are in distinct composants. A similar argument will show that any two distinct bottom accessible points are in distinct composants.

Let a and b be distinct top accessible points. Suppose $a = \bigcap_{i \in \omega_0} C_i(a_i)$ and $b = \bigcap_{i \in \omega_0} C_i(b_i)$. Let α and β be arcs above P_s with $\alpha \cap P_s = a$ and $\beta \cap P_s = b$. We can suppose without loss of generality that for each $i \in \omega_0$, $\alpha \cap C_i^*$ and $\beta \cap C_i^*$ are connected.

Claim. The subcontinuum M of P_s irreducible between a and b contains both p and q (i.e., for each $i \in \omega_0$ and sufficiently large $j \in \omega_0$ the subchain of C_j between $C_j(a_j)$ and $C_j(b_j)$ has links in each of $C_i(0)$ and $C_i(n_i)$, where $C_i(n_i)$ is the last link of C_i).

Proof of claim. For each $i \in \omega_0$ there exists $k \in \omega_0$ such that, for $j \geqq k$, $\mathrm{cl}(\{C_j(n) \mid C_j(n) \cap \alpha \neq \varnothing\}^*) \subseteqq C_i(a_i)$ and $\mathrm{cl}(\{C_j(m) \mid C_j(m) \cap \beta \neq \varnothing\}^*) \subseteqq C_i(b_i)$. Choose i large enough that there are at least two links of C_i between $C_i(a_i)$ and $C_i(b_i)$, and $k > i$ so that the above condition holds and it takes at last three links of C_k to span between nonadjacent links of C_i or to reach from $C_k(0)$ to $C_i(1)$ or to reach from $C_k(n_k)$ to $C_i(n_i - 1)$.

Consider $j > k$ such that j is even. Then $\{C_j(n) \mid C_j(n) \cap \alpha \neq \varnothing\}$ and $\{C_j(m) \mid C_j(m) \cap \beta \neq \varnothing\}$ are separated by several links of C_j. Suppose $\{C_j(n) \mid C_j(n) \cap \alpha \neq \varnothing\}$ comes first in C_j. Then because C_{j+1} is descending in C_j and α, β lie above P_s, $C_{j+1}(a_{j+1})$ is in the maximal subchain of C_{j+1} with no links reaching past $\{C_j(n) \mid C_j(n) \cap \alpha \neq \varnothing\}^*$. But $C_{j+1}(b_{j+1})$ is not in any of the links of C_j up to this point (or in fact at least three links beyond), so by crookedness of C_{j+1} in C_j there is a link γ of C_{j+1} between $C_{j+1}(a_{j+1})$ and $C_{j+1}(b_{j+1})$ with $\gamma \subseteqq C_j(1) \subseteqq C_i(0)$.

Similarly, if j is odd, there is a link δ of C_{j+1} between $C_{j+1}(a_{j+1})$ and $C_{j+1}(b_{j+1})$ with $\delta \subseteqq C_{j+1}(n_{j+1} - 1) \subseteqq C_i(n_i)$. Thus the subcontinuum M of P_s irreducible between a and b contains both p and q. Hence $M = P_s$, and a and b are in different composants of P_s.

Similarly any two bottom accessible points of P_s are in different composants. By Theorem 3.1 of [5] if top and bottom accessible points of P_s are in the same composant C of P_s then either $p \in C$ or $q \in C$. Thus the top and bottom accessible points must be the same and be either p or q. So any two distinct accessible points of P_s are in different composants.

It follows from [8] that, though P_s has $c = 2^{\omega_0}$ distinct accessible composants, there exists some component of P_s which is not accessible.

2. Other embeddings. We will now show how to obtain $c = 2^{\omega_0}$ distinct embeddings of the pseudo-arc in the plane. These will be distinguished by use of prime ends and accessibility. First however we will describe $c = 2^{\omega_0}$ distinct 0-dimensional closed subsets of the unit circle, S^1, which will be associated with these embeddings.

Let $X_j = e^{\pi i/j}$ for $j = 1, 2, \cdots$. (This is the only place in the paper where i is not an integer or finite ordinal. Here of course $i = \sqrt{-1}$. In all later discussion, we return to letting i be an integer or ordinal.) Let $X_0 = 1$. This is a simple sequence which divides S^1 into a countable number of open intervals. For any subset A of $\{1, 2, \cdots\}$, let C_A be the closed set consisting of $\{X_i\}_{i \in \omega_0}$ together with a Cantor set in the open interval between X_i and X_{i+1} for each $i \in A$. We shall describe how to embed a pseudo-arc P_A in the plane such that its space of prime ends is homeomorphic to S^1 by a homeomorphism h, where for each open interval I of $S^1 - C_A$ all accessible points which correspond to prime ends in $h^{-1}(I)$ are in the same composant of P_A, and accessible points which correspond to prime ends in different intervals are in different composants of P_A. Thus if A and B are distinct subsets of $\{1, 2, \cdots\}$ then P_A and P_B are inequivalently embedded in the plane.

For each basic Cantor set C in C_A, $C = \bigcap_{i \in \omega_0} C(i)$ where each $C(i)$ is a finite collection of closed intervals in S^1 and $C(i + 1)$ is obtained by removing open intervals from the middle of each component of $C(i)$. Order the set of all endpoints of components of $C(i)$'s such that each endpoint of a component of $C(i)$ comes before each endpoint of a component of $C(i + 1)$ which is not also an endpoint of a component of $C(i)$.

Let $\{y_i\}_{i \in \omega_0}$ be a well-ordering of the set of all end points of components of $S^1 - C_A$ such that:

(1) $y_0 = X_1$ and $y_1 = X_0$.

(2) For each brsic Cantor set C in C_A the restriction of the well-ordering of $\{y_i\}_{i \in \omega_0}$ to points in C is the ordering described above.

(3) Both X_j and X_{j+1} come before any point of a Cantor set between these two points.

Let C_0 be a chain in E^2 running straight across horizontally from a point z_0 (in link L_0) to a point z_1 (in link L_1). (Consistent with our previous notation, subscripts will not indicate adjacent links but will rather indicate points z contained in these links.) Suppose inductively that chain C_{2i} has been formed with distinct nonadjacent links $L_0, L_1, \cdots, L_{i+1}$ specified so that the ordering of the L_j's along the chain corresponds to the ordering of $\{y_j\}_{j\leqq i+1}$ in S^1 going from X_1 to X_0 clockwise. Suppose also that points z_j have been specified in each L_j with $\mathrm{st}(z_j, C_{2i}) = L_j$. We will now describe how to form chains C_{2i+1} (refining C_{2i}) and C_{2i+2} (refining C_{2i+1}).

Think of chain C_{2i} as straightened out horizontally with z_0 on the left and z_1 on the right. Let $\{W_n\}_{n\leqq i+2}$ be the ordering of $\{y_j\}_{j\leqq i+2}$ induced by the order of the points in S^1 from X_1 to X_0 clockwise. Let μ be a bijection such that $W_n = y_{\mu(n)}$ for each $n \leqq i + 2$. In C_{2i} chain C_{2i+1} is a chain (see Figure 2) from z_0 to z_1 which starts

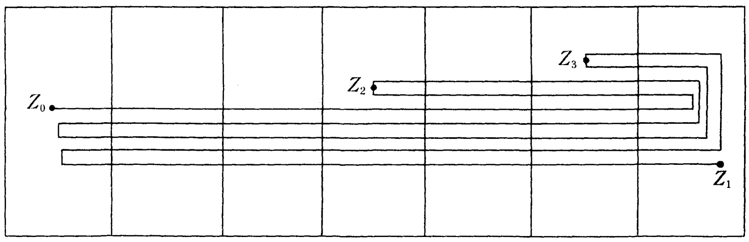

FIGURE 2
One possible cofiguration of the nerve of C_3 in C_2 is shown.

by running straight from L_0 to L_1, then consists of segments D_n, for $1 < n \leqq i + 2$, such that (for $\mu(n) \neq i + 2$):

(1) D_n runs straight from L_1 to $L_{\mu(n)}$ above all previous parts of C_{2i+1}, straight back to L_1 above all previous parts of C_{2i+1}, straight to L_0 below all previous parts of C_{2i+1}, then straight back to L_1 below all previous parts of C_{2i+1}.

(2) The bend D_n in $L_{\mu(n)}$ contains $z_{\mu(n)}$, where $W_n = Y_{\mu(n)}$.

(3) D_n intersects only D_{n-1} and D_{n+1}, each of which it intersects in an end link.

If y_{i+2} is a point of a basic Cantor set C of C_A and is either the leftmost point of C or the left end one of the intevals removed in forming C (by the $C(i)$'s), then $D_{\tilde{n}}$ (where $\mu(\tilde{n}) = i + 2$) satisfies conditions (1) and (3) with L_{i+2} being the link of C_{2i} immediately after $L_{\mu(\tilde{n}-1)}$. Choose z_{i+2} in the bend of $D_{\tilde{n}}$ in $L_{\mu(\tilde{n}-1)}$ (and not in either adjacent link of C_{2i+1}). Otherwise do the same with L_{i+2} chosen to be the link of C_{2i} immediately before $L_{\mu(\tilde{n}+1)}$. The chain C_{2i+1} is the union of the D_n's and the initial straight segment from L_0 to L_1.

To get chain C_{2i+2} think of straightening C_{2i+1} out horizontally with z_0 on the left, and consider the set Γ of links of C_{2i+1} which are either end links of the D_n's, links where the bends of the D_n's occur, or end links of C_{2i+1}. In each subchain of C_{2i+1} connecting consecutive elements in Γ, place a crooked descending chain going between the two ends (and if a z_n is in such a subchain place it in the appropriate end link of the crooked chain). This can be done so that the underlying point sets of crooked chains in adjacent subchains intersect exactly in an end link. Chain C_{2i+2} will be the union of these small crooked chains.

Note that, while C_{2i+1} is not crooked in C_{2i}, nor is C_{2i+2} in C_{2i+1}, chain C_{2i+2} is crooked in C_{2i}. If we do this so that the mesh of the chains gets arbitrarily small, then the intersection is a pseudoarc P_A [2]. By construction, each z_i is accessible, and different z_i's lie in different composants of P_A.

Let h be a homeomorphism between the space of prime ends of P_A and S^1 such that $h(\tilde{z}_i) = y_i$ for each $i \in \omega_0$, where $\tilde{z}_i$ is the prime end associated with the accessible point z_i. Suppose p and q are accessible points of P_A with associated prime ends $\tilde{p}$ and $\tilde{q}$, where $h(\tilde{p})$ and $h(\tilde{q})$ lie in the same component I of $S^1 - C_A$. Let a (respectively b) be the accessible point whose associated prime end $\tilde{a}$ (resp. $\tilde{b}$) is mapped by h to the largest (smallest) endopoint of I in the counterclockwise ordering $(0, 2\pi]$ of S^1.

Claim. Each of p and q is in the same composant of P_A as the point a.

Proof. Let α and β be disjoint rays to infinity which intersect P_A only at their endpoints a and b respectively. Let π be a ray, disjoint from α and β, which intersects P_A only in its endpoint p. We may assume that, for each $i \in \omega_0$, $\pi \cap C_i^*$ is connected (as are also $\alpha \cap C_i^*$ and $\beta \cap C_i^*$). If $a = y_{m_1}$ and $b = y_{m_2}$ choose N bigger than both m_1 and m_2 and such that the sets $\{l \in C_{2N} | \alpha \cap l \neq \varnothing\}^*$, $\{l \in C_2 | \pi \cap l \neq \varnothing\}^*$, and $\{l \in C_{2N} | \beta \cap l \neq \varnothing\}^*$ are disjoint. For each $n > 2N$, let M_n be the minimum subchain of C_n containing both $\{l \in C_n | \alpha \cap l \neq \varnothing\}$ and $\{l \in C_n | \pi \cap l \neq \varnothing\}$. Then for each $n > 2N$, $\mathrm{cl}(M_{n+1}^*) \subseteq \mathrm{cl}(M_n^*)$, by our construction of the C_i's (and since there are no other y_j's between a and b). Thus $M = \bigcap_{n>2N} \mathrm{cl}(M_n^*)$ is a proper subcontinuum of P_A containing both a and p. Similarly, there is a proper subcontinuum of P_A containing both a and q.

By the above claim, the fact that all of the y_i's are in different composants, and Theorem 3.1 of [5], we get that p and q are in different composants of P_A if $h(\tilde{p})$ and $h(\tilde{q})$ are in different components of $S^1 - C_A$.

If we use the above procedure to construct pseudo-arcs, but stop introducing new z_n's and L_n's at some point, we can obtain for each positive integer i a pseudo-arc in the plane with exactly i composants accessible.

3. Questions. Though our P_A's are all embedded differently in E^2, any two contain equivalently embedded subcontinua (e.g., ones containing z_1). This leads us to the following question.

Question 1. Do every two pseudo-arcs in the plane contain equivalently embedded subcontinua? (A comparison of subcontinua of P_S with subcontinua of P_A might be useful here.)

The following is also of interest.

Question 2. Are there $c = 2^{\omega_0}$ distinct embeddings of the pseudo-circle in E^2? of every hereditarily indecomposable plane continuum?

We know that, though there are embeddings of the pseudo-arc with $c = 2^{\omega_0}$ distinct accessible composants, there are also always inaccessible composants [8]. Of the embeddings we have described, P_S is the only one with the property that any two accessible points are in distinct composants. Is there any other embedding with this property?

Michel Smith has recently announced results analogous to these.

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