PACIFIC JOURNAL OF MATHEMATICS
Vol. 93, No. 1, 1981

POWER SERIES RINGS OVER DISCRETE VALUATION RINGS

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If V is a discrete valuation ring with Krull dimension m, it is shown that the power series ring $V[[x_1, \cdots, x_n]]$ has Krull dimension $mn + 1$.

Throughout the paper all rings are assumed to be commutative with identity and the ring R is not considered to be a prime ideal of R. In [1] the author defines a ring to have the SFT (strong finite type) property if for each ideal A of R there exists a finitely generated ideal B and a positive integer k such that $B \subseteqq A$ and $a^k \in B$ for each $a \in A$. It is shown in [1, Theorem 1] that if R fails to have the SFT-property then the power series ring $R[\![Y]\!]$ has infinite Krull dimension. On the other hand, if D is a Prüfer domain with $\dim D = m$ and if D has the SFT-property then $\dim D[\![Y]\!] = m+1$ [2, Theorem 3.8]. Recall that a valuation ring V with finite Krull dimension is discrete if and only if $P \neq P^2$ for each nonzero prime ideal P of V [5, pp. 190-192]. A valuation ring V has the SFT-property if and only if it is discrete [2, Proposition 3.1]. Thus, if V is a valuation ring and $\dim V = m$ then either V is discrete and $\dim V[\![Y]\!] = m + 1$ (this specific result was proved by Fields in [4, Theorem 2.7]) or V is nondiscrete and $\dim V[\![Y]\!] = \infty$. For $\dim R = m$ the author asks in [1, p. 303] if either $\dim R[\![Y]\!] = m + 1$ or $\dim R[\![Y]\!] = \infty$. We show that the answer is no for ring $V[\![x_1, \cdots, x_{n-1}]\!]$, where V is a discrete valuation ring with $\dim V \geqq 2$. Specifically, we prove the following theorem.

THEOREM. *If V is a discrete valuation ring with Krull dimension m then the power series ring $V[\![x_1, \cdots, x_n]\!]$ has Krull dimension $mn + 1$.*

Proof. The proof is by induction on m and the case $m = 1$ is well-known since, in this case, V is Noetherian (cf. Lemma 2.6 of [4]). Thus assume that $m \geqq 2$, that the theorem holds if $\dim V = m - 1$, let $\dim V = m$, and suppose that $(0) = P_0 \subset P_1 \subset P_2 \subseteqq \cdots \subseteqq P_m$ is the set of prime ideals of V. Throughout the proof X denotes the set $\{x_1, \cdots, x_n\}$ of analytic indeterminates over V, $V[\![X]\!]$ denotes the power series ring $V[\![x_1, \cdots, x_n]\!]$, $p \in P_1 \backslash P_1^2$, $W = V_{P_1}$, $U = V/P_1$, $F = W/P_1W$ and, even though $P_1 = P_1W$, we write $\mathscr{P}$ to denote the ideal P_1W. We note that W is a rank one discrete valuation ring with maximal ideal $\mathscr{P} = pW$, F is the quotient field of U, and for each integer $k \geqq 1$

we have $P_1^{k+1} \subseteq p^k P_1$. If $\xi \in (W[\![X]\!])_{W\setminus(0)}$ then there exists a nonzero element a in W such that $a\xi \in W[\![X]\!]$. But then $pa\xi \in V[\![X]\!]$ and $pa \in V$ so $\xi \in (V[\![X]\!])_{V\setminus(0)}$. This shows that $(W[\![X]\!])_{W\setminus(0)} \subseteq (V[\![X]\!])_{V\setminus(0)}$ and the reverse containment is obvious so equality holds. It follows that the correspondence $Q \to Q \cap V[\![X]\!]$ is a bijection from the set

$$\{Q \in \operatorname{Spec}(W[\![X]\!]) \mid Q \cap W = (0)\}$$

to the set $\{Q' \in \operatorname{Spec}(V[\![X]\!]) \mid Q' \cap V = (0)\}$ which preserves set containment. Thus, if $Q \in \operatorname{Spec}(W[\![X]\!])$ and $Q \cap W = (0)$, then $\operatorname{rank} Q = \operatorname{rank}(Q \cap V[\![X]\!])$ and it follows that $\operatorname{rank} Q' \leqq n$ for each $Q' \in \operatorname{Spec}(V[\![X]\!])$ such that $Q' \cap V = (0)$.

Let $(0) \subset Q_1 \subset \cdots \subset Q_t = P_m + (X)$ be a maximal chain of prime ideals of $V[\![X]\!]$ and choose k so that $Q_k \cap V = (0)$ while $Q_{k+1} \cap V \neq (0)$. Then, as we have already observed, $k = \operatorname{rank} Q_k \leqq n$. Since $p \in Q_{k+1}$ we have $(P_1[\![X]\!])^2 \subseteq P_1^2[\![X]\!] \subseteq pP_1[\![X]\!] \subseteq Q_{k+1}$ and hence $Q_{k+1} \supseteq P_1[\![X]\!]$.

We first consider the case in which $Q_{k+1} \neq P_1[\![X]\!]$. It follows from Theorem 3.14 of [3] that there exist elements $\lambda_1 = x_1, \lambda_2, \cdots, \lambda_n$ in $x_1F[\![x_1]\!]$ such that the $U[\![x_1]\!]$-homomorphism $\phi: U[\![X]\!] \to U[\![\lambda_1, \cdots, \lambda_n]\!]$ determined by $\phi(x_i) = \lambda_i$, $1 \leqq i \leqq n$, is an isomorphism. But ϕ extends to an $F[\![x_1]\!]$-epimorphism $\bar{\phi}: F[\![X]\!] \to F[\![x_1]\!]$ and if $\bar{Q}$ is the kernel of $\bar{\phi}$ then depth $\bar{Q} = 1$, $\operatorname{rank} \bar{Q} = n - 1$ [6, Corollary 1, p. 218], and $\bar{Q} \cap U[\![X]\!] = (0)$. Since $F[\![X]\!] = (W/\mathscr{P})[\![X]\!] \cong W[\![X]\!]/\mathscr{P}[\![X]\!]$ and $U[\![X]\!] \cong V[\![X]\!]/P_1[\![X]\!]$, $\bar{Q}$ determines a prime ideal Q of $W[\![X]\!]$ such that depth $Q = 1$, $\operatorname{rank}(Q/\mathscr{P}[\![X]\!]) = n - 1$, and $Q \cap V[\![X]\!] = P_1[\![X]\!]$. Therefore, $\operatorname{rank} Q \geqq n$ and, since $\dim W[\![X]\!] = n + 1$, it follows that $\operatorname{rank} Q = n$. If we choose $f_1, \cdots, f_{n-1} \in XW[\![X]\!]$ such that the corresponding elements $\bar{f}_1, \cdots, \bar{f}_{n-1}$ in $F[\![X]\!]$ form a regular system of parameters for $(F[\![X]\!])_{\bar{Q}}$, then $\{f_1, \cdots, f_{n-1}, p\}$ is a regular system of parameters for $(W[\![X]\!])_Q$ and the ideal $N_i' = (f_1, \cdots, f_i)(W[\![X]\!])_Q$ is a prime ideal of $(W[\![X]\!])_Q$ for $1 \leqq i \leqq n - 1$ (cf. Corollary 1, p. 302 and Theorem 26, p. 303 of [6]). In particular, $N_{n-1} = N_{n-1}' \cap W[\![X]\!]$ is a prime ideal of $W[\![X]\!]$ such that $\operatorname{rank} N_{n-1} = n - 1$, $N_{n-1} \subset Q$, and $N_{n-1} \cap W = (0)$. We now have $P_1[\![X]\!] = Q \cap V[\![X]\!] \supset N_{n-1} \cap V[\![X]\!]$ and $\operatorname{rank}\ (N_{n-1} \cap V[\![X]\!]) = \operatorname{rank} N_{n-1} = n - 1$—that is, $\operatorname{rank}\ P_1[\![X]\!] \geqq n$. Therefore, $k + 1 = \operatorname{rank} Q_{k+1} \geqq 1 + \operatorname{rank} P_1[\![X]\!]) \geqq n + 1$. We have already seen that $k \leqq n$, so $k = n$ and $\operatorname{rank}(Q_{k+1}/P_1[\![X]\!] = 1$. Thus, $P_1[\![X]\!]/P_1[\![X]\!] \subset Q_{k+1}/P_1[\![X]\!] \subset \cdots \subset Q_t/P_1[\![X]\!]$ is a maximal chain of prime ideals in $V[\![X]\!]/P_1[\![X]\!] \cong U[\![X]\!]$ of length $t-k$. By assumption $t-k = (m-1)n+1$ and since $k = n$ this implies that $t = mn + 1$.

We now consider the case in which $Q_{k+1} = P_1[\![X]\!]$. It follows from the previous argument that $n \leqq \operatorname{rank} P_1[\![X]\!] = \operatorname{rank} Q_{k+1} = k + 1$. We will show that equality holds. Let $\mathscr{V}$ be a valuation overring of $V[\![X]\!]$ with prime ideals $(0) \subset Q_1' \subset \cdots \subset Q_{k+1}'$ such that $Q_i' \cap V[\![X]\!] = Q_i$

for each i. Since $Q_k \cap V = (0)$ we may assume that $Q'_{k+1} = \text{rad}(p\mathscr{V})$ and, by localizing if necessary, we assume that Q'_{k+1} is the maximal ideal of $\mathscr{V}$. We wish to show that $\mathscr{V} \supseteqq W[\![X]\!]$. If this is not the case then there exists $h \in W[\![X]\!]$ such that $h^{-1} \in Q'_{k+1}$. If $f = ph$ then $f \in P_1[\![X]\!]$, $h^{-1} = p/f$, and there exists an integer s such that $h^{-s} = p^s/f^s = p\zeta$ for some $\zeta \in \mathscr{V}$. But $f^s \in (P_1[\![X]\!])^s \subseteqq p^{s-1}P_1[\![X]\!]$ so we have $p\zeta = p^s/p^{s-1}f_1$ for some $f_1 \in P_1[\![X]\!]$. Therefore, $1/f_1 = \zeta \in \mathscr{V}$ contrary to the assumption that $P_1[\![X]\!] \subseteqq Q'_{k+1}$. It follows that $W[\![X]\!] \subseteqq \mathscr{V}$ and if $Q''_i = Q'_i \cap W[\![X]\!]$ for $1 \leqq i \leqq k+1$ then $(0) \subset Q''_1 \subset \cdots \subset Q''_{k+1}$ is a chain of prime ideals of $W[\![X]\!]$ such that $Q''_i \cap V[\![X]\!] = Q_i$. In particular, $Q''_{k+1} \cap V[\![X]\!] = P_1[\![X]\!]$. Since $[\mathscr{P} + (X)] \cap V[\![X]\!] = P_1 + (X)$ it follows that Q''_{k+1} is not maximal in $W[\![X]\!]$. Thus, $n+1 > \text{rank}\, Q''_{k+1} \geqq k+1$ — that is, $k < n$. It follows that $k = n-1$ and this together with the previous argument shows that, in either case, rank $P_1[\![X]\!] = n$. We now have that $P_1[\![X]\!]/P_1[\![X]\!] \subset Q_{k+2}/P_1[\![X]\!] \subset \cdots \subset Q_t/P_1[\![X]\!]$ is a maximal chain of prime ideals in $V[\![X]\!]/P_1[\![X]\!] \cong U[\![X]\!]$ of length $t-(k+1) = t-n$. By assumption, $t - n = (m-1)n + 1$, so $t = mn + 1$.

REMARK. The proof of the theorem shows that if $(0) = P_0 \subset P_1 \subset \cdots \subset P_m$ is the set of prime ideals of a discrete valuation ring V then each of the prime ideals $P_i[\![x_1, \cdots, x_n]\!]$ can be included in a maximal chain of prime ideals of $V[\![x_1, \cdots, x_n]\!]$ and for $0 < i < m$ we have rank $(P_i[\![x_1, \cdots, x_n]\!]/P_{i-1}[\![x_1, \cdots, x_n]\!]) = n$.

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Received March 19, 1980 and in revised form August 25, 1980.

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