PACIFIC JOURNAL OF MATHEMATICS
Vol. 93, No. 2, 1981

C^*-ALGEBRAS ASSOCIATED WITH IRRATIONAL ROTATIONS

MARC A. RIEFFEL

For any irrational number α let A_α be the transformation group C^*-algebra for the action of the integers on the circle by powers of the rotation by angle $2\pi\alpha$. It is known that A_α is simple and has a unique normalized trace, τ. We show that for every β in $(\boldsymbol{Z} + \boldsymbol{Z}\alpha) \cap [0, 1]$ there is a projection p in A_α with $\tau(p) = \beta$. When this fact is combined with the very recent result of Pimsner and Voiculescu that if p is any projection in A_α then $\tau(p)$ must be in the above set, one can immediately show that, except for some obvious redundancies, the A_α are not isomorphic for different α. Moreover, we show that A_α and A_β are strongly Morita equivalent exactly if α and β are in the same orbit under the action of GL (2, $\boldsymbol{Z}$) on irrational numbers.

0. Introduction. Let α be an irrational number, and let S denote the rotation by angle $2\pi\alpha$ on the circle, $\boldsymbol{T}$. Then the group of integers, $\boldsymbol{Z}$, acts as a transformation group on $\boldsymbol{T}$ by means of powers of S, and we can form the corresponding transformation group C^*-algebra, A_α, as defined in [8, 19, 30]. If we view S as also acting on functions on $\boldsymbol{T}$, and if $C(\boldsymbol{T})$ denotes the algebra of continuous complex-valued functions on $\boldsymbol{T}$, then S acts as an automorphism of $C(\boldsymbol{T})$. This gives an action of $\boldsymbol{Z}$ as a group of automorphisms of $C(\boldsymbol{T})$, and A_α is just the crossed product algebra for this action [19, 30]. A convenient concrete realization of A_α consists of the norm-closed *-algebra of operators on $L^2(\boldsymbol{T})$ generated by S together with all the pointwise multiplication operators, M_f, for $f \in C(\boldsymbol{T})$. It is known [8, 19, 22, 30] that A_α is a simple C^*-algebra (with identity element) not of type I, and that A_α has a unique normalized trace, τ. In fact, on the dense *-subalgebra $C_c(\boldsymbol{Z}, \boldsymbol{T}, \alpha)$ consisting of finite sums of the form $\Sigma M_{f_n} S^n$ the trace is given by

$$\tau(\Sigma M_{f_n} S^n) = \int_T f_0(t)dt ,$$

where dt is Lebesgue measure on the circle normalized to give the circle unit measure. (We remark that Theorem 1.1 of [27] can be used to show that this dense subalgebra itself is also simple.)

Little else has been known about the A_α. In particular, it has not been known whether or not the A_α are isomorphic as α varies. An interesting question raised in 7.3 of [8], and again recently in [22], is whether the A_α contain any projections. But in fact, shortly after [8] appeared, R. T. Powers showed in unpublished work that there are self-adjoint elements in the A_α which have disconnected spectrum, from which one can infer that the A_α contain proper projections.

The main contribution of this paper is to show how to describe very explicitly some projections in the A_α—so explicitly that it is then obvious what value the trace has on them. Specifically, we show:

THEOREM 1. *For each* $\beta \in (\boldsymbol{Z} + \boldsymbol{Z}\alpha) \cap [0, 1]$ *there is a projection* p *in* A_α *such that* $\tau(p) = \beta$.

This result was announced in [26], together with the conjecture that the trace of any projection in A_α must be in $(\boldsymbol{Z} + \boldsymbol{Z}\alpha) \cap [0, 1]$. I was essentially through writing up this work when I received the fascinating preprint [20] of M. Pimsner and D. Voiculescu in which they show that the above conjecture is true. Their ingenious method of proof consists of showing that A_α can be embedded in one of the special AF algebras constructed by E. G. Effros and C. L. Shen [10] whose K_0 group is $\boldsymbol{Z} + \boldsymbol{Z}\alpha$, ordered as a subgroup of the real line $\boldsymbol{R}$. This fact, together with the results of the present paper, show that the range of the trace on the projections in A_α is exactly $(\boldsymbol{Z} + \boldsymbol{Z}\alpha) \cap [0, 1]$. And this, in turn, settles the isomorphism question. Specifically, as also stated in [20]:

THEOREM 2. *If* α *and* β *are irrational numbers in the interval* $[0, 1/2]$, *and if* A_α *and* A_β *are isomorphic, then* $\alpha = \beta$. *If* α *is any irrational number, with fractional part* $\{\alpha\}$, *let* $\beta = \{\alpha\}$ *or* $1 - \{\alpha\}$ *depending on which is in* $[0, 1/2]$. *Then* A_α *and* A_β *are isomorphic.*

In § 1 we also point out that a trivial modification of the result of Pimsner and Voiculescu also settles the isomorphism question for the algebras of $n \times n$ matrices over the A_α. Specifically, if M_n denotes the algebra of complex $n \times n$ matrices, then:

THEOREM 3. *Let* α *and* β *be irrational numbers in* $[0, 1/2]$, *and let* m *and* n *be positive integers. If* $M_m \otimes A_\alpha$ *is isomorphic to* $M_n \otimes A_\beta$, *then* $m = n$ *and* $\alpha = \beta$.

Finally, in § 2 we show how our results together with those of Pimsner and Voiculescu can also be used to settle the question of when the A_α are strongly Morita equivalent, as defined in [24]. The main result is:

THEOREM 4. *The algebras A_α and A_β are strongly Morita equivalent if and only if α and β are in thesame orbit of the action of* GL(2, $\boldsymbol{Z}$) *on irrational numbers by linear fractional transformations.*

We conclude this paper by pointing out the implications of these theorems for the transformation group C^*-algebras for flows on the torus at irrational angle, and also a curious consequence for functions of a real variable.

There still remains much that is unknown about the A_α. Among the few other facts which are known, are that the A_α are strongly amenable, hence amenable and nuclear—see [28] by J. Rosenberg. I am also familiar with unpublished work of P. Green in which he shows that the group of invertible elements in an A_α is not connected, so that the A_α are not themselves AF algebras. This result also has just appeared at the end of [3]. During second corrections of this paper I received the preprint [21] of Pimsner and Voiculescu in which they show that the K_0 group of A_α is $\boldsymbol{Z} + \boldsymbol{Z}\alpha$. They also compute the K_1 group[1]. Also, a very recent combination of arguments of S. Popa and myself [34] show that the strong Ext group of A_α is $\boldsymbol{Z} + \boldsymbol{Z}$.

The A_α occur in a variety of situations. They are exactly the C^*-algebras generated by any pair of unitary operators U and V which satisfy $UV = \lambda VU$ where $\lambda = \exp(-2\pi i\alpha)$. They can be defined as the C^*-algebras corresponding to appropriate cocycles on $\boldsymbol{Z} \times \boldsymbol{Z}$ as in [30]. They are exactly the simple C^*-algebras on which the torus group $\boldsymbol{T}^2$ has ergodic actions [1, 18, 33]. They occur as the simple non-finite-dimensional quotients of the group C^*-algebra of the Heisenberg group over $\boldsymbol{Z}$, that is, the group of 3×3 upper triangular matrices with entries in $\boldsymbol{Z}$ and ones on the diagonal [16]. They occur as the quotients by the commutator ideal of certain C^*-algebras associated to one-parameter semigroups in [7] (see also [11]). They are Morita equivalent to the transformation group C^*-algebras for flows on the torus at irrational angles. (It was Phil Green who pointed out to me that this is one consequence of the main theorem of [24], and his results in [15] can be used to give more information about the relation between these algebras.) Consequently, the A_α are strongly Morita equivalent to certain simple quotients of the group C^*-algebras of various solvable Lie groups (see closing comments in [12, 14]). The A_α are also related to the work of A. Connes [5] concerning operator algebras associated with foliations[2].

I am very indebted to R. T. Powers for having pointed out to me at an early stage the benefits of being able to calculate the

[1] See also [**32**].

[2] See also [**31**].

trace on projections, namely that if B is a separable C^*-algebra with unique normalized trace, then the range of the trace on the projections in B is a countable subset of the interval $[0, 1]$ which is an isomorphism invariant of B. I would also like to thank B. Blackadar and P. Green for helpful comments.

1. Projections. For ease of notation we will view the elements of $C(\boldsymbol{T})$ as continuous functions on the real line, $\boldsymbol{R}$, which are periodic of period 1. Thus S just becomes the shift $S(f)(t) = f(t - \alpha)$ for $f \in C(\boldsymbol{T})$ and $t \in \boldsymbol{R}$. Notice that $SM_f = M_{S(f)}S$. We will say that an element of A_α is supported on $\{-1, 0, 1\}$ if it is of the form

$$M_h S^{-1} + M_f + M_g S$$

for $h, f, g \in C(\boldsymbol{T})$. We have the following slight refinement of Theorem 1:

THEOREM 1.1. *For every $\beta \in (\boldsymbol{Z} + \boldsymbol{Z}\alpha) \cap [0, 1]$ there is a projection p in A_α, supported on $\{-1, 0, 1\}$, such that $\tau(p) = \beta$.*

Proof. Suppose that p is a projection supported on $\{-1, 0, 1\}$, and expressed, as above, in terms of h, f, g. Then from the fact that p is self-adjoint it is easily seen that f is real-valued, and that $h = S^*(\bar{g})$. Combining this with the fact that p is idempotent, one obtains:

(1) $g(t)g(t - \alpha) = 0$,

(2) $g(t)[1 - f(t) - f(t - \alpha)] = 0$,

(3) $f(t)[1 - f(t)] = |g(t)|^2 + |g(t + \alpha)|^2$,

for $t \in \boldsymbol{R}$. Conversely, it is easily seen that if f and g are elements of $C(\boldsymbol{T})$ which satisfy these equations, and if we let $h = S^*(\bar{g})$, then the corresponding element of A_α will be a projection. Closer examination then shows that there are myriad choices of f and g which satisfy these relations.

Since translation by α is the same on $C(\boldsymbol{T})$ as translation by the fractional part of α, we assume now that $\alpha \in [0, 1]$. Furthermore since S^* is translation by $1 - \alpha$, so that $A_\alpha \cong A_{1-\alpha}$, we can assume that $\alpha \in [0, 1/2]$. With this assumption, let us show first how to construct a projection p such that $\tau(p) = \alpha$. For this, let f be almost the characteristic function of $[0, \alpha]$, but rounded at the ends in a somewhat careful way. Specifically, we notice that equation (3) says that if $f(t)$ is not 0 or 1, then either $g(t)$ or $g(t + \alpha)$ is nonzero (while equation (1) says that not both can be nonzero simultaneously). Then equation (2) says that if $g(t) \neq 0$, then $f(t) + f(t - \alpha) = 1$. Choose any $\varepsilon > 0$ such that $\varepsilon < \alpha$ and $\alpha + \varepsilon < 1/2$. On $[0, \varepsilon]$ let f be any continuous function with values in $[0, 1]$ and with $f(0) = 0$ and $f(\varepsilon)=1$. On $[\alpha, \alpha+\varepsilon]$ define f by $f(t) = 1-f(t-\alpha)$, while on $[\varepsilon, \alpha]$ and $[\alpha + \varepsilon, 1]$ let f have values 1 and 0 respectively. Finally, on $[\alpha, \alpha + \varepsilon]$ define g by

$$g(t) = (f(t)(1 - f(t)))^{1/2} ,$$

and let g have value zero elsewhere on $[0, 1]$. Then f and g satisfy relations (1), (2) and (3) above and so define a projection, whose trace is $\int_0^1 f(t)dt = \alpha$.

To handle the general case, note first that, for any positive integer m, the algebra $C(\boldsymbol{T})$ contains the algebra $C_m(\boldsymbol{T})$ of continuous functions on $\boldsymbol{R}$ periodic of period $1/m$. On $C_m(\boldsymbol{T})$ the shift by α looks like the shift on $C(\boldsymbol{T})$ by $\{m\alpha\}$, the fractional part of $m\alpha$. What this means is that $A_{\{m\alpha\}}$ is embedded as a subalgebra of A_α, with the same identity element. The restriction to $A_{\{m\alpha\}}$ of the trace on A_α will be the trace on $A_{\{m\alpha\}}$, and so a projection in $A_{\{m\alpha\}}$ of trace $\{m\alpha\}$, constructed as above, will be a projection in A_α of same trace. Furthermore, elements of $A_{\{m\alpha\}}$ which are supported on $\{-1, 0, 1\}$ will also be supported there when viewed as elements of A_α.

Finally, we must treat values of form $\{-m\alpha\}$ for m positive. But for these it suffices to find projections of form $1 - \{-m\alpha\} = \{m\alpha\}$, and this is handled above. $\square$

If we combine this theorem with that of Pimsner and Voiculescu [20] described earlier, we obtain:

THEOREM 1.2. *The range of the trace on projections in* A_α *is exactly* $(\boldsymbol{Z} + \boldsymbol{Z}\alpha) \cap [0, 1]$.

To view this result in a wider context, let p and q be projections in a C^*-algebra A. We say they are unitarily equivalent if there is a unitary u in A such that $q = upu^*$. It can be shown that if $\|p - q\| < 1$, then p and q are unitarily equivalent [19]. If A is separable, it then follows that there is only a countable number of unitary equivalence classes of projections in A. Now any trace on A will be constant on unitary equivalence classes, and so the range of the trace when restricted to projections will be a countable set of positive numbers. If A has a unique normalized trace, then the range of this trace on projections will be an isomorphism invariant for A. All of this was pointed out to me by Robert T. Powers.

Now if $(\boldsymbol{Z} + \boldsymbol{Z}\alpha) \cap [0, 1] = (\boldsymbol{Z} + \boldsymbol{Z}\beta) \cap [0, 1]$, with $\alpha, \beta \in [0, 1]$, then a quick calculation shows that $\beta = \alpha$ or $1 - \alpha$. Since, as noted above, $A_\alpha \cong A_{1-\alpha}$ (and, of course, $A_\alpha \cong A_{\alpha+n}$ for all $n \in \boldsymbol{Z}$), we see that we have arrived at a proof of Theorem 2.

We turn now to the proof of Theorem 3. Let B_α denote the AF algebra constructed by Effros and Shen [10] whose K_0 group is $\boldsymbol{Z} + \boldsymbol{Z}\alpha$, and into which Pimsner and Voiculescu [20] show that A_α can be embedded (with same identity element). As they emphasize, B_α has a unique normalized trace whose range on projections is $(\boldsymbol{Z} + \boldsymbol{Z}\alpha) \cap [0, 1]$. Now $M_n \otimes B_\alpha$ will have the same K_0 group as B_α, and will also have a unique normalized trace, but this trace is easily seen now to have $(n^{-1}(\boldsymbol{Z} + \boldsymbol{Z}\alpha)) \cap [0, 1]$ as its range on projections. Since $M_n \otimes A_\alpha$ can be embedded in $M_n \otimes B_\beta$, it follows that the range of the trace for $M_n \otimes A_\alpha$ on projections must be contained in $(n^{-1}(\boldsymbol{Z} + \boldsymbol{Z}\alpha)) \cap [0, 1]$. But if $0 < j + k\alpha < n$, and if we let m denote the integer part of $j + k\alpha$, then $(j - m) + k\alpha$ is in $[0, 1]$ so that there is a projection, q, in A_α with $\tau(q) = (j - m) + k\alpha$. Since $m < n$, we can form a projection in $M_n \otimes A_\alpha$ which has q as one diagonal entry, 1's in m other diagonal entries, and 0's elsewhere. It is clear that the normalized trace for $M_n \otimes A_\alpha$ on this projection will be $n^{-1}(j + k\alpha)$. Consequently:

PROPOSITION 1.3. *The range of the normalized trace for $M_n \otimes A_\alpha$ on projections is exactly $(n^{-1}(\boldsymbol{Z} + \boldsymbol{Z}\alpha)) \cap [0, 1]$.*

Proof of Theorem 3. The range of the trace of $M_n \otimes A_\alpha$ and $M_n \otimes A_\beta$ on projections must clearly contain $1/m$ and $1/n$ respectively. From Proposition 1.3 it follows that $m = n$. Then again from Proposition 1.3, $n^{-1}\alpha = n^{-1}(p + q\beta)$ and $n^{-1}\beta = n^{-1}(r + s\alpha)$. It follows that $\alpha = \beta$. □

For the purposes of the next section, let us now interpret the above results at the level of K_0 groups, as defined in [9]. Let A be a C^*-algebra which has a faithful trace, τ. Then $K_0(A)$ will be a partially ordered group for the reasons given in [6, 12]. Furthermore, τ defines an evident homomorphism, $\hat{\tau}$, from $K_0(A)$ to $\boldsymbol{R}$, and $\hat{\tau}$ will be order preserving. From the earlier results one quickly obtains:

PROPOSITION 1.4. *As ordered group, $\hat{\tau}(K_0(A_\alpha))$ is just $\boldsymbol{Z} + \boldsymbol{Z}\alpha$ ordered as a subgroup of $\boldsymbol{R}$.*

As mentioned in the introduction, Pimsner and Voiculescu have gone on to show [21] that $\hat{\tau}$ is in fact an isomorphism of $K_0(A_\alpha)$ with $\boldsymbol{Z} + \boldsymbol{Z}\alpha$.

2. Morita equivalence. Let G be a locally compact group, and let H and K be closed subgroups of G. Then G acts by translation on G/H and G/K, and we can restrict this action to K and H respectively, so that K acts on G/H while H acts on G/K. The main theorem of [24] then says that the corresponding transformation group C^*-algebras $C^*(K, G/H)$ and $C^*(H, G/K)$ are strongly Morita equivalent.

If we apply the above to the case in which $G = \boldsymbol{R}$, $H = \boldsymbol{Z}$ and $K = \boldsymbol{Z}\alpha$, we find that $C^*(\boldsymbol{Z}\alpha, \boldsymbol{R}/\boldsymbol{Z})$ is strongly Morita equivalent to $C^*(\boldsymbol{Z}, \boldsymbol{R}/\boldsymbol{Z}\alpha)$. Now the first of these algebras is just A_α. But if we apply the homeomorphism $t \to t\alpha^{-1}$ to $\boldsymbol{R}$, we find that the second of these algebras is isomorphic to $C^*(\boldsymbol{Z}\alpha^{-1}, \boldsymbol{R}/\boldsymbol{Z})$. That is, A_α is strongly Morita equivalent to $A_{(\alpha^{-1})}$. (Of course, if we want to restrict to α in $[0, 1]$ we need to take the fractional part of α^{-1}, but for present purposes it is simpler not to make this restriction.)

As indicated earlier, A_α is obviously isomorphic to $A_{(\alpha+n)}$ for any $n \in \boldsymbol{Z}$. Let GL$(2, \boldsymbol{Z})$ denote the group of 2×2 matrices with entries in $\boldsymbol{Z}$ and with determinant ± 1, and let GL$(2, \boldsymbol{Z})$ act on the set of irrational numbers by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \alpha = \frac{a\alpha + b}{c\alpha + d} .$$

It is well-known (see Appendix B of [17]) that GL$(2, \boldsymbol{Z})$ is generated by the matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} .$$

But these are just the matrices which carry α to α^{-1}, and $\alpha + 1$ respectively. It follows that if α and β are irrational numbers which are in the same orbit of the action of GL$(2, \boldsymbol{Z})$, then A_α and A_β are strongly Morita equivalent. We will now see that by using the results of Pimsner and Voiculescu we can show the converse, thus obtaining a proof of Theorem 4.

If A and B are C^*-algebras with identity elements which are strongly Morita equivalent, then they are stably isomorphic [4], and from this it is known that A and B will have isomorphic K_0 groups. Now, as mentioned earlier, traces on a C^*-algebra define homomorphisms from the K_0 group of the algebra into $\boldsymbol{R}$. For C^*-algebras which are strongly Morita equivalent and have unique traces, the ranges of the corresponding homomorphisms from the K_0 groups will be isomorphic as groups. But note from Proposition 1.4 that the $\hat{\tau}(K_0(A_\alpha))$, as abstract groups, are all isomorphic anyway for different α. So in order to gain significant information, what we need to show is that for algebras which are strongly Morita equivalent, the isomorphisms which one obtains between the $\hat{\tau}(K_0(A_\alpha))$ are in fact order isomorphisms for the order obtained from being subgroups of $\boldsymbol{R}$. To do this we must carefully relate traces to Morita equivalence.

Recall [4] that by a corner of a C^*-algebra C with identity element we mean a subalgebra of form pCp where p is a projection in C, and that a corner is said to be full if it is not contained in any proper two-sided ideal. Now for C^*-algebras with identity elements, strong Morita equivalence is essentially the same as purely algebraic Morita equivalence. In particular, in analogy with 22.7 of [2], we have:

PROPOSITION 2.1. *If C and D are C^*-algebras which are strongly Morita equivalent, and if they both have identity elements, then each is a full corner of the algebra of $n \times n$ matrices over the other, for suitable n.*

Proof. Let X be a C-D-equivalence bimodule (i.e., imprimitivity bimodule—see 6.10 of [23]). By the definition of X, the range of $\langle\ ,\ \rangle_D$ spans a dense ideal of D. But since D has an identity element, this range must in fact coincide with D. Consequently, we can find $2n$ elements, $x_1, \cdots, x_n, y_1, \cdots, y_n$ of X such that

$$\Sigma\langle x_i, y_i\rangle_D = 1 .$$

Let M_n denote the algebra of $n \times n$ complex matrices and let $E = M_n \otimes C$. Consider X^n as an E-D-equivalence bimodule in the evident way, and let $x = \{x_1, \cdots, x_n\}$ and $y = \{y_1, \cdots, y_n\}$, which are elements of X^n. Then $\langle x, y\rangle_D = 1$. Consequently, $\langle y, x\rangle_D = 1$ also, so that

$$\begin{aligned} 1 &= \langle x, y\rangle_D\langle y, x\rangle_D = \langle x, y\langle y, x\rangle_D\rangle_D \\ &= \langle x, \langle y, y\rangle_E x\rangle_D = \langle z, z\rangle_D , \end{aligned}$$

where $z = \langle y, y\rangle_E^{1/2}x$. Then

$$\begin{aligned} \langle z, z\rangle_E\langle z, z\rangle_E &= \langle\langle z, z\rangle_E z, z\rangle_E \\ &= \langle z\langle z, z\rangle_D, z\rangle_E = \langle z, z\rangle_E , \end{aligned}$$

so that $\langle z, z\rangle_E$ is a projection, which we will denote by p. Simple calculations show that the map ϕ of D into E defined by $\phi(d) = \langle zd, z\rangle_E$ is a $*$-homomorphism which is injective and into the corner pEp. Finally, since the range of $\langle\ ,\ \rangle_E$ is dense in E, the corner pEp will be densely spanned by elements of form $p\langle x, y\rangle_E p$ for x, $y \in X^n$. But a simple calculation shows that

$$p\langle x, y\rangle_E p = \phi(\langle z, x\rangle_D\langle y, z\rangle_D) .$$

Thus the range of ϕ is exactly the corner pEp. By reversing the roles of C and D one finds in the same way that C is isomorphic to a corner in matrices over D. It is easily seen that the corners must be full. □

Now if C and D are C^*-algebras and if X is a C-D-equivalence bimodule, then every trace on C can be induced by X to give a trace on D. For the general case of possibly unbounded traces, this is implicit in Proposition 28 of [14]. But in the present case of C^*-algebras with identity elements and of finite traces, the situation is very simple:

PROPOSITION 2.2. *Let C and D be C^*-algebras with identity elements, and let X be a C-D-equivalence bimodule. Then there is a bijection between the (nonnormalized) finite traces on C and those on D, under which to a trace on C there is associated a trace τ_X on D satisfying*

$$\tau_X(\langle x, y\rangle_D) = \tau(\langle y, x\rangle_C)$$

for all $x, y \in X$.

Proof. Since the span of the range of $\langle\ ,\ \rangle_D$ is all of D, it is clear that, if τ_X exists, then τ_X is uniquely determined by the above condition. Let n, z, E and ϕ be defined as in the proof of Proposition 2.1. Let τ also denote the corresponding (nonnormalized) trace on $M_n \otimes C = E$, and let τ_X be the trace on D defined by $\tau_X(d) = \tau(\phi(d))$. We show that τ_X satisfies the above condition. Let $x, y \in X$, and view them as the elements $(x, 0, \cdots, 0)$ and $(y, 0, \cdots, 0)$ of X^n, so that

$$\tau(\langle y, x\rangle_E) = \tau(\langle y, x\rangle_C) .$$

Then

$$\begin{aligned}\tau_X(\langle x, y\rangle_D) &= \tau(\langle z\langle x, y\rangle_D, z\rangle_E)\\ &= \tau(\langle\langle z, x\rangle_E y, z\rangle_E) = \tau(\langle z, x\rangle_E\langle y, z\rangle_E)\\ &= \tau(\langle y, z\rangle_E\langle z, x\rangle_E) = \tau(\langle y\langle z, z\rangle_D, x\rangle_E)\\ &= \tau(\langle y, x\rangle_E) = \tau(\langle y, x\rangle_C) .\end{aligned}$$

□

Let C, D and X still be as above, and let A be the linking algebra for X as defined on page 350 of [4]. If τ is a trace on C and if τ_X is defined as in Proposition 2.2, then a straightforward calculation shows that the functional on A defined by using τ and τ_X to evaluate on the diagonal of elements of A will be a trace. In fact one quickly sees in this way that:

PROPOSITION 2.3. *Let C, D and X be as above, and let A be the linking algebra for X. Then each trace on C has a unique extension to a trace on A. The restriction to D of this trace on A will be τ_X.*

By construction, C and D sit as complementary full corners of the linking algebra A. We recall that if ψ is any homomorphism between C^*-algebras (possibly not preserving identity elements), then ψ induces a homomorphism between the corresponding K_0 groups. This homomorphism is described in [9], and is denoted by $\tilde{\psi}_*$. We now need the following fact, which is undoubtedly familiar to other workers in this area:

PROPOSITION 2.4. *Let A be a C^*-algebra with identity element, let pAp be a full corner of A, and let ψ be the injection of pAp into A. Then $\tilde{\psi}_*$ is an isomorphism of $K_0(pAp)$ with $K_0(A)$.*

Proof. View $X = pA$ as a pAp-A-equivalence bimodule (6.8 of [23]). Then, as in the proof of Proposition 2.1, we can find $a_1, \cdots, a_n \in A$ such that $\Sigma a_i^* p a_i = 1$. Let ϕ be the corresponding map of A into $M_n(pAp)$, so that $\phi(a) = (pa_i a a_j^* p)_{i,j}$, and ϕ maps A onto the corner of $M_n(pAp)$ defined by the projection $(pa_i a_j^* p)_{i,j} = P$. Let V be the element of $M_n(A)$ whose first column consists of $pa_1, \cdots, pa_n$, and all of whose other entries are 0. Then a simple calculation shows that $VV^* = P$, while V^*V is the matrix with 1 in the upper left corner, and 0 elsewhere. Thus, "conjugation" of $M_n(A)$ by V carries $\phi(A)$ onto the corner of $M_n(A)$ consisting of the matrices all of whose entries are zero except that in the upper left corner. If we view ψ as giving also the inclusion of $M_n(pAp)$ into $M_n(A)$, we see in this way that $\tilde{\psi}_* \circ \tilde{\phi}_*$ is the identity map on $K_0(A)$. (We use here the fact that, as remarked in [9] immediately after the proof of Lemma 3.6, it does not matter whether one uses unitary or Murrey-von Neumann equivalence in defining K_0.) It follows that $\tilde{\psi}_*$ is surjective. Thus we have shown that the inclusion map of a full corner into an algebra induces a surjection of K_0 groups. But $\phi(A)$ is a full corner of $M_n(pAp)$, and ϕ is an isomorphism of A with $\phi(A)$. It follows that $\tilde{\phi}_*$ is surjective. Since $\tilde{\psi}_* \circ \tilde{\phi}_*$ is an isomorphism, it follows that $\tilde{\psi}_*$ must be injective, and so is an isomorphism. □

Now let again X be a C-D-equivalence bimodule, and let A be the linking algebra for X. Let $\tilde{\psi}_*$ and $\tilde{\theta}_*$ denote the isomorphisms of $K_0(C)$ and $K_0(D)$ with $K_0(A)$ obtained from the inclusions of C and D as corners of A. We will let Φ_X denote the isomorphism $(\tilde{\theta}_*)^{-1} \circ \tilde{\psi}_*$ of $K_0(C)$ with $K_0(D)$. (We remark in passing that by this means one can see that the Picard group of a C^*-algebra B, as defined in [4], will act as a group of automorphisms of $K_0(B)$.)

PROPOSITION 2.5. *Let C and D be C^*-algebras with identity, let X be a C-D-equivalence bimodule, let τ be a finite trace on C and let τ_X be the corresponding (nonnormalized) trace on D defined above. Let Φ_X denote the isomorphism of $K_0(C)$ onto $K_0(D)$ determined by X as above, and let $\hat{\tau}$ and $\hat{\tau}_X$ be the homomorphisms of $K_0(C)$ and $K_0(D)$ into $\boldsymbol{R}$ determined by τ and τ_X. Then*

$$\hat{\tau}_X \circ \Phi_X = \hat{\tau} .$$

Proof. This follows immediately from the definitions and the fact that τ_X is the restriction to D of the unique extension of τ to the linking algebra. □

COROLLARY 2.6. *Let C, D and X be as above, let τ be a trace on C, and let τ_X be the corresponding trace on D. Then the ranges of $\hat{\tau}$ and $\hat{\tau}_X$ are the same.*

Proof of Theorem 4. Suppose that A_α and A_β are strongly Morita equivalent, with equivalence bimodule X. Let τ be the normalized trace on A_α, and let τ_X be the corresponding (nonnormalized) trace on A_β, so that $\hat{\tau}_X(K_0(A_\beta)) = \hat{\tau}(K_0(A_\alpha))$. Now τ_X differs from the normalized trace on A_β only by a scalar multiple. From this and Proposition 1.4 it follows that there is a positive real number r such that $\boldsymbol{Z} + \boldsymbol{Z}\beta = r(\boldsymbol{Z} + \boldsymbol{Z}\alpha)$. In particular, there are $j, k, m, n \in \boldsymbol{Z}$ such that $j + k\beta = r$ and $1 = r(m + n\alpha)$. On eliminating r from these equations one finds that α and β are in the same orbit of $\mathrm{GL}(2, \boldsymbol{Z})$. This is a special case of the fact that if $\boldsymbol{Z} + \boldsymbol{Z}\alpha$ and $\boldsymbol{Z} + \boldsymbol{Z}\beta$ are isomorphic as ordered groups, then α and β are in the same orbit of $\mathrm{GL}(2, \boldsymbol{Z})$, as mentioned in [7, 10] and shown in Lemma 4.7 of [29]. □

We remark that the situation described in the first two paragraphs of this section is also interesting at the von Neumann algebra level. Specifically, let M and N denote the von Neumann algebras on $L^2(\boldsymbol{R})$ generated by $C^*(\boldsymbol{Z}\alpha, \boldsymbol{R}/\boldsymbol{Z})$ and $C^*(\boldsymbol{Z}, \boldsymbol{R}/\boldsymbol{Z}\alpha)$ respectively. Then M and N are finite factors which are each other's commutants. Thus the coupling constant between them is defined, and a simple calculation shows that this coupling constant is just α. I plan to discuss this matter and its generalizations in a future paper [35, 36].

Let $\boldsymbol{R}$ act on the torus $\boldsymbol{T}^2$ by the flow at an irrational angle, α, and let C_α denote the corresponding transformation group C^*-

algebra. As mentioned in the introduction, Philip Green pointed out to me some years ago that one consequence of the main theorem of [24] is that C_α is Morita equivalent to A_α, if the bookkeeping is done correctly. It is well-known that the flow at an irrational angle is the "flow under the constant function" corresponding to the rotation on $\boldsymbol{T}$ by angle α, and Phil Green has shown, in as of yet unpublished work, that quite generally the transformation group C^*-algebra for the flow under a constant function is strongly Morita equivalent to that for the original transformation[3]. Moreover, from the results in [15] he can conclude even more, namely that the C^*-algebra for the flow is isomorphic to the tensor product of the algebra of compact operators with the C^*-algebra for the transformation. In particular, C_α will be stable [4]. It follows then from [4] that if C_α and C_β are strongly Morita equivalent, then they are in fact isomorphic.

Now the group of automorphisms of $\boldsymbol{T}^2$ is $\mathrm{GL}(2, \boldsymbol{Z})$, via its evident action as automorphisms of $\boldsymbol{Z}^2$, the dual group of $\boldsymbol{T}^2$. Furthermore, the corresponding action of $\mathrm{GL}(2, \boldsymbol{Z})$ on the one-parameter subgroups of $\boldsymbol{T}^2$ of irrational slope is according to the action on irrational numbers by fractional linear transformations described earlier. It follows that C_α and C_β are isomorphic if α and β are in the same orbit under the action of $\mathrm{GL}(2, \boldsymbol{Z})$. With hindsight, this might be viewed as the reason that the corresponding A_α and A_β are strongly Morita equivalent. Now if α and β are not in the same orbit of $\mathrm{GL}(2, \boldsymbol{Z})$ then we have seen that A_α and A_β are not strongly Morita equivalent. Consequently:

THEOREM 2.7. *The algebras C_α and C_β are isomorphic if and only if α and β are in the same orbit of* $\mathrm{GL}(2, \boldsymbol{Z})$. *If α and β are not in the same orbit, then C_α and C_β are not even strongly Morita equivalent.*

We conclude with a curiosity. By specializing the formulas of [24], the strong Morita equivalence of A_α with $A_{(\alpha^{-1})}$ can be described quite explicitly. Specifically, $C_c(\boldsymbol{R})$, the space of continuous functions of compact support on $\boldsymbol{R}$, forms the natural equivalence (i.e., imprimitivity) bimodule between the pre-C^*-algebras $C_c(\boldsymbol{Z}\alpha, \boldsymbol{R}/\boldsymbol{Z})$ and $C_c(\boldsymbol{Z}, \boldsymbol{R}/\boldsymbol{Z}\alpha)$. The actions of these algebras on $C_c(\boldsymbol{R})$ come from the corresponding "covariant representations" obtained from translation on $C_c(\boldsymbol{R})$ by $\boldsymbol{Z}\alpha$ and $\boldsymbol{Z}$, and pointwise multiplication by functions of period 1 and α respectively. If we denote the above algebras by C and D respectively, as in [24], then the algebra-valued inner products are given by

$$\langle F, G\rangle_C(m, r) = \Sigma_n F(r - n)\bar{G}(r - n - m\alpha)$$

[3] See also [**37**].

$$\langle F, G\rangle_D(m, r) = \Sigma_n \bar{F}(r - n\alpha)G(r - n\alpha - m)$$

for $F, G \in C_c(\boldsymbol{R})$, $m \in \boldsymbol{Z}$, $r \in \boldsymbol{R}$, where we write $f(m, r)$ instead of the $f_m(r)$ used earlier in this paper.

Let us consider projections in C which are of the form $\langle F, F\rangle_C$. First, notice that there are many of them. For according to Proposition 2.1, D can be embedded as a corner in $n \times n$ matrices over C. But in the present situation something special happens, namely that in one direction matrices of size 1×1 will work. To see this, assume that $\alpha \in [0, 1]$, and let G be any nonnegative function in $C_c(\boldsymbol{R})$ which is supported in an interval of length strictly smaller than 1, but is strictly positive on an interval of length greater than α. Then from the first of these conditions it follows that $\langle G, G\rangle_D(m, r) = 0$ if $m \neq 0$, whereas from the second condition it follows that $\langle G, G\rangle_D(0, r) > 0$ for all $r \in \boldsymbol{R}$. In other words, $\langle G, G\rangle_D$ is in the (Cartan) subalgebra $C(\boldsymbol{R}/\boldsymbol{Z}\alpha)$ of D, and is invertible there. Let $H = G*(\langle G, G\rangle_D)^{-1/2}$. Then $H \in C_c(\boldsymbol{R})$ and $\langle H, H\rangle_D = 1_D$. As seen earlier, it follows that $\langle H, H\rangle_C$ is a projection in C. In fact, it was exactly this observation which led me to discover the projections described in § 1. Now, again as seen earlier, the map $f \to \langle H*f, H\rangle_C$ will be an isomorphism of D onto a corner of C (except for the fact that C is not complete). In particular if p is any projection in D, then $\langle H*p, H*p\rangle_C$ will be a projection in C. Since we saw in § 1 that D contains many projections, of different sizes, it follows that many $(F, F)_C$ are projections, of many sizes.

There is a simple abstract characterization of those F which give projections:

PROPOSITION 2.8. *Let X be an A-B-equivalence bimodule and let $x \in X$. Then $\langle x, x\rangle_A$ is a projection* iff $x\langle x, x\rangle_B = x$.

Proof. Suppose this last equation holds. Then from the fact that $x\langle x, x\rangle_B = \langle x, x\rangle_A x$ it is easily seen that $\langle x, x\rangle_A$ is idempotent, and so is a projection since it is self-adjoint. Conversely, suppose that $\langle x, x\rangle_A$ is a projection. Then a simple calculation shows that

$$\langle x\langle x, x\rangle_B - x, x\langle x, x\rangle_B - x\rangle_A = 0 ,$$

so that $x\langle x, x\rangle_B = x$. $\square$

Now the trace of $\langle F, F\rangle_C$ is given by

$$\begin{aligned}\tau(\langle F, F\rangle_C) &= \int_0^1 \Sigma_n F(r - n)\bar{F}(r - n)dr \\ &= \int_{-\infty}^{\infty} |F(r)|^2 dr .\end{aligned}$$

Putting together this observation with Theorem 1.2 and Proposition 2.11 we obtain:

PROPOSITION 2.9. *Let α be an irrational number, and let F be an element of $C_c(\boldsymbol{R})$ which satisfies the functional equation*

$$F(r) = \Sigma_{m,n} F(r-m)\bar{F}(r-m-n\alpha)F(r-n\alpha) .$$

(There are many such F.) Then

$$\int_{-\infty}^{\infty} |F(r)|^2 dr \in (\boldsymbol{Z} + \boldsymbol{Z}\alpha) \cap [0, 1] .$$

It is an interesting challenge to find a proof of this result concerning functions of a real variable which does not use C^*-algebra techniques.

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Received January 2, 1980. This work was partially supported by National Science Foundation grant MCS 77-13070.

UNIVERSITY OF CALIFORNIA
BERKELEY, CA 94720