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A CHARACTERIZATION OF LOCALLY MACAULAY COMPLETIONS

CRAIG HUNEKE

The purpose of this note is to prove the following theorem.

THEOREM 1.1. Let (R, m) be a Noetherian local ring of dimension $d \geqq 1$ and depth $d-1$. By $\hat{R}$ denote the completion of R in the m-adic topology. Then the following are equivalent:

(1) $\hat{R}$ is equidimensional and satisfies Serre's property S_{d-1}

(2) $H_m^{d-1}(R)$ has finite length

(3) There exists an $N > 0$ such that if $x_1, \cdots, x_d$ is a sequence of elements R with $\mathrm{ht}(x_{i_1}, \cdots, x_{i_j}) = j$ for all j-elements subsets of $\{1, \cdots, n\}$, $1 \leqq j \leqq n$, and if $m_i \geqq N$, $1 \leqq i \leqq d$, then $x_1^{m_1}, \cdots, x_d^{m_d}$ is an unconditioned d-sequence.

Recall the local ring (S, N) is *equidimensional* if for every minimal prime divisor p of zero, $\dim S/p = \dim S$.

Serre's property S_k is that

$$\operatorname{depth} R_p \geqq \min [\operatorname{ht} p, k]$$

for all primes p.

We will always denote the local cohomology functor by $H_m^j(\_\_)$ ([1]).

We recall the definition of a d-sequence due to this author [3].

DEFINITION 0.1. A system of elements $x_1, \cdots, x_d$ in a commutative ring R is said to be a d-sequence if

(1) $x_i \notin (x_1, \cdots, \hat{x}_i, \cdots, x_d)$

(2) $((x_1, \cdots, x_i): x_{i+1}x_k) = ((x_1, \cdots, x_i): x_k)$ for $k \geqq i+1$ and $i \geqq 0$.

A d-sequence is said to be unconditioned if any permutation of it remains a d-sequence.

These have been studied extensively by this author and have been useful to determine the "analytic" properties of ideals generated by them. In [3] the following was skown:

PROPOSITION. *Let (R, m) be a local Noetherian ring. Then R is Buchsbaum (see [10] for a definition and discussion) if and only if every system of parameters forms a d-sequence.*

Thus Theorem 1.1 may be seen as a related result, characterizing rings in which "almost all" s.o.p.'s form a d-sequence. Independent of this characterization of rings with "lots" of d-sequences, Theorem 1.1 is the generalization of a result due to Steven McAdam [7] which in turn is related to a characterization of unmixed 2-dimensional local rings proved by Ratliff [8].

Let (R, m) be a 2-dimensional local domain and let b, c be a system of parameters. By $S(b, c, n)$ denote the least k such that

$$(b^n: c^k) = (b^n: c^{k+1}) .$$

Recall a local ring R is said to be *unmixed* if for each prime divisor p of (0) in $\hat{R}$, $\dim \hat{R}/p = \dim \hat{R}$.

Ratliff showed, [8],

PROPOSITION. *The following are equivalent for a 2-dimensional local domain*

(1) *R is unmixed.*

(2) *$S(b, c, \_)$ is bounded.*

(3) *$R^{(1)} = \bigcap_{\mathrm{ht}\, p=1} R_p$ is a finite R-module.*

McAdam discussed this and obtained the following improvement:

PROPOSITION [5]. *Let (R, m) be as above. Then the following are equivalent:*

(1) *R is unmixed*, i.e., *for all prime divisors p of* (0) *in $\hat{R}$*, $\dim \hat{R}/p = \dim \hat{R} = 2$.

(2) *$R^{(1)}$ is a finite R-module.*

(3) *There exists an N such that for every* s.o.p. *x, y*

$$S(x, y, \_) \leqq N .$$

In particular, (3) is equivalent to saying for all $n \geqq N$ that $(x^n: y^n) = (x^n: y^{2n})$ and this is equivalent (in this case) to saying x^n, y^n form a d-sequence.

To see our statement (1) is equivalent to (1) of the above proposition, note that if $\dim R = 2$ and R is a domain, then to say R is unmixed is precisely to say $\hat{R}$ satisfies S_1 and is equidimensional.

Finally, we will show that $R^{(1)}/R$ is isomorphic to $H_m^1(R)$ in this case, and show that $R^{(1)}/R$ has finite length if and only if $R^{(1)}$ is a finitely generated R-module. Hence our Theorem 1.1 is the exact generalization of the above proposition of McAdam.

1. Proof of Theorem 1.1. For details on local cohomology we refer the reader to [1]. We note the following facts.

(1) Since $\operatorname{depth} R = d - 1$, $H_m^i(R) = 0$ if $i < d - 1$.

(2) There is a canonical isomorphism, $H_m^{d-1}(R) \cong H_{\hat{m}}^{d-1}(\hat{R})$.

(3) If S is a complete regular local ring mapping onto $\hat{R}$ (see [6]) and M is the maximal ideal of S, then $H_m^{d-1}(R) \cong H_M^{d-1}(\hat{R})$ where $\hat{R}$ is regarded as an S-module.

(4) If S is chosen as in (3), $e = \dim S$, and we let $E = H_M^e(S/M) =$ an injective hull of S/M, then

$$\operatorname{Hom}_S(H_m^j(R), E) \cong \operatorname{Ext}_S^{e-j}(\hat{R}, S)$$

and $H_m^j(R) \cong \operatorname{Hom}_S(\operatorname{Ext}_S^{e-j}(\hat{R}, S), E)$. This is local duality.

(5) We may compute $H_m^{d-1}(R)$ as follows: let $x_1, \cdots, x_d$ be an s.o.p., and consider the complex,

$$\bigoplus_{i<j} R_{x_1,\cdots,\hat{x}_i,\cdots,\hat{x}_j,\cdots,x_n} \longrightarrow \bigoplus_i R_{x_1,\cdots,\hat{x}_i,\cdots,x_n} \longrightarrow R_{x_1,\cdots,x_n} \longrightarrow 0$$

where the subscripts denote localization at the elements subscripted. Then $H_m^{d-1}(R)$ is isomorphic to the middle homology of this complex. If we denote by $\operatorname{syz}(x_1, \cdots, x_d)$ the module defined by K/L where $K \subseteq R^d$ is the module of syzygies of $x_1, \cdots, x_d$ and L is the submodule of syzygies which come from the trivial ones given by the Koszul relations, then

$$H_m^{d-1}(R) \cong \varinjlim \operatorname{syz}(x_1^{n_1}, \cdots, x_d^{n_d})$$

where if $m_i \geqq n_i$, the map

$$\operatorname{syz}(x_1^{n_1}, \cdots, x_d^{n_d}) \longrightarrow \operatorname{syz}(x_1^{m_1}, \cdots, x_d^{m_d})$$

is defined by mapping a syzygy $(r_1, \cdots, r_d)$ of $(x_1^{n_1}, \cdots, x_d^{n_d})$ to the syzygy $(r_1 x_2^{m_2-n_2} \cdots x_d^{m_d-n_d}, \cdots, r_d x_1^{m_1-n_1} \cdots x_{d-1}^{m_{d-1}-n_{d-1}})$ of $(x_1^{m_1}, \cdots, x_d^{m_d})$. We now turn to the proof of Theorem 1.1.

The fact (1) if and only if (2) holds is well-known but we give the details here for completeness.

We first observe that $H_m^{d-1}(R)$ has finite length if and only if $\operatorname{Hom}_S(H_m^{d-1}(R), E) \cong \operatorname{Ext}_S^{e-(d-1)}(\hat{R}, S)$ has finite length. (See [5].)

If p is a prime in S and $\hat{R} \cong S/I$, then if $p \not\supseteq I$

$$(\operatorname{Ext}_S^{e-(d-1)}(\hat{R}, S))_p = 0\ .$$

Hence, $\operatorname{Ext}_S^{e-(d-1)}(\hat{R}, S)$ has finite length if and only if

$$(\operatorname{Ext}_S^{e-(d-1)}(\hat{R}, S))_p = \operatorname{Ext}_{S_p}^{e-(d-1)}((\hat{R}_p, S_p) = 0 \quad \text{for all} \quad p \supseteq I\ , \quad p \neq M\ .$$

If $i < d-1$, then since $\operatorname{depth} \hat{R} = \operatorname{depth} R = d-1$, we see

$$H_{\hat{m}}^i(\hat{R}) = H_M^i(\hat{R}) = 0$$

and so

$$\operatorname{Ext}_S^{e-i}(\hat{R}, S) = 0$$

or, otherwise put,

$$\operatorname{Ext}^k_{S_p}(\hat{R}_p, S_p) = 0$$

for all $k \geqq e - (d-1)$ if and only if $H^{d-1}_m(R)$ has finite length. (Note for $k > e$, $\operatorname{Ext}^k_S(M, S) = 0$ for all M.)

Since S_p is regular,

$$\operatorname{Sup}_n \{\operatorname{Ext}^n_{S_p}(\hat{R}_p, S_p) \neq 0\} + \operatorname{depth} \hat{R}_p = \dim S_p . \quad \text{(See [9.])}$$

From this we may conclude that $H^{d-1}_m(R)$ has finite length if and only if $\operatorname{depth}(\hat{R})_p > \dim S_p - (e - (d-1))$ i.e., if and only if

$$\operatorname{depth}(\hat{R})_p \geqq \dim S_p - \dim S + \dim \hat{R} .$$

We claim that

$$\dim S_p - \dim S + \dim \hat{R} \geqq \dim (\hat{R})_p$$

in any case. For since S is regular, $\dim S = \dim S_p + \dim S/p$ and so the left side is just

$$-\dim S/p + \dim \hat{R} .$$

Thus it is enough to show

$$\dim \hat{R} \geqq \dim S/p + \dim (\hat{R})_p$$

but this clearly always holds since p contains I.

Thus we have shown $H^{d-1}_m(R)$ has finite length if and only if

$$(*) \qquad \operatorname{depth}(\hat{R})_p \geqq \dim S_p - \dim S + \dim \hat{R} \geqq \dim (\hat{R})_p .$$

We claim these last two inequalities occur if and only if $\hat{R}$ satisfies S_{d-1} and is equidimensional.

If (*) occurs then clearly $(\hat{R})_p$ must be Cohen-Macaulay for all $p \neq \hat{m}$, and since $\operatorname{depth} \hat{R} = d - 1$, this shows $\hat{R}$ satisfies S_{d-1}. Since we must have

$$\dim (\hat{R})_p = \dim S_p - \dim S + \dim \hat{R}$$

in this case, the work above shows that for all $p \supseteqq I$,

$$\dim \hat{R} = \dim S/p + \dim (\hat{R})_p ,$$

and this shows $\hat{R}$ is equidimensional.

Conversely, since $\hat{R}$ is catenary, if $\hat{R}$ satisfies S_{d-1} and is equidimensional then

$$(\mathrm{a}) \qquad \operatorname{depth}(\hat{R})_p = \dim (\hat{R})_p$$

for all primes $p \neq \hat{m}$, and

$$\text{(b)} \qquad \dim \hat{R} = \dim S/p + \dim (\hat{R})_p$$

for all primes p. Thus in this case (*) holds and so $H_m^{d-1}(R)$ has finite length.

We now show (2) if and only if (3). Assume (2). Then there is a N such that $m^N H_m^{d-1}(R) = 0$. It was shown in [2] that if $R \to S$ faithfully flat and $x_1, \cdots, x_n \in R$ then these elements form a d-sequence in R if and only if they form a d-sequence in S. Thus we may work in $\hat{R}$ and assume R is complete for the remainder of this implication. By (1), R is locally Cohen-Macaulay on the punctured spectrum, i.e., R satisfies Serre's condition S_{d-1}.

Now let $x_1, \cdots, x_d$ be in R such that $\operatorname{ht}(x_{j_1}, \cdots, x_{j_i}) = i$ for each i, $1 \leqq i \leqq d$.

Then since R satisfies S_{d-1}, $x_{i_1}, \cdots, x_{i_{d-1}}$ form an R-sequence for any $d-1$ of $\{x_1, \cdots, x_d\}$. Hence to show (3) it is enough to show for $m_i \geqq N$ that

$$((x_1^{m_1}, \cdots, \hat{x}_i, \cdots, x_d^{m_d}) : x_i^{2m_i}) = ((x_1^{m_1}, \cdots, \hat{x}_i, \cdots, x_d^{m_d}) : x_i^{m_i}) .$$

Since we may rearrange the x_i we may assume $i = d$. Suppose $(r_1, \cdots, r_d)$ is a syzygy of $(x_1^{m_1}, \cdots, x_{d-1}^{m_{d-1}}, x_d^{2m_d})$. Since $m^N H_m^{d-1}(R) = 0$ we see that $x_d^{m_d}$ must kill the image of this syzygy in $H_m^{d-1}(R)$.

By the construction (5) above we see this means that

$$(r_1 x_d^{m_d}(x_2, \cdots, x_d)^M, \cdots, r_d x_d^{m_d}(x_1, \cdots, x_{d-1})^M)$$

becomes a trivial syzygy of

$$(x_1^{m_1+M}, \cdots, x_{d-1}^{m_{d-1}+M}, x_d^{2m_d+M}) .$$

In particular,

$$r_d x_d^{m_d}(x_1, \cdots, x_{d-1})^M \in (x_1^{m_1+M}, \cdots, x_{d-1}^{m_{d-1}+M}) .$$

As $x_1, \cdots, x_{d-1}$ forms an R-sequence, this shows (see [4]) that

$$r_d x_d^{m_d} \in (x_1^{m_1}, \cdots, x_{d-1}^{m_{d-1}})$$

which shows (3).

Now assume (3) and let us show (2). First, we show,

LEMMA 1.1. *Let (R, m) be a local Noetherian ring of dimension. d. Suppose for every $x_1, \cdots, x_d$ in m such that height $(x_1, \cdots, x_j) = j$, there exist integers $m_1, \cdots, m_d \geqq 1$ such that $x_1^{m_1}, \cdots, x_d^{m_d}$ form a d-sequence. Then R_p is Cohen-Macaulay for all $p \neq m$.*

Proof. Let p be a minimal prime in R with R_p not Cohen-Macaulay. If height $p = n$, choose $a_1, \cdots, a_n$ in p such that height

$(a_1, \cdots, a_i) = i$. Complete $a_1, \cdots, a_n$ to a system of parameters $a_1, \cdots, a_n, a_{n+1}, \cdots, a_d$ of R with $\mathrm{ht}(a_1, \cdots, a_i) = i$. Since p is the minimal prime which is not Cohen-Macaulay, we may assume p is associated to $(a_1, \cdots, a_i)$ with $i < n$. Let $m_1, \cdots, m_d$ be chosen so that $a_1^{m_1}, \cdots, a_d^{m_d}$ form a d-sequence. Then p is still associated to $a_1^{m_1}, \cdots, a_i^{m_i}$. By [3],

$$(a_1^{m_1}, \cdots, a_i^{m_i}) = ((a_1^{m_1}, \cdots, a_i^{m_i}): a_{i+1}^{m_{i+1}}) \cap (a_1^{m_1}, \cdots, a_d^{m_d}) .$$

Now since $(a_1^{m_1}, \cdots, a_d^{m_d})$ is primary to m, this decomposition shows that p is associated to $((a_1^{m_1}, \cdots, a_i^{m_i}): a_{i+1}^{m_{i+1}})$. However $a_{i+1}^{m_{i+1}} \in p$ and $a_{i+1}^{m_{i+1}}$ is not a zero divisor modulo $((a_1^{m_1}, \cdots, a_i^{m_i}): a_{i+1}^{m_{i+1}})$. This contradiction proves the lemma.

Now assume (3). By Lemma 1.1 R satisfies S_{d-1}. (Note we may *not* assume $\hat{R}$ satisfies S_{d-1}!)

Hence if $x_1, \cdots, x_d$ are chosen so that height $(x_{j_1}, \cdots, x_{j_i}) = i$ for all $1 \leqq i \leqq d$, to show $H_m^{d-1}(R) = 0$ it is enough to show in this case that if such an $x_1, \cdots, x_d$ are a d-sequence, then

$$\mathrm{syz}\,(x_1, \cdots, x_d) \longrightarrow \mathrm{syz}\,(x_1, \cdots, x_{d-1}, x_d^2)$$

is onto. For if we can show this, then it is clear that the map

$$\mathrm{syz}\,(x_1^N, \cdots, x_d^N) \longrightarrow H_m^{d-1}(R)$$

will be onto, where N is as in (3). This will show $H_m^{d-1}(R)$ is finitely generated; as $H_m^{d-1}(R)$ satisfies the descending chain condition, this will show (2).

So let $(r_1, \cdots, r_d)$ be a syzygy of $x_1, \cdots, x_{d-1}, x_d^2$. Then since

$$r_d \in ((x_1, \cdots, x_{d-1}): x_d^2) = ((x_1, \cdots, x_{d-1}): x_d)$$

we see

$$0 = r_d x_d + \sum_{j=1}^{d-1} s_i x_i \,, \qquad \text{and hence}$$

$$(r_1 - s_1 x_d)x_1 + \cdots + (r_{d-1} - s_{d-1}x_d)x_{d-1} = 0 .$$

Thus, $(r_1 - s_1x_d, \cdots, r_{d-1} - s_{d-1}x_d, 0)$ is a syzygy of $(x_1, \cdots, x_{d-1}, x_d^2)$. Since $x_1, \cdots, x_{d-1}$ will form an R-sequence, this syzygy of $(x_1, \cdots, x_{d-1}, x_d^2)$ will be trivial. Hence the image of $(s_1, \cdots, s_{d-1}, r_d)$ in $\mathrm{syz}\,(x_1, \cdots, x_d)$ will map onto $(r_1, \cdots, r_d) \in \mathrm{syz}\,(x_1, \cdots, x_d^2)$. This finishes the proof of Theorem 1.1.

Finally, we wish to relate condition (2) of Theorem 1.1 to the finiteness of $R^{(1)}$. To this end, let (R, m) be a 2-dimensional Noetherian local domain and let $S = R^{(1)} = \bigcap R_p$ taken over all height one primes p. If t is in S, then $J = \{r \in R \mid rt \in R\}$ is not contained in any height one prime and is thus primary to m. Hence if x, y is an s.o.p., $x^k \in J$ for some k. Then $x^k t = r \in R$ and so $t = r/x^k$. Thus $J = (x^k : r)$

is primary to m, and so $y^m \in J$ for some J which shows $r \in (x^k : y^m)$ for some m. Thus (see McAdam [7]), $S = \{r/x^k \mid r \in (x^k : y^m) \text{ some } k, m\}$. (The converse is easy to see; i.e., such r/x^k are indeed in R_p for all height one primes p.)

Now $H^1_m(R)$ in this case is the middle homology of

$$R \longrightarrow R_x \oplus R_y \longrightarrow R_{xy} \longrightarrow 0 .$$

That is, if

$$\{(r/x^k, s/y^e) \mid r/x^k - s/y^e = 0\} = N$$

and $M = \{(r, r) \mid r \in R\}$ then

$$H^1_m(R) \cong N/M .$$

(Note $r/x^c + s/y^e = 0$ if and only if $ry^e + sx^k = 0$ since R is a domain.)

We map S onto $H^1_m(R)$ as follows: if $t \in S$, let $g(t) = (t, t) \in N/M$. The discussion above shows $t \in R_x \cap R_y$ and so the map $g(\_)$ makes sense. This map is clearly onto since

$$S = \{r/x^k \mid r \in (x^k : y^m) \text{ for some } k, m\} .$$

The kernel is the set of $t \in S$ such that $(t, t) \in M$; this is precisely if $t \in R$.

We have therefore shown

$$H^1_m(R) \cong S/R .$$

Now if S is finitely generated over R, then $H^1_m(R)$ is also and so it has finite length. Conversely, if $H^1_m(R) = S/R$ has finite length, then S is obviously a finite R-module.

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UNIVERSITY OF MICHIGAN
ANN ARBOR, MI 48109