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ULTRASPHERICAL EXPANSIONS AND PSEUDO ANALYTIC FUNCTIONS

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This paper takes up the function theoretic approach to the study of ultraspherical expansions, their conjugates, the associated elliptic equations, and first order systems. The theory of pseudo analytic functions and Bergman-Gilbert type integral operators are employed, and the relation between these two approaches is examined. Throughout, results obtained are analogs of well known theorems from the theory of analytic functions of a single complex variable, and the related study of harmonic functions and Fourier series.

The study of trigonometric series, analytic functions, Laplace's equation, and the Cauchy-Riemann system are all in a sense equivalent. Since this study has proven to be one of the most fruitful in mathematics, and since Laplace's equation is just one specific elliptic partial differential equation, analogous developments should be expected for more general elliptic equations. In particular, it is natural to hope for a relationship with analytic functions corresponding to that found in the case of harmonic functions, $u=\mathrm{Re}(f)$, which has proved so useful in the study of Laplace's equation and expansions in the associated special functions (trigonometric series).

For u the solution of an elliptic equation more general than Laplace's, two approaches are apparent:

(1) Generalize the operation of "taking the real part". That is, find bounded linear operators which transform analytic functions to solutions u. Such results have been obtained, in particular, by Bergman [3] and Gilbert [14], where *integral operators* are developed to provide the transformation from analytic functions to solutions of corresponding elliptic equations.

(2) Generalize function theory. That is, extend function theory so that solutions u of elliptic equations can be obtained as $u=\mathrm{Re}(f)$, where f is a *"pseudo" analytic function* sharing many of the properties associated with classical analytic functions of a single complex variable. Bers [4], and Vekua [24] have developed such an approach.

In the pursuit of function theoretic results in this context, the case of ultraspherical expansions is of particular interest, since in this case both approaches (1) and (2) are directly applicable and intimately related. Using them, we obtain function theoretic results for ultraspherical expansions, adding to the extensive work of Muchenhaupt and Stein [19] on the subject.

The ultraspherical (or Gegenbauer) polynomials $C_n^\mu(x)$, $\mu > 0$, of degree n are defined by the generating relation

$$(1.1)\qquad (1 - 2zx + z^2)^{-\mu} = \sum_{n=0}^{\infty} z^n C_n^\mu(x) .$$

The sequence $\{C_n^\mu(\cos\theta)\}_{n=0}^{\infty}$ is orthogonal and complete over $(0, \pi)$ with respect to the measure $\sin^{2\mu}\theta d\theta$. Thus a function $f(\theta)$ defined on $(0, \pi)$ has the expansion

$$(1.2)\qquad f(\theta) \sim \sum_{n=0}^{\infty} a_n C_n^\mu(\cos\theta) ,$$

where

$$a_n = \frac{\int_0^\pi f(\theta) C_n^\mu(\cos\theta) \sin^{2\mu}\theta d\theta}{\int_0^\pi [C_n^\mu(\cos\theta)]^2 \sin^{2\mu}\theta d\theta} = \frac{2^{2\mu-1} n!(n+\mu)\Gamma^2(\mu)}{\pi\Gamma(n+2\mu)} \int_0^\pi f(\theta) C_n^\mu(\cos\theta) \sin^{2\mu}\theta d\theta .$$

The Abel sums

$$(1.3)\qquad U(r, \theta) = \sum_{n=0}^{\infty} a_n r^n C_n^\mu(\cos\theta)$$

of the series (1.2) arise as solutions of the partial differential equation

$$(1.4)\qquad U_{xx} + U_{yy} + \frac{2\mu}{y} U_y = 0 ,$$

$(x, y) = (r\cos\theta,\ r\sin\theta)$, and are called generalized axisymmetric potentials (see [26] for a discussion of this equation and physical applications). Corresponding to equation (1.3) is the first order system

$$(1.5)\qquad \begin{aligned} U_x - V_y &= \frac{2\mu}{y} V \\ U_y + V_x &= 0 , \end{aligned}$$

defining the conjugate V of U [19, p. 19]. V satisfies the "conjugate equation"

$$(1.6)\qquad V_{xx} + V_{yy} + \frac{2\mu}{y} V_y - \frac{2\mu}{y^2} V = 0 .$$

Letting $u = U$, $v = y^{2\mu} V$, if U and V satisfy (1.5), then, u, v are solutions of

(1.7) $$u_x = y^{-2\mu} v_y \text{ and } u_y = -y^{-2\mu} v_x .$$

If U and V are solutions of the equations (1.5), then $F = U + iV$ is a pseudo analytic function of the first kind. We call such functions μ-pseudo analytic. Since $[1, y^{-2\mu}]$ is a generating pair for the system (1.5) the function $u + iv$ is the corresponding pseudo analytic function of the second kind (cf. the equations of mixed type example [9, pp. 389-390]). It is easy to show that if U is a solution of equation (1.4) then there exists a function V related to U by the system (1.5), and V is unique up to a function of y alone. Specifically, given U and V satisfying (1.5), the functions U and V^* satisfy (1.5) if and only if $V^*(x, y) = V(x, y) + g(y)$, where g satisfies $g' + (2\mu/y)g = 0$. For convenience, if U is given by the series expansion (1.3) following [2], [19] we choose as *the* conjugate V related to U by (1.5) the following:

(1.8) $$V(r, \theta) = 2\mu \sum_{n=1}^{\infty} \frac{1}{n + 2\mu} a_n r^n C_{n-1}^{\mu+1}(\cos\theta) \sin\theta .$$

Gilbert [15] has developed an integral operator which tranforms analytic functions f to generalized axisymmetric potentials U. If

$$U(r, \theta) = \sum_{n=1}^{\infty} \frac{n!}{\Gamma(n + 2\mu)} a_n r^n C_n^{\mu}(\cos\theta) ,$$

then this A_μ operator gives

$$\begin{aligned} U(r, \theta) &= A_\mu(f) \\ &= \frac{\Gamma(2\mu)}{2^{2\mu-1}\Gamma^2(\mu)} \int_0^{\pi} f(r\cos\theta + ir\sin\theta\cos t) \sin^{2\mu-1} t\,dt , \end{aligned}$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Heins [18] traces the origin of such integral representations back to Poisson [21]. An explicit integral representation for the inverse transform A_μ^{-1} was first obtained in [15], and is

$$f(z) = A_\mu^{-1}(U)$$

(1.9) $$= \int_0^{\pi} U(r, t) K(z/r, \cos t) \sin t\,dt, \quad |z| < r ,$$

where

$$K(\sigma, \xi) = \frac{\Gamma^2(\mu + 1)}{\pi} \frac{(1 - \xi^2)^{\mu-1/2}(1 - \sigma^2)}{(1 - 2\xi\sigma + \sigma^2)^{\mu+1}} .$$

Our first objective is to examine the relationship between the A_μ integral operator and μ-pseudo analytic functions. Also, for $\mu = 0$, (1.4) and the conjugate equation (1.6) each reduce to Laplace's equation, and the system (1.5) becomes the Cauchy-Riemann system. Thus, given a proper formulation, all results regarding ultraspherical expansions, their conjugates, and μ-pseudo analytic functions should in the limit as $\mu \to 0$ reduce to classical function theory.

2. Integral operator for the conjugate; the limit $\mu \to 0$. Throughout, we shall restrict attention to ultraspherical series having real coefficients. Since

$$\lim_{\mu\to 0} \mu^{-1} C_n^{\mu}(x) = \frac{2}{n} T_n(x) , \tag{2.1}$$

where T_n is the Tchebychev polynomial of degree n (see [23, p. 178]), it is natural to renormalize the A_μ integral as follows: let

$$L_\mu = \frac{2^{2\mu-1}}{\mu\Gamma(2\mu)} A_\mu .$$

If

$$U(r, \theta) = \frac{2^{2\mu-1}}{\mu} \sum_{n=0}^{\infty} \frac{n}{\Gamma(n+2\mu)} a_n r^n C_n^{\mu}(\cos\theta) , \tag{2.2}$$

then

$$\begin{aligned} U(r, \theta) &= L_\mu(f) \\ &= \frac{1}{\mu\Gamma^2(\mu)} \int_0^{\pi} f(r\cos\theta + ir\sin\theta\cos t) \sin^{2\mu-1} t\,dt , \end{aligned}$$

where

$$f(z) = \sum_{n=0}^{\infty} a_n z^n . \tag{2.3}$$

We call f the L_μ associate of U. The limit (2.1) then yields

$$\begin{aligned} \lim_{\mu\to 0} L_\mu(z^n) &= \lim_{\mu\to 0} \frac{2^{2\mu-1} n!}{\mu\Gamma(n+2\mu)} r^n C_n^{\mu}(\cos\theta) \\ &= r^n T_n(\cos\theta) \\ &= \operatorname{Re}(z^n) . \end{aligned} \tag{2.4}$$

Since $|L_\mu(z^n) - \operatorname{Re}(z^n)| \sim g(n)$ on $|z| \leq 1$, where $[g(n)]^{1/n} \to 1$ as $n \to \infty$, we have the following

THEOREM 2.1. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$, *where* a_n *are real, and* $\overline{\lim} |a_n|^{1/n} = 1/R$, $R > 1$. *Then*

$$\lim_{\mu\to 0} L_\mu(f) = \operatorname{Re}(f)$$

uniformly on $|z| \leqq 1$.

Proof.

$$|L_\mu(f) - \operatorname{Re}(f)| \leqq \sum_{n=0}^{\infty} \left| \frac{2^{2\mu-1} n!}{\mu \Gamma(n+2\mu)} C_n^\mu(\cos\theta) - \cos n\theta \right| |a_n| r^n .$$

By the result [23, p. 182]

$$C_n^\mu(\cos\theta) = \sum_{m=0}^{n} \alpha_m \alpha_{n-m} \cos(n-2m)\theta ,$$

where

$$\alpha_m = \frac{\Gamma(\mu+m)}{\Gamma(\mu) m!} .$$

It is easy to see that

$$\lim_{\mu\to 0} \frac{\alpha_m \alpha_{n-m}}{\mu} = \begin{cases} 0 & \text{if } m \neq 0, n \\ 1/n & \text{if } m = 0, n . \end{cases}$$

Thus

$$\begin{aligned} &\left| \frac{2^{2\mu-1} n!}{\mu \Gamma(n+2\mu)} C_n^\mu(\cos\theta) - \cos n\theta \right| \\ &\leqq \left| \frac{2^{2\mu-1} n!}{\mu \Gamma(n+2\mu)} C_n^\mu(\cos\theta) - \frac{2^{2\mu} n!}{\mu \Gamma(n+2\mu)} \alpha_0 \alpha_n \cos n\theta \right| \\ &\quad + \left| \frac{2^{2\mu} n!}{\mu \Gamma(n+2\mu)} \alpha_0 \alpha_n \cos n\theta - \cos n\theta \right| \\ &\leqq \frac{2^{2\mu-1}}{\mu} \sum_{m=1}^{n-1} \frac{n!}{\Gamma(n+2\mu)} \alpha_m \alpha_{n-m} + \left| \frac{2^{2\mu} n! \, \alpha_0 \alpha_n}{\mu \Gamma(n+2\mu)} - 1 \right| , \end{aligned}$$

and for $0 < \mu < 1$, this is

$$\leqq n^2 \left[\frac{2^{2\mu-1}}{\mu \Gamma^2(\mu)} + \frac{1}{n^2} \left| \frac{2^{2\mu} \Gamma(n+\mu)}{\mu \Gamma(\mu) \Gamma(n+2\mu)} - 1 \right| \right] .$$

Now, since $\Gamma(n+\mu)/\Gamma(n+2\mu) \sim (n+\mu)^{-\mu}$ as $n \to \infty$, and as $\mu \to 0$, $\mu\Gamma(\mu) \to 1$, there exists an N_1 such that

$$\frac{2^{2\mu-1}}{\mu \Gamma^2(\mu)} + \frac{1}{n^2} \left| \frac{2^{2\mu} \Gamma(n+\mu)}{\mu \Gamma(\mu) \Gamma(n+2\mu)} - 1 \right| \leqq c ,$$

for all $n > N_1$ and all $\mu \in [0, 1]$, where c is a constant (eg., 3 will do). Thus given $\varepsilon > 0$, there exists an $N_2 > N_1$ such that

$$\sum_{n=N_2}^{\infty} \left| \frac{2^{2\mu-1} n!}{\mu \Gamma(n+2\mu)} C_n^\mu(\cos\theta) - \cos n\theta \right| |a_n| r^n < \varepsilon$$

for all $\mu \in [0, 1]$, $\theta \in [0, \pi]$, and $0 \leqq r \leqq 1$. The convergence (2.4) of the first N_2 terms then completes the proof.

The conjugate V of U given by (2.2) has the series expansion

$$V(r, \theta) = 2^{2\mu} \sum_{n=1}^{\infty} \frac{n!}{\Gamma(n + 2\mu + 1)} a_n r^n C_{n-1}^{\mu+1}(\cos \theta) \sin \theta . \tag{2.5}$$

Bers and Gelbart [8, p. 177] have given an integral representation for conjugate functions which yields the definition of a "conjugate" integral operator L_μ^* that transforms the L_μ associate f of U to V. Appealing to this result, the following is immediate:

If U is given by the series (2.2), then a function connected with U by the equations (1.5) is

$$V(r, \theta) = -\frac{i}{\mu \Gamma^2(\mu)} \int_0^\pi f(r \cos \theta + ir \sin \theta \cos t) \cos t \sin^{2\mu-1} t dt .$$

We further verify directly that this expression has the series expansion (2.5) above. An integration by parts shows that

$$\begin{aligned}
&-\frac{i}{\mu \Gamma^2(\mu)} \int_0^\pi (r \cos \theta + ir \sin \theta \cos t)^n \cos t \sin^{2\mu-1} t dt \\
&= \frac{nr \sin \theta}{2\Gamma^2(\mu + 1)} \int_0^\pi (r \cos \theta + ir \sin \theta \cos t)^{n-1} \sin^{2\mu+1} t dt \\
&= \frac{n}{2}(\mu + 1) r \sin \theta L_{\mu+1}(z^{n-1}) \\
&= \frac{2^{2\mu} n!}{\Gamma(n + 2\mu + 1)} r^n C_{n-1}^{\mu+1}(\cos \theta) \sin \theta ,
\end{aligned}$$

which yields the result. Also, since

$$|C_n^\mu(x)| \leqq \frac{\Gamma(n + 2\mu)}{n! \Gamma(2\mu)} , \tag{2.6}$$

the three series (2.2), (2.3), and (2.5) all have the same radius of convergence, i.e., they all converge absolutely and uniformly on compact subsets of $|z| < R$, where $\overline{\lim} |a_n|^{1/n} = R^{-1}$. Thus, defining

$$L_\mu^*(f) = -\frac{i}{\mu \Gamma^2(\mu)} \int_0^\pi f(r \cos \theta + ir \sin \theta \sin t) \cos t \sin^{2\mu-1} t dt , \tag{2.7}$$

on interchanging the order of summation and integration we find that if f has the Taylor series expansion (2.3), then $L_\mu^*(f)$ is given by the series (2.5). (Regarding the conjugate operator, see also [14, p. 189]). Further, the L_μ and L_μ^* integral representations for U and V obtain throughout the common disk $|z| < R$.

In view of Theorem 1, it is natural to expect that $L_\mu^*(f) \to \mathrm{Im}(f)$

as $\mu \to 0$. Now,

$$\lim_{\mu\to 0} C_{n-1}^{\mu+1}(\cos\theta) = C_{n-1}^{1}(\cos\theta) = \frac{\sin n\,\theta}{\sin\theta}$$

(see [23, p. 187]). Thus

$$\lim_{\mu\to 0} L_\mu^*(z^n) = \lim_{\mu\to 0} \frac{2^{2\mu} n!}{\Gamma(n+2\mu+1)} a_n C_{n-1}^{\mu+1}(\cos\theta)\sin\theta \qquad (2.8)$$

$$= r^n \sin n\theta$$

$$= \operatorname{Im}(z^n)\ .$$

Again, since $|L_\mu^*(z^n) - \operatorname{Im}(z^n)| \sim g(n)$ on $|z| \leqq 1$, where $[g(n)]^{1/n} \to 1$ as $n \to \infty$, we have the desired result for functions which are analytic on a closed disk:

THEOREM 2.2. *Let* $f(z) = \sum_{n=0}^\infty a_n z^n$, *where* a_n *are real and* $\overline{\lim}_{n\to\infty} |a_n|^{1/n} = 1/R$, $R > 1$. *Then*

$$\lim_{\mu\to 0} L_\mu^*(f) = \operatorname{Im}(f)$$

uniformly on $|z| \leqq 1$.

Proof. The generating relation (1.1) yields

$$\frac{z\sin\theta}{(1-2z\cos\theta+z^2)^{\mu+1}} = \sum_{n=1}^\infty z^n C_{n-1}^{\mu+1}(\cos\theta)\sin\theta .$$

Subtracting

$$\frac{z\sin\theta}{1-2z\cos\theta+z^2} = \sum_{n=1}^\infty z^n \sin n\theta\ ,$$

and noting that each series converges uniformly on compact subsets of $|z| < 1$, we have for $0 < \rho < 1$, $n = 1, 2, \cdots$,

$$C_{n-1}^{\mu+1}(\cos\theta) - \sin n\theta$$
$$= \frac{1}{2\pi i}\int_{|z|=\rho} \frac{z\sin\theta[1-(1-2z\cos\theta+z^2)^\mu]}{(1-2z\cos\theta+z^2)^{\mu+1}} \frac{dz}{z^{n+1}}\ .$$

Thus

$$|C_{n-1}^{\mu+1}(\cos\theta)\sin\phi - \sin n\theta| \leqq \frac{C}{\rho^{n-1}} \qquad (2.10)$$

for all $\theta \in [0, 2\pi]$, and $\mu \in [0, 1]$, where C is a constant depending only on ρ. Now,

$$|L^*(f) - \operatorname{Im}(f)|$$
$$\leqq \sum_{n=1}^{\infty} \left| \frac{2^{2\mu} n!}{\Gamma(n+2\mu+1)} a_n r^n C_{n-1}^{\mu+1}(\cos\theta)\sin\theta - a_n r^n \sin n\theta \right|$$
$$\leqq \sum_{n=1}^{\infty} |a_n| r^n \Big[\left| \frac{2^{2\mu} n!}{\Gamma(n+2\mu+1)} C_{n-1}^{\mu+1}(\cos\theta)\sin\theta - C_{n-1}^{\mu+1}(\cos\theta)\sin\theta \right| + |C_{n-1}^{\mu+1}(\cos\theta)\sin\theta - \sin n\theta| \Big] .$$

Let $\varepsilon > 0$. Choose $\rho \in (0, 1)$ sufficiently near 1 so that $\sum_{n=0}^{\infty} |a_n| \rho^{-n}$ converges. Then by the result (2.10), there exists an N_1 such that

$$\sum_{n=N_1}^{\infty} |a_n| r^n |C_{n-1}^{\mu+1}(\cos\theta)\sin\theta - \sin n\theta| < \varepsilon$$

for all $r \in [0, 1]$, $\theta \in [0, 2\pi]$. Further, using the bound (2.6), we have

$$\left| \frac{2^{2\mu} n!}{\Gamma(n+2\mu+1)} C_{n-1}^{\mu+1}(\cos\theta)\sin\theta - C_{n-1}^{\mu+1}(\cos\theta)\sin\theta \right|$$
$$\leqq \left| \frac{2^{2\mu} n!}{\Gamma(n+2\mu+1)} - 1 \right| \frac{\Gamma(n+2\mu+1)}{(n-1)!\,\Gamma(n+2\mu)}$$
$$\leqq \frac{(2^{2\mu}+1)(n+1)(n+2)}{\Gamma(2\mu+2)} .$$

Thus there exists an N_2 such that

$$\sum_{n=N_2}^{\infty} |a_n| r^n \left| \frac{2^{2\mu} n!}{\Gamma(n+2\mu+1)} C_{n-1}^{\mu+1}(\cos\theta)\sin\theta - C_{n-1}^{\mu+1}(\cos\theta)\sin\theta \right| < \varepsilon .$$

The convergence (2.8) of the first $N = \max(N_1, N_2)$ terms then completes the proof.

Define

$$\mathscr{L}_\mu = L_\mu + iL_\mu^* .$$

If U and V are conjugate functions given by (2.2) and (2.5), then $F = U + iV$ is a μ-pseudo analytic function of the first kind, and

$$F(r, \theta) = \mathscr{L}_\mu(f)$$
$$= \frac{1}{\mu\Gamma^2(\mu)} \int_0^\pi f(r\cos\theta + ir\sin\theta\cos t)\ (1 + \cos t)\sin^{2\mu-1} t\,dt , \tag{2.11}$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n$. If F is regular on $|z| < 1$ (i.e., U and V are C^2 solutions of the system (1.5)), then the $\mathscr{L}_\mu$ associate f of F is analytic on $|z| < 1$, and the integral representation (2.11) holds throughout the disk $|z| < 1$.

As an immediate consequence of Theorems 1 and 2, we have the

COROLLARY. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$, *where* a_n *are real, and* $\overline{\lim} |a_n|^{1/n} = 1/R$, $R > 1$. *Then*

$$\lim_{\mu \to 0} \mathscr{L}_\mu(f) = f$$

uniformly on $|z| \leqq 1$.

Thus analytic functions of a single complex variable appear as a limiting case of μ-pseudo analytic functions.

With the exception of an explicit representation for the inverse transform $(L_\mu^*)^{-1}$, our discussion of integral operators which generalize the operations Re and Im, and examination of their relationship with pseudo analytic functions is complete. The development of an explicit representation for $(L_\mu^*)^{-1}$ will be postponed to the next section, since (as will be shown) both this transform and L_μ^{-1} are closely related to the Poisson integral.

3. Function theoretic results on a disk. Since ultraspherical expansions and their conjugates enjoy the same relation with μ-pseudo analytic functions that is found between trigonometric series and analytic functions, it is natural to expect analogs of classical function theoretic results on a disk. Such results have been obtained by Muckenhaupt and Stein [19] and include, in particular, a Poisson integral formula for U, conjugate kernel, analog of Fatou's theorem, Riese's theorem ($U \to V$ is a bounded operator on L^p), and H^p theory. Our objectives in this section are to examine the relationship between the Poisson integral formula and the transform L_μ^{-1}, obtain new integral representations for the Poisson kernel and conjugate kernel, develop an explicit integral representation for $(L_\mu^*)^{-1}$, and to give an extension of Privaloff's theorem [20] to ultraspherical expansions and their conjugates.

If $U(\theta)$ has ultraspherical expansion

$$U(\theta) = \frac{2^{2\mu-1}}{\mu} \sum_{n=0}^{\infty} \frac{n!}{\Gamma(n+2\mu)} a_n C_n^\mu (\cos\theta) ,$$

then the Abel sum of this series

$$U(r, \theta) = \frac{2^{2\mu-1}}{\mu} \sum_{n=0}^{\infty} \frac{n!}{\Gamma(n+2\mu)} a_n r^n C_n (\cos\theta) \tag{3.1}$$

is given by the "Poisson integral"

$$U(r, \theta) = \int_0^\pi U(\phi) P(r, \theta, \phi) \sin^{2\mu}\phi d\phi , \tag{3.2}$$

where the "Poisson kernel" P is

$$P(r,\theta,\phi)=\frac{\Gamma^2(\mu)2^{2\mu-1}}{\pi}\sum_{n=0}^{\infty}\frac{(n+\mu)n!}{\Gamma(n+2\mu)}r^nC_n^\mu(\cos\theta)C_n^\mu(\cos\phi)$$
$$=\frac{\mu(1-r^2)}{\pi}\int_0^\pi\frac{\sin^{2\mu-1}t}{[1-2r(\cos\theta\cos\phi+\sin\theta\sin\phi\cos t)+r^2]^{\mu+1}}dt$$

(see [19, p. 25]). Gilbert's inverse transform L_μ^{-1} is easily obtained from the Poisson integral formula (3.2). Letting $\theta=0$ in (3.2) and analytically continuing r to complex values z throughout the disk $|z|<1$ yields

$$U(z,0)=\int_0^\pi U(\phi)P(z,0,\phi)\sin^{2\mu}\phi d\phi\ , \tag{3.3}$$

where

$$P(z,0,\phi)=\frac{\mu}{\pi}\frac{1-z^2}{(1-2z\cos\phi+z^2)^{\mu+1}}\int_0^\pi\sin^{2\mu-1}tdt$$
$$=\frac{\mu\Gamma(\mu)\Gamma(1/2)}{\pi\Gamma(\mu+1/2)}\frac{1-z^2}{(1-2z\cos\phi+z^2)^{\mu+1}}\ .$$

Since $C_n^\mu(1)=\Gamma(n+2\mu)/n!\Gamma(2\mu)$, we have

$$U(z,0)=\frac{2^{2\mu-1}}{\mu\Gamma(2\mu)}\sum_{n=0}^{\infty}a_nz^n=\frac{2^{2\mu+1}}{\mu\Gamma(2\mu)}f(z)\ ,$$

where f is the L_μ associate of $U(r,\theta)$. Thus (3.3) yields for $|z|<1$,

$$f(z)=\frac{\Gamma^2(\mu+1)}{\pi}\int_0^\pi U(\phi)\frac{1-z^2}{(1-2z\cos\phi+z^2)^{\mu+1}}\sin^{2\mu}\phi d\phi\ , \tag{3.3a}$$

which is the expression (1.9) for L_μ^{-1} in the case where the integral is taken over the unit half circle.

Conversely, one obtains the Poisson integral formula by applying the L_μ transform to the expression for L_μ^{-1}. Letting $r=1$ in (1.9), we have for $|z|<1$

$$\begin{aligned}L_\mu[f(z)]&=L_\mu\left[\frac{\Gamma^2(\mu+1)}{\pi}\int_0^\pi U(1,\phi)\frac{1-z^2}{(1-2z\cos\phi+z^2)^{\mu+1}}\sin^{2\mu}\phi d\phi\right]\\&=\frac{\Gamma^2(\mu+1)}{\pi}\int_0^\pi U(1,\phi)L_\mu\left[\frac{1-z^2}{(1-2z\cos\phi+z^2)^{\mu+1}}\right]\sin^{2\mu}\phi d\phi\ .\end{aligned} \tag{3.4}$$

Now,

$$\frac{1-z^2}{(1-2z\cos\phi+z^2)^{\mu+1}}=\sum_{n=0}^{\infty}\frac{n+\mu}{\mu}z^nC_n^\mu(\cos\phi)\ , \tag{3.5}$$

and the series converges uniformly for $|z|\leqq\rho<1$, $\phi\in[0,\pi]$. Applying the L_μ transform termwise to this series yields

$$L_\mu\left[\frac{1-z^2}{(1-2z\cos\phi+z^2)^{\mu+1}}\right]=\frac{2^{2\mu-1}}{\mu^2}\sum_{n=0}^{\infty}\frac{(n+\mu)n!}{\Gamma(n+2\mu)}r^nC_n^\mu(\cos\phi)C_n^\mu(\cos\theta)$$
$$=\frac{\pi}{\Gamma^2(\mu+1)}P(r,\theta,\phi)\,.$$

Since $U(r,\theta)=L_\mu[f(z)]$, (3.4) is the Poisson integral formula for U. Note that this also yields an additional integral representation for the Poisson kernel:

$$P(r,\theta,\phi)$$
$$=\frac{\mu}{\pi}\int_0^\pi\frac{1-(x+iy\cos t)^2}{[1-2(x+iy\cos t)\cos\phi+(x+iy\cos t)^2]^{\mu+1}}\sin^{2\mu-1}t\,dt\,, \tag{3.6}$$

where $(x,y)=(r\cos\theta,\ r\sin\theta)$.

Again let $U(r,\theta)$ be given by the series (3.1). Then the conjugate $V(r,\theta)$ of U has on the unit disk the representation

$$V(r,\theta)=\int_0^\pi U(1,\phi)Q(r,\theta,\phi)\sin^{2\mu}\phi\,d\phi\,,$$

where Q is the "conjugate Poisson kernel"

$$Q(r,\theta,\phi)=\frac{2^{2\mu}\mu\Gamma^2(\mu)}{\pi}\sum_{n=0}^{\infty}\frac{n!(n+\mu)}{\Gamma(n+2\mu+1)}r^nC_n^\mu(\cos\phi)C_{n-1}^{\mu+1}(\cos\theta)\sin\theta$$

(see [19, p. 35]). On applying the L_μ^* transform to the generating relation (3.5), a routine computation yields an integral representation for Q similar to that found above for P:

$$Q(r,\theta,\phi)$$
$$=\frac{-i\mu}{\pi}\int_0^\pi\frac{1-(x+iy\cos t)}{[1-2(x+iy\cos t)\cos\phi+(x+iy\cos t)^2]^{\mu+1}}\cos t\sin^{2\mu-1}t\,dt\,. \tag{3.7}$$

Expressed in terms of the L_μ, L_μ^* integral operators, the representations (3.6) and (3.7) become

$$P(r,\theta,\phi)=\frac{\Gamma(\mu+1)}{\pi}L_\mu\left[\frac{1-z^2}{(1-2z\cos\phi+z^2)^{\mu+1}}\right],$$

and

$$Q(r,\theta,\phi)=\frac{\Gamma(\mu+1)}{\pi}L_\mu^*\left[\frac{1-z^2}{(1-2z\cos\phi+z^2)^{\mu+1}}\right].$$

Thus, for $0\leqq r<1$, $\theta\in[0,2\pi]$,

$$F(r,\theta)=\int_0^\pi U(1,\phi)[P(r,\theta,\phi)+iQ(r,\theta,\phi)]\sin^{2\mu}\phi\,d\phi\,, \tag{3.8}$$

where $F=U+iV$, and

$$P(r, \theta, \phi) + iQ(r, \theta, \phi) = \frac{\Gamma(\mu+1)}{\pi} \mathscr{L}_\mu\left[\frac{1-z^2}{(1-2z\cos\phi+z^2)^{\mu+1}}\right].$$

This result provides an analog of the Schwartz formula

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{it})\frac{e^{it}+z}{e^{it}-z}dt \tag{3.9}$$

which recovers an analytic function f from its real part u. Notice that on restricting $\theta = 0$ in (3.8) and analytically continuing r to complex values ζ in $|\zeta| < 1$, the integral reduces to the expression for the inverse transform L_μ^{-1}. This integral representation for L_μ^{-1} might also be viewed as analogous to Schwartz's formula. For L_μ provides a transformation from analytic functions to solutions of an elliptic equation—in fact, is a continuous extension of the operation Re. And Schwartz's formula is nothing more than an explicit integral representation for the transform Re^{-1}. This observation leads to the following analog of Privaloff's theorem [20]:

THEOREM 3. *If*

$$U(\theta) = \frac{2^{2\mu-1}}{\mu}\sum_{n=0}^{\infty}\frac{n!}{\Gamma(n+2\mu)}a_n C_n^\mu(\cos\theta)$$

satisfies a Lipschitz condition of order α, $0 < \mu < \alpha < 1$, *on* $[0, \pi]$, *then its conjugate*

$$V(\theta) = 2^{2\mu}\sum_{n=0}^{\infty}\frac{n!}{\Gamma(n+2\mu+1)}a_n C_{n-1}^{\mu+1}(\cos\theta)\sin\theta$$

satisfies a Lipschizt condition of order $\alpha - \mu$ *on* $[0, \pi]$.

Proof. Hardy and Littlewood [17] have shown the following: an analytic function $f(z)$ satisfies $|f'(z)| < C/(1-|z|)^{1-\alpha}$ throughout the disk $|z| < 1$ if and only if f satisfies a Lipschitz condition of order α on $|z| \leqq 1$. On applying this result to the Schwartz formula, a proof of the classical theorem of Privaloff obtains (see [16, p. 414]). In an entirely similar manner, differentiating the expression (3.3a) for L_μ^{-1} and appealing to the result of Hardy and Littlewood one obtains a Lipschitz condition of order $\alpha - \mu$ on the L_μ associate f of U (for details see [12]). That is, for $f(z) = \sum_{n=0}^{\infty} a_n z^n$ we have

$$|f(z_1) - f(z_2)| \leqq K|z_1 - z_2|^{\alpha-\mu}|z_1|, \ |z_2| < 1 .$$

Since

$$V(r, \theta) = L_\mu^*(f)$$
$$= -\frac{i}{\mu\Gamma^2(\mu)}\int_0^\pi f(r\cos\theta + ir\sin\theta\cos t)\cos t \sin^{2\mu-1}t dt ,$$

it is then easy to see that for $0 \leqq r_1, r_2 \leqq 1$,

$$| V(r_1, \theta_1) - V(r_2, \theta_2)| \leqq \frac{\pi K}{\Gamma^2(\mu+1)}[(r_1\cos\theta_1 - r_2\cos\theta_2)^2 + (r_1\sin\theta_1 - r_2\sin\theta_2)^2]^{(\alpha-\mu)/2} .$$

In particular,

$$| V(1, \theta_1) - V(1, \theta_2)| \leqq \frac{\pi K}{\Gamma^2(\mu+1)} |\theta_1 - \theta_2|^{\alpha-\mu} .$$

COROLLARY. *Let* $F(z) = U(r, \theta) + iV(r, \theta)$ *be* μ*-pseudoanalytic on the open disk* $|z|<1$. *If* U *is continuous on the closed disk* $|z| \leqq 1$, *and*

$$| U(1, \theta_1) - U(1, \theta_2)| \leqq K|\theta_1 - \theta_2|^\alpha,\ 0 < \mu < \alpha < 1 ,$$

then F *is continuous on the closed disk* $|z| \leqq 1$, *there satisfying a Lipschitz condition of order* $\alpha - \mu$, i.e.,

$$|F(z_1) - F(z_2)| \leqq M|z_1 - z_2|^{\alpha-\mu}, \quad |z_1|, |z_2| \leqq 1 .$$

Proof. As in the proof of the previous theorem, the hypotheses on U imply that the L_μ associate f of U satisfies a Lipschitz condition of order $\alpha - \mu$ on the closed disk $|z| \leqq 1$. Since

$$F(z) = \mathscr{L}_\mu(f) = \frac{1}{\mu\Gamma^2(\mu)}\int_0^\pi f(x + iy\cos t)(1 + \cos t)\sin^{2\mu-1}t dt,$$

the result is immediate.

Let $U(r, \theta)$ be given by the series (3.1), where $|a_n|^{1/n} \to 1$ as $n \to \infty$. In [12] it was shown that if, further, on the boundary circle U satisfies

$$(3.10) \qquad | U(1, \theta_1) - U(1, \theta_2)| \leqq \lambda(|\theta_1 - \theta_2|) ,$$

where

$$\int_0^{\pi/2} \frac{\lambda(t)}{t^{\mu+1}} dt < \infty ,$$

then $f(z) = L_\mu^{-1}(U)$ is bounded on $|z| \leqq 1$. Thus $F = \mathscr{L}_\mu(f)$ is bounded on $|z| \leqq 1$, and we see that condition (3.10) is sufficient to insure that the conjugate V of U is bounded on $|z| \leqq 1$. We note that this result, as well as those of the above theorem and corollary, reduce in the limit as $\mu \to 0$ to classical function theoretic results which are known to be best possible.

It is also possible to express U interior to a disk in terms of an integral of its conjugate V over the boundary circle. Such a result leads immediately to an explicit integral formula for the transform $(L_\mu^*)^{-1}$. Define

$$Q^*(r, \theta, \phi) = \frac{2^{2\mu}\Gamma^2(\mu+1)}{\pi\mu} \sum_{n=1}^{\infty} \frac{(n+\mu)(n-1)!}{\Gamma(n+2\mu)} \cdot r^n C_n^\mu(\cos\theta) C_{n-1}^{\mu+1}(\cos\phi)\sin\phi \,.$$

Then for all $r \in [0, 1)$, $\theta \in [0, 2\pi]$, we have

$$U(r, \theta) = \int_0^\pi V(1, \phi) Q^*(r, \theta, \phi) \sin^{2\mu}\phi d\phi + U_0 \,, \tag{3.11}$$

where $U_0 = U(0, \theta)$. This result is easily obtained by termwise integration of the series expansion (2.5) for V. Now, restricting $\theta = 0$ in this expression and analytically continuing r to complex values z, $|z| < 1$, yields

$$U(z, 0) = \int_0^\pi V(1, \phi) Q^*(z, 0, \phi) \sin^{2\mu}\phi d\phi + U_0 \,,$$

where

$$Q^*(z, 0, \phi) = \frac{2^{2\mu}\Gamma^2(\mu+1)}{\pi\mu\Gamma(2\mu)} \sum_{n=1}^{\infty} \frac{n+\mu}{n} \cdot z^n C_{n-1}^{\mu+1}(\cos\phi)\sin\phi \,.$$

Using the generating relation (3.5) we have

$$\frac{1}{\mu+1}\sum_{n=1}^{\infty}\frac{n+\mu}{n} z^n C_{n-1}^{\mu+1}(\cos\phi) = \int_0^z \frac{1-\zeta^2}{(1-2\zeta\cos\phi+\zeta^2)^{\mu+2}} d\zeta \,.$$

Further, $U(z, 0) = 2^{1-2\mu}\mu\Gamma(2\mu) f(z)$, where f is the L_μ^* associate of V. Thus we have the explicit integral representation for the inverse transform

$$\begin{aligned} f(z) &= (L^*)^{-1}(V) \\ &= a_0 + \int_0^\pi V(1, \phi) H^*(z, \phi) \sin^{2\mu+1}\phi d\phi \,, \end{aligned}$$

where

$$H^*(z, \phi) = \frac{2(\mu+1)}{\Gamma^2(\mu+1)} \int_0^z \frac{1-\zeta^2}{(1-2\zeta\cos\phi+\zeta^2)^{\mu+2}} d\zeta \,.$$

4. Singularities. As is well known, if u is a function which is harmonic in a simply connected region Ω, then the conjugate v of u is also harmonic throughout Ω, and thus $u = \mathrm{Re}(f)$ where $f(z)$ is analytic in Ω. We next obtain the analogous result for generalized axisymmetric potentials, their conjugates, and μ-pseudo analytic functions.

A function U is said to be a *regular* generalized axisymmetric potential on a region Ω if it is a C^2 solution of equation (1.4) throughout $\Omega \backslash \{y = 0\}$, and $\partial U(x, 0) \backslash \partial y = 0$ for all $(x, 0) \in \Omega$. If U is regular in a neighborhood of the origin, it has the expansion (2.2). The conjugate V of U is said to be regular in a region Ω if it is a C^2 solution of the system (1.5) throughout $\Omega \backslash \{y = 0\}$ and $V(x, 0) = 0$ for all $(x, 0) \in \Omega$. If V is regular in a neighborhood of the origin, it has the expansion (2.5). We say that a region Ω is *axiconvex* if it is connected and $(x, y) \in \Omega$ implies $(x, \lambda y) \in \Omega$ for all $\lambda \in [-1, 1]$.

The following result is owing to Gilbert [15]:

Let U be a generalized axisymmetric potential and f be its L_μ associate. Then $(x, \pm y)$ is a singularity of U iff $x \pm iy$ is a singularity of f.

In particular, the axiconvex regions of regularity of U and f coincide (this result has also been obtained by Erdélyi [10]). Hence given U regular on an axiconvex region Ω, we have

$$V(r, \theta) = L_\mu^*(f) = -\frac{i}{\mu \Gamma^2(\mu)} \int_0^\pi f(r\cos\theta + ir\sin\theta\cos t) \cdot \cos t \sin^{2\mu-1} t dt$$

where f is analytic on Ω. Thus it is immediate that V is of class C^2 throughout Ω and $V(x, 0) = 0$. Further, using the identity

$$\int_0^\pi f'(x + iy\cos t) \sin^{2\mu+1} t dt = -\frac{2\mu i}{y} \int_0^\pi f(x + iy\cos t) \cos t \sin^{2\mu-1} t dt ,$$

it is easy to verify that V is a solution of the system (1.5) throughout $\Omega \backslash \{y = 0\}$. Thus V is also regular on Ω. We state this result as a theorem.

THEOREM 4.1. *If U is a generalized axisymmetric potential which is regular on an axiconvex region Ω, then its conjugate V is also regular on Ω.*

The following result is also immediate.

THEOREM 4.2. *F is μ-pseudo analytic on an axiconvex region*

Ω if and only if its $\mathscr{L}_\mu$ associate f is analytic throughout Ω.

Proof. If F is μ-pseudo analytic on Ω then U is regular on Ω, and by Gilbert's result f is analytic on Ω. Conversely, if f is analytic then $U = L_\mu(f)$ is regular on Ω, and by the last theorem its conjugate V is regular on Ω. Thus $F = U + iV = \mathscr{L}_\mu(f)$ is μ-pseudo analytic on Ω.

5. Polynomial approximation. One of the most striking roles the transformation $u = \mathrm{Re}\,(f)$ plays in the study of harmonic functions lies in the theory of uniform polynomial approximation. While Fourier series techniques yield the uuiform harmonic polynomial $\sum_{k=0}^{n} r^k\ (a_n \cos k\theta + b_k \sin k\,\theta)$ approximation to harmonic functions on a disk, they do not provide an adequate tool for dealing with such problems on more general regions. But in the latter case results obtain immediately using the operation Re to draw on the theory of uniform polynomial approximation for analytic functions, a rich and highly developed area.

Similarly, when dealing with more general elliptic equations, separation of variables and Fourier series expansion yields approximations on certain regions having a very simple geometry. Thus, for example, separation of variables for the equation (1.4) expressed in polar coordinates leads to the Gegenbauer polynomials, and the uniform approximation via partial sums of the series (2.2) to solutions which are regular on a disk. It is natural to expect similar approximations on regions having a more complicated geometry. Just as in the case of harmonic functions, using integral operators which generalize the operations Re and Im, we shall obtain such results with ease.

Bers has developed a theory of "formal power" approximation for pseudo analytic functions which closely resembles the theory of polynomial approximation for analytic functions of a single complex variable. Recall that $u + iv$ is a μ-pseudo analytic function of the second kind if and only if u and v satisfy the system (1.7). In this case, Bers' formal power $Z^{(n)}(\lambda, 0, z)$ of degree n, centered at 0, with *real* coefficient λ is given by

$$(5.1)\qquad \begin{aligned} Z^{(n)}(\lambda, 0, z) &= \lambda Z^{(n)}(1, 0, z) \\ &= \lambda \sum_{j=0}^{n} \binom{n}{j} x^{n-j} i^j Y^{(j)}(y)\ , \end{aligned}$$

where

$$Y^{(0)}(y) = 1,\ Y^{(1)}(y) = \int_0^y \eta^{2\mu} d\eta$$

$$Y^{(2)}(y) = 2\int_0^y \frac{Y^{(1)}(\eta)}{\eta^{2\mu}} d\eta, \quad Y^{(3)}(y) = 3\int_0^y \eta^{2\mu} Y^{(2)}(\eta) d\eta, \cdots$$

(cf. [9, pp. 389–90]). The formal powers $Z^{(n)}(\lambda, 0, z)$ given above are μ-pseudo analytic functions of the second kind. That is, writing

$$Z^{(n)}(\lambda, 0, z) = u_n + iv_n, \; u_n, v_n \text{ real},$$

then u_n and v_n satisfy the system (1.7). Since $[1, iy^{-2\mu}]$ is a generating pair for μ-pseudo analytic functions (see [9, p. 389]), the corresponding formal powers of the first kind are

$$W^{(n)}(\lambda, 0, z) = u_n + iy^{-2\mu} v_n .$$

These powers are in fact homogeneous polynomials of degree n in x and y. For convenience let $W^{(k)}(z) = W^{(k)}(1, 0, z)$. A μ-pseudo analytic polynomial of degree n is then a sum $\sum_{k=0}^{n} a_k W^{(k)}(z)$, where $a_n \neq 0$. The polynomials $W^{(n)}$ may also be computed as the $\mathscr{L}_\mu$ transforms of the powers z^n. That is,

$$(5.2) \qquad \begin{aligned} W^{(n)}\left(\frac{2^{2\mu}}{\Gamma(2\mu + 1)}, 0, z\right) &= \mathscr{L}_\mu(z^n) \\ &= \frac{1}{\mu\Gamma^2(\mu)}\int_0^\pi (x + iy\cos t)^n (1 + \cos t)\sin^{2\mu-1} t\, dt \end{aligned}$$

(cf. [8, pp. 175–177]). By either evaluating the integral (5.2) or proceeding inductively according to (5.1) the first few formal powers are easily computed as

$$\begin{aligned} W^{(0)}(z) &= 1 \\ W^{(1)}(z) &= \frac{1}{2\mu + 1}[(2\mu + 1)x + iy] \\ W^{(2)}(z) &= \frac{1}{2\mu + 1}[(2\mu + 1)x^2 - y^2 + i2xy] \\ W^{(3)}(z) &= \frac{3}{(2\mu + 3)(2\mu + 1)} \\ &\quad \times \left\{\frac{(2\mu + 3)(2\mu + 1)}{3} x^3 - (2\mu + 3)x^2y^2 + i[(2\mu + 3)x^2y - y^3]\right\} . \end{aligned}$$

Notice that $W^{(n)}(z) \to z^n$ as $\mu \to 0$.

Bers has obtained an analog of Runge's theorem regarding the uniform approximation of pseudo analytic functions by formal powers [4, p. 119]. We consider the rate of convergence of the approximating polynomials in the case of μ-pseudo analytic functions.

Let $\mathrm{Cl}(\Omega)$ denote the closure of a region Ω. We say F is μ-pseudo analytic on $\mathrm{Cl}(\Omega)$ if it is μ-pseudo analytic throughout some larger region $\Omega' \supset \Omega$.

THEOREM 5.1. *Let Ω be a bounded axiconvex region. If F is μ-pseudo analytic on* $\mathrm{Cl}(\Omega)$, *then there exist μ-pseudo analytic polynomials*

$$W_n(z) = \sum_{k=0}^{n} a_k W^{(k)}(z)$$

such that

$$\| F - W_n \| \leqq \frac{M}{R^n},$$

where $\| \cdot \|$ denotes the sup norm over $\mathrm{Cl}(\Omega)$, *M is a constant independent of n, and $R > 1$.*

Proof. By Theorem 4.2 the $\mathscr{L}_\mu$ associate f of F is analytic on $\mathrm{Cl}(\Omega)$. Thus there exist polynomials $P_n(z) = \sum_{k=0}^{n} b_k z^k$ such that

$$\max_{z \in \mathrm{Cl}(\Omega)} | f(z) - p_n(z) | = \| f - p_n \| \leqq \frac{M}{R^n},$$

where $R > 1$ and M is a constant independent of n (see [25, pp. 75-76]). Now

$$\begin{aligned} \mathscr{L}_\mu[p_n(z)] &= \sum_{k=0}^{n} b_k \mathscr{L}_\mu(z^k) \\ &= \frac{2^{2\mu}}{\Gamma(2\mu + 1)} \sum_{k=0}^{n} b_k W^{(k)}(z) \end{aligned}$$

are μ-pseudo analytic polynomials of degree n—call them $W_n(z)$. Then

$$\begin{aligned} \| F - W_n \| &= \| \mathscr{L}_\mu(f - p_n) \| \\ &\leqq \frac{2\sqrt{\pi}}{\Gamma(\mu + 1)\Gamma(\mu + 1/2)} \| f - p_n \| \\ &\leqq \frac{2\sqrt{\pi}}{\Gamma(\mu + 1)\Gamma(\mu + 1/2)} \frac{M}{R^n}. \end{aligned}$$

While the general theory of pseudo analytic functions enjoys an analog of Runge's polynomial approximation theorem, with the exception of the preceding result, no information regarding the rate of convergence of the formal power approximants seems to have been obtained. When proceeding in analogy with classical function theory, severe difficulties are encountered owing to the fact that pseudo analytic functions do not form an algebra (i.e., products of pseudo analytic functions need not be pseudo analytic). Nevertheless, the preceeding theorem, as well as close analogies with function theory, suggests that formal power approximants to functions which are regular in the closure of a region may in general converge at a geometric rate.

We next obtain results which are more constructive in nature. It is possible to introduce analogs of the Faber polynomials. Recall that for a bounded simply connected Ω, if $w=\phi(z)=z+a_0+a_1z^{-1}+\cdots$ maps the compliment of Ω onto the circular region $|w|>\rho$, then the Faber polynomials $f_n(z)$ of degree n for Ω are defined by the generating relation

$$\frac{t\psi'(t)}{\psi(t)-z}=\sum_{n=0}^{\infty}f_n(z)t^{-n},$$

where $\psi=\phi^{-1}$ (see [22, p. 130]). We define

$$F_n=\mathscr{L}_\mu(f_n)$$

and call F_n the μ-pseudo analytic Faber polynomials for Ω.

THEOREM 5.2. *Let Ω be a bounded axiconvex region, $w=\phi(z)=z+a_0+a_1z^{-1}+\cdots$ map the compliment of Ω onto the circular region $|w|>\rho$, and let $\psi=\phi^{-1}$. If $F(x, y)$ is μ-pseudo analytic on the region B_R bounded by $\Gamma_R=\{z: |\phi(z)|=R\}$, then F can be expanded into a series of μ-pseudo analytic Faber polynomials*

$$F(x, y)=\sum_{n=0}^{\infty}a_nF_n(x, y), \tag{5.3}$$

where

$$a_n=\frac{\mu\Gamma(2\mu)}{2^{2\mu}\pi i}\int_{|z|=r}F(\psi(z), 0)z^{-k-1}dz,\ \rho<r<R. \tag{5.4}$$

Further

$$\limsup_{n\to\infty}|a_n|^{1/n}=1/R,$$

and the series (4.3) *converges uniformly on* Cl (Ω) *at a geometric rate.*

Proof. Since F is μ-pseudo analytic on B_R, Theorem 3.2 insures its $\mathscr{L}_\mu$ associate f is analytic on B_R. Thus

$$f(z)=\sum_{n=0}^{\infty}a_nf_n(z), \tag{5.5}$$

where f_n are the Faber polynomials for Ω, and

$$b_n = \frac{1}{2\pi i}\int_{|z|=r} f(\psi(z))z^{-k-1}dz, \ \rho < r < R$$

(see [22, p. 138]). Also, $|b_n|^{1/n} \to 1/R$, and for $z \in \partial\Omega$, $|f_n(z)|^{1/n} \to \rho$. Taking the $\mathscr{L}_\mu$ transform of (5.5) we have

$$F(x, y) = \sum_{n=0}^{\infty} a_n F_n(x, y) . \tag{5.6}$$

Let $\|\cdot\|$ denote the sup norm on Cl(Ω). Since

$$\|F_n\| = \|\mathscr{L}_\mu(f_n)\| \leqq c\|f_n\| ,$$

we have

$$\lim_{n\to\infty} \|F_n\|^{1/n} \leqq \lim_{n\to\infty} \|f_n\|^{1/n} \leqq \rho .$$

Since $|a_n|^{1/n} \to R > \rho$, the series (5.6) converges uniformly on Cl(Ω). Also restricting $F(x, y) = U(x, y) + iV(x, y)$ to the x-axis and analytically continuing to complex variables yields

$$F(z, 0) = U(z, 0) = \frac{2^{2\mu-1}}{\mu\Gamma(2\mu)} f(z) ,$$

where f is the $\mathscr{L}_\mu$ associate of F. Thus the expression (5.4) obtains for the coefficients a_n.

It is also possible to obtain uniform polynomial approximation to μ-pseudo analytic functions via polynomial interpolation. Let Ω be an axiconvex region and $C_n = \{z_k = x_k + iy_k\}_{k=0}^{n}$ be a set of $n+1$ points on $\partial\Omega$. Define

$$V(C_n) = \det[W^{(k)}(z_j)]_{k,j=0}^{n} ,$$

and

$$V_k(z; C_n) = V(C_n)|_{z_k=z} .$$

If $V(C_n) \neq 0$, and F is any function defined on $\partial\Omega$, then

$$L_n(z; C_n; F) = \sum_{k=0}^{n} F(z_k) \frac{V_k(z; C_n)}{V(C_n)}$$

is the unique μ-pseudo analytic polynomial of degree n which interpolates to F on C_n. Further, it is easy to show that for $n=1, 2, 3\cdots$ there exist $C_n \subset \partial\Omega$ such that $V(C_n) \neq 0$. The result is clearly true for $n = 1$ i.e., if $C_1 = \{z_0, z_1\}$ where $z_0 \neq z_1$, then

$$V(C_1) = \begin{vmatrix} 1 & W^{(1)}(z_0) \\ 1 & W^{(2)}(z_0) \end{vmatrix} = x_1 - x_0 + \frac{i}{2\mu+1}(y_1 - y_0) \neq 0 .$$

Suppose there exist $C_{n-1} = \{z_0, z_1, \cdots, z_{n-1}\} \subset \partial\Omega$ such that $V(C_{n-1}) \neq 0$. Then letting $C_n = \{z_0, \cdots, z_{n-1}, z_n\}$, we have on expanding $V(C_n)$ by the first row,

$$V(C_n) = V(C_{n-1}) W^{(n)}(z_n) + b_{n-1} W^{(n-1)}(z_n) + \cdots + b_0 ,$$

where the coefficients b_k are functions of $z_0, z_1, \cdots, z_{n-1}$. By hypothesis $V(C_{n-1}) \neq 0$, and thus $V(C_n)$ is a μ-pseudo analytic polynomial of degree n in z_n. Since the zeros of a nonconstant pseudo analytic function are isolated [4, p. 18] we easily have a $z_n \in \partial\Omega$ such that $V(C_n) \neq 0$.

THEOREM 5.3. *Let Ω be a bounded axiconvex region and C_n^* maximize $|V(C_n)|$ over* Cl (Ω). *Then*

$$\lim_{n\to\infty} L_n(z; C_n^*; F) = F(z)$$

uniformly on Cl (Ω) *for every function F which is μ-pseudo analytic on* Cl (Ω). *Further, the convergence is at a geometric rate,* i.e.,

$$\| F - L_n \| \leqq \frac{(n+2)M}{R^n}$$

where $\|\cdot\|$ denotes the sup norm over Ω and the constants $M, R>1$ are as given in Theorem 5.1.

Proof. By Theorem 5.1 there exist μ-pseudo analytic polynomials p_n of degree n such that

$$\| F - p_n \| \leqq \frac{M}{R^n} .$$

Since $V(C_n^*) \neq 0$, the μ-pseudo analytic polynomial of degree n assuming $n+1$ given values at the points $z_0, z_1, \cdots, z_n \in C_n^*$ is unique. Thus

$$L_n(z; C_n^*; p_n) = \sum_{k=0}^{n} p_n(z_k) \frac{V_k(z; C_n^*)}{V(C_n^*)} \equiv p_n(z) ,$$

and

$$\begin{aligned} |F(z) - L_n(z; C_n^*; F)| &\leqq |F(z) - p_n(z)| + |p_n(z) - L_n(z; C_n^*; F)| \\ &= |F(z) - p_n(z)| + \left| \sum_{k=0}^{n} [p_n(z_k) - F(z_k)] \frac{V_k(z; C_n^*)}{V(C_n^*)} \right| \\ &\leqq \frac{M}{R^n}\left(1 + \sum_{k=0}^{n} \frac{V_k(z; C_n^*)}{V(C_n^*)}\right) \\ &\leqq \frac{(n+2)M}{R^n} . \end{aligned}$$

By taking real and imaginary parts of the μ-pseudo analytic functions involved, the previous theorems yield results regarding the uniform approximation to solutions of the elliptic equations (1.4) and (1.6) by polynomial solutions of the equations. For further results regarding the approximation of solutions of equation (1.4) see [11] and [13].

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