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HOMOLOGY 3-SPHERES WHICH ADMIT NO PL INVOLUTIONS

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An infinite family of irreducible homology 3-spheres is constructed, each member of which admits no PL involutions.

1. Introduction. In Problem 3.24 of [6] H. Hilden and J. Montesinos ask whether every homology 3-sphere is the double branched covering of a knot in S^3. The interest in this question lies in the fact that there is an algorithm, due to J. Birman and H. Hilden [1], for deciding whether such a 3-manifold is homeomorphic to S^3. In addition, the Smith Conjecture for homotopy 3-spheres [4] implies that every homotopy 3-sphere of this type must be homeomorphic to S^3.

In this paper an infinite family of irreducible homology 3-spheres is exhibited which admit no PL involutions. This gives a negative answer to the above question since the nontrivial covering translation of a branched double cover is a PL involution.

2. Preliminaries. We shall work throughout in the PL category.

A *knot* K is an oriented simple closed curve in the oriented 3-sphere S^3 which does not bound a disk. The *exterior* $Q = Q(K)$ is the closure of the complement of a regular neighborhood of K. A *meridian* $\mu = \mu(K)$ of K is an oriented simple closed curve in ∂Q which bounds a disk in $S^3 - \operatorname{Int} Q$ and has linking number $+1$ with K. A *longitude* $\lambda = \lambda(K)$ of K is an oriented simple closed curve in ∂Q such that λ bounds a surface in Q and $\lambda \sim K$ in $S^3 - \operatorname{Int} Q$. ("$\sim$" means "is homologous to").

K is $\pm$ *amphicheiral* if there is an orientation reversing homeomorphism g of S^3 such that $g(K) = \pm K$. K is *invertible* if there is an orientation preserving homeomorphism g of S^3 such that $g(K) = -K$.

For the definitions of simple knot, torus knot, and fibered knot we refer to [8]. For the definitions of irreducible 3-manifold, incompressible surface, and of parallel surfaces in a 3-manifold we refer to [5]. Note that a knot K is simple if and only if every incompressible torus in $Q(K)$ is parallel to $\partial Q(K)$. If K is simple and $Q(K)$ contains an incompressible annulus which is not parallel to an annulus in $\partial Q(K)$, then K is a torus knot [3].

Suppose h is an involution on a homology 3-sphere M. Then by Smith theory [2] the fixed point set Fix $\langle h \rangle$ is homeomorphic to S^0

or S^2 if h reverses orientation and is empty or homeomorphic to S^1 if h preserves orientation.

3. The construction. Let K_0 and K_1 be knots. Let $Q_i = Q(K_i)$, $\mu_i = \mu(K_i)$, and $\lambda_i = \lambda(K_i)$, $i = 0, 1$. We construct $M = M(K_0, K_1)$ by identifying ∂Q_0 and ∂Q_1 so that $\mu_0 = \lambda_1$ and $\lambda_0 = -\mu_1$. We denote $Q_0 \cap Q_1$ by T and μ_0, λ_0 by α, β, respectively. Note that M is an irreducible homology 3-sphere and that T is incompressible in M.

LEMMA 3.1. *If K_0 and K_1 are simple knots, other than torus knots, then every incompressible torus in $M(K_0, K_1)$ is isotopic to T.*

Proof. Let T' be an incompressible torus in M. Isotop T' so that T and T' are in general position and meet in a minimal number of components.

Suppose some component J of $T \cap T'$ bounds a disk D' in T'. We may assume $D' \cap T = \partial D'$. By the incompressibility of T, $\partial D' = \partial D$ for some disk D in T. By the irreducibility of M, $D \cup D'$ bounds a 3-cell B in M. So T' can be isotoped by pushing D' across B and off D to remove at least J from $T \cap T'$. This contradicts minimality and so cannot happen. A similar argument shows that no component of $T \cap T'$ bounds a disk in T.

Thus if $T \cap T' \neq \varnothing$, $T' \cap Q_i$ consists of incompressible annuli. Let A' be such an annulus in Q_0. Since K_0 is simple and not a torus knot, A' is parallel in Q_0 to an annulus A in T. Therefore T' can be isotoped by pushing A' across the solid torus bounded by $A \cup A'$ and off A to remove at least ∂A from $T \cap T'$. By minimality this cannot occur.

Thus T' lies in some Q_i. Since K_i is simple, T' is parallel to T and we are done.

4. Involutions on $M(K_0, K_1)$. An involution h on $M(K_0, K_1)$ is *good* if $h(Q_i) = Q_i$, $i = 0, 1$, Fix $\langle h \rangle$ and T are in general position, $h(\alpha) \sim \pm\alpha$, and $h(\beta) \sim \pm\beta$.

LEMMA 4.1. *Let K_0 and K_1 be simple knots, other than torus knots, such that Q_0 and Q_1 are not homeomorphic. Then every involution of $M(K_0, K_1)$ is conjugate to a good involution.*

Proof. By Theorem 1 of Tollefson [1] and Lemma 3.1 there is an isotopy f_t of M such that $f_0 = id$, $f_1(T)$ and Fix $\langle h \rangle$ are in general position, and either $h(f_1(T)) = f_1(T)$ or $h(f_1(T)) \cap f_1(T) = \varnothing$. Let $h' = f_1^{-1} \circ h \circ f_1$. Then either $h'(T) = T$ or $h'(T) \cap T = \varnothing$.

Suppose $h'(T) \cap T = \varnothing$. We may assume $h'(T) \subset \operatorname{Int} Q_0$. If

$h(Q_0) \subset \operatorname{Int} Q_0$, then $Q_0 = h^2(Q_0) \subset \operatorname{Int} h(Q_0) \subset \operatorname{Int} h^2(Q_0) = \operatorname{Int} Q_0$, which is absurd. Thus $Q_1 \subset \operatorname{Int} h(Q_0)$. But since ∂Q_1 is parallel to $\partial h(Q_0)$ in $h(Q_0)$, Q_0 and Q_1 are homeomorphic, a contradiction. Therefore $h'(T) = T$ and so $h'(Q_i) = Q_i$.

Finally $h(\alpha) = h(\mu_0) = h(\lambda_1) \sim \pm\lambda_1 = \pm\alpha$ and similarly $h(\beta) \sim \pm\beta$.

LEMMA 4.2. *Suppose K_0 is non-amphicheiral. Then every good involution on $M(K_0, K_1)$ is orientation preserving.*

Proof. $h(\beta) \sim \pm\beta$ implies that $h(\mu_0) \sim \pm\mu_0$ and thus that the orientation reversing homeomorphism $h|Q_0$ can be extended to an orientation reversing homeomorphism g of S^3 such that $g(K_0) = \pm K_0$, a contradiction.

LEMMA 4.3. *Suppose K_1 is non-invertible. If h is a good, orientation preserving invoution on $M(K_0, K_1)$, then* Fix $\langle h \rangle \cap T = \emptyset$.

Proof. Suppose not. Then Fix $\langle h \rangle$ is a simple closed curve meeting T transversely in finitely many points $x_1, \cdots, x_n$. Let T^* be the orbit space of T under $h|T$. The projection $q: T \to T^*$ is a 2-fold covering branched over $x_1^*, \cdots, x_n^*$, where $x_i^* = q(x_i)$. An Euler characteristic argument shows that T^* is a 2-sphere and $n = 4$.

Let γ^* and δ^* be arcs in T^* such that γ^* joins x_1^* and x_2^*, δ^* joins x_2^* and x_3^*, and each misses the other two branch points. Then $\gamma = q^{-1}(\gamma^*)$ and $\delta = q^{-1}(\delta^*)$ are simple closed curves meeting transversely in the single point x_2. After choosing orientations, γ and δ form a basis for $H_1(T)$. Moreover $h(\gamma) \sim -\gamma$ and $h(\delta) \sim -\delta$. It follows that $h(\mu_1) \sim -\mu_1$ and $h(\lambda_1) \sim -\lambda_1$. Then $h|Q_1$ can be extended to an orientation preserving homeomorphism g of S^3 such that $g(K_1) = -K_1$, a contradiction.

LEMMA 4.4. *Let h be an orientation preserving free involution on a torus T. Let $\alpha \cup \beta$ be a pair of simple closed curves in T which meet transversely in a single point. Then $\alpha \cup \beta$ can be isotoped so that either*

(i) *$h(\alpha) = \alpha$ and $h(\beta) \cap \beta = \emptyset$, or*

(ii) *$h(\beta) = \beta$ and $h(\alpha) \cap \alpha = \emptyset$, or*

(iii) *$h(\alpha) \cap \alpha = \emptyset = h(\beta) \cap \beta$.*

Proof. Note that h induces the identity on $H_1(T)$. Isotop $\alpha \cup \beta$ so that $h(\alpha) \cap \alpha$ is minimal.

Suppose $h(\alpha) \cap \alpha \neq \emptyset$. Since $h(\alpha) \sim \alpha$ there is a disk D in T with $\partial D = \gamma \cup \delta$, where γ and δ are arcs in α and $h(\alpha)$, respectively, and $(\alpha \cup h(\alpha)) \cap \operatorname{Int} D = \emptyset$. Suppose $h(D) \cap D = \emptyset$. Then α can be isotoped by pushing γ across D and off δ to obtain a new curve having four fewer intersection points with its image. This contradicts minimality and so does not occur. Suppose $h(D) \cap D$ is a single point p. Then α can be isotoped by pushing γ across D and off $\delta - p$ to obtain a curve having two fewer intersections with its image. So this cannot happen. Therefore $h(D) \cap D$ consists of two points p and q. In fact $h(\alpha) \cap \alpha = \{p, q\}$. Isotop α by pushing γ across D to δ. Then $h(\alpha) = \alpha$.

Now isotop β, keeping α pointwise fixed, so that $h(\alpha) \cap \beta$ is a single point. (This is only necessary if $h(\alpha) \cap \alpha = \emptyset$.) Then isotop β, keeping α and $h(\alpha)$ setwise fixed, so that $h(\beta) \cap \beta$ is minimal. As in the case of α above, the result will be that either $h(\beta) \cap \beta = \emptyset$ or that β can be isotoped so that $h(\beta) = \beta$. This can be done keeping α and $h(\alpha)$ setwise fixed because the analogous disk D used in the isotopies meets each of α and $h(\alpha)$ in at most a point of $\gamma \cap \delta$ or an arc with one endpoint in each of $\operatorname{Int}(\gamma)$ and $\operatorname{Int}(\delta)$.

LEMMA 4.5. *Let h be a good orientation preserving involution on $M(K_0, K_1)$ such that* $\operatorname{Fix}\langle h\rangle \cap T = \emptyset$. *Then* $\operatorname{Fix}\langle h\rangle = \emptyset$ *and $\alpha \cup \beta$ can be isotoped so that $h(\alpha) \cap \alpha = \emptyset = h(\beta) \cap \beta$.*

Proof. We may assume that $\alpha \cup \beta$ satisfies one of the three possible outcomes of Lemma 4.4. Suppose (i) is true. Then $h|Q_0$ can be extended to an involution g on S^3 with $K_0 \subset \operatorname{Fix}\langle g\rangle$. By Smith theory $K_0 = \operatorname{Fix}\langle g\rangle$. By the period two Smith Conjecture [14] K_0 is unknotted, a contradiction. A similar argument rules out (ii). Thus (iii) holds. If $\operatorname{Fix}\langle h\rangle \neq \emptyset$, then $\operatorname{Fix}\langle h\rangle \subset \operatorname{Int} Q_i$ for some i. Then the homology 3-sphere $M(K_i, K_i)$ admits an involution g with $\operatorname{Fix}\langle g\rangle$ homeomorphic to $S^1 \cup S^1$. This contradicts Smith theory, so $\operatorname{Fix}\langle h\rangle = \emptyset$.

LEMMA 4.6. *Suppose K_0 has a unique isotopy class of incompressible spanning surface. If h is a good, orientation preserving free involution on $M(K_0, K_1)$, then K_0 is a fibered knot.*

Proof. Let Q_0^* be the orbit space of Q_0 under h. Let $q\colon Q_0 \to Q_0^*$ be the quotient map and set $\mu_0^* = q(\mu_0)$, $\lambda_0^* = q(\lambda_0)$, and $T^* = q(T)$. Let $i\colon T^* \to Q_0^*$ be the inclusion map. Choose an oriented simple closed curve ξ which meets λ_0^* transversely in a single point. It follows from Lemma 4.5 that μ_0^* and λ_0^* meet transversely in two points, so $\mu_0^* = 2\xi + k\lambda_0^*$. (We now confuse curves in T^* with their homology classes.)

Claim. $H_1(Q_0^*) \cong \boldsymbol{Z}$ and is generated by ξ.

Since ∂Q_0^* is a torus, $H_1(Q_0^*)$ is infinite. This fact, together with the exact sequence

$$1 \longrightarrow \pi_1(Q_0) \xrightarrow{q_*} \pi_1(Q_0^*) \xrightarrow{\rho} \boldsymbol{Z}_2 \longrightarrow 1$$

implies that

$$q_*[\pi_1(Q_0), \pi_1(Q_0)] = [\pi_1(Q_0^*), \pi_1(Q_0^*)] .$$

Hence we have the exact sequence $0 \to H_1(Q_0) \overset{q_*}{\to} H_1(Q_0^*) \overset{\rho}{\to} \boldsymbol{Z}_2 \to 0$. So $H_1(Q_0^*)$ is either $\boldsymbol{Z}$ or $\boldsymbol{Z} \oplus \boldsymbol{Z}_2$. Suppose $H_1(Q_0^*) \cong \boldsymbol{Z} \oplus \boldsymbol{Z}_2$ with generators γ, δ for $\boldsymbol{Z}$, $\boldsymbol{Z}_2$, respectively. Then $i_*(\xi) = m\gamma + n\delta$. So $\gamma = i_*q_*(\mu_0) = i_*(\mu_0^*) = i_*(2\xi) = 2m\gamma + 2n\delta = 2m\gamma$, which is impossible. Thus $H_1(Q_0^*) \cong \boldsymbol{Z}$ with generator γ. Then $i_*(\xi) = m\gamma$ and $2\gamma = i_*q_*(\mu_0) = i_*(\mu_0^*) = i_*(2\xi) = 2m\gamma$ implies $m = 1$. This establishes the claim.

Now choose a map $f\colon Q_0^* \to S^1$ which realizes the epimorphism $\pi_1(Q_0^*) \to \boldsymbol{Z}$. Modify f on ∂Q_0^* so that $(f \mid T^*)^{-1}(p) = \lambda_0^*$ for some point p in S^1. Using standard surgery techniques (as in Lemma 6.5 of [5]) modify f on $\operatorname{Int} Q_0^*$ so that some component F^* of $f^{-1}(p)$ is an incompressible surface with $\partial F^* = \lambda_0^*$. Since $\pi_1(F^*) \leqq [\pi_1(Q_0^*), \pi_1(Q_0^*)] \leqq q_*\pi_1(Q_0)$, $f^{-1}(F^*)$ consists of two disjoint incompressible surfaces F_0 and F_1 which are interchanged by h. Since $\partial F_i \sim \lambda_0$ in T, the F_i are spanning surfaces for K_0 and so by assumption are isotopic. By Lemma 5.3 of [13] they cobound a product $F \times [0, 1]$ in Q_0. Since $Q_0 = (F \times [0, 1]) \cup h(F \times [0, 1])$ and $(F \times [0, 1]) \cap h(F \times [0, 1]) = F_0 \cup F_1$, K_0 is a fibered knot.

5. The examples.

THEOREM 5.1. *There is an infinite family of pairwise non-homeomorphic irreducible homology* 3-*spheres each of which admits no* PL *involutions.*

Proof. To construct one such example, it is sufficient, by the results of the previous section, to find simple knots K_0 and K_1, other than torus knots, having non-homeomorphic exteriors, such that K_0 is non-amphicheiral, has a unique isotopy class of incompressible spanning surface, and is not fibered, and K_1 is non-invertible.

Let K_0 be a twist knot [8, p. 112] with q twists, $q \leqq -2$. K_0 has bridge number 2 and so is simple [10]. K_0 has signature -2 and is therefore non-amphicheiral [8, p. 217]. K_0 has Alexander polynomial $qt^2 - (2q + 1)t + q$ and is therefore nonfibered [8, p. 326]; so K_0 is not a torus knot. By Lyon [7] K_0 has a unique isotopy type of incompressible spanning surface.

Let K_1 be the (3, 5, 7) pretzel knot [12]. K_1 has genus one and is therefore prime [9]. Since K_1 has bridge number 3 this implies [10] that K_1 is simple. Trotter [12] has shown that K_1 is non-invertible. K_1 has Alexander polynomial $18t^2 - 35t + 18$ and so is not a torus knot and has exterior not homeomorphic to that of K_0.

An infinite family of different examples is obtained by letting K_0 range over all twist knots with $q \leqq -2$ twists. No two of these are homeomorphic since, by Lemma 3.1, any homeomorphism between $M(K_0, K_1)$ and $M(K_0', K_1)$ could be deformed so that it carries Q_0 homeomorphically onto Q_0'. However, these are distinguished by the Alexander polynomials of K_0 and K_0'.

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