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IDENTIFICATION SPACES AND UNIQUE UNIFORMITY

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Properties of T_0-identification spaces and uniform identification spaces are used to obtain necessary and sufficient conditions for topological spaces to have a unique compatible uniformity.

1. Introduction. The major results in this paper are five characterizations of completely regular spaces with a unique compatible uniformity. All prior results of this type assume the space to be Tychonoff, i.e., completely regular and Hausdorff. In our approach we introduce a uniform identification space and develop some of its properties in order to demonstrate a $1-1$ correspondence between the family of compatible uniformities on a completely regular space and the family of compatible uniformities on its T_0-identification space.

Section 2 contains background on T_0-identification spaces and several new features of such spaces which we use later in the paper. In § 3 we present the main aspects of uniform identification spaces which lead to the order isomorphism in Theorem 3.5. In § 4, which contains the major theorem of the paper, we sketch the development of five characterizations of Tychonoff spaces with a unique compatible uniformity and then prove that each of these characterizations has a parallel for completely regular spaces.

2. T_0-identification spaces. In 1936 M. H. Stone proved that every topological space can be made into a T_0-space by identifying points with the same closure. A complete statement of Stone's work, with additional properties not included in Stone's paper, can be found in Theorem 14.2 in [8]. Briefly, given a topological space $(X, \mathscr{T})$, define $x \sim y$ iff $\overline{\{x\}} = \overline{\{y\}}$. Then $\sim$ is an equivalence relation on X and the quotient space $(Y, \mathscr{V})$ is a T_0-space. For $x \in X$, let D_x be the member of Y containing x. Then $f: X \to Y$ by $f(x) = D_x$ is a continuous, open and closed map onto Y.

Throughout this section $(Y, \mathscr{V})$ will be the T_0-identification space of $(X, \mathscr{T})$, and D_x and f will be as designated in the two preceding sentences.

A topological space $(X, \mathscr{T})$ is said to be *C-embedded* (in the terminology of [3], *normally embedded*) in the space $(Z, \mathscr{U})$ if every real-valued, continuous function on X has a continuous extension to Z, possibly through a homeomorphism of $(X, \mathscr{T})$ onto a subspace of $(Z, \mathscr{U})$. Such a homeomorphism is called an *embedding* of $(X, \mathscr{T})$

into $(Z, \mathscr{U})$. Also, $(X, \mathscr{T})$ is said to be *densely embedded* in $(Z, \mathscr{U})$ if there is an embedding f of $(X, \mathscr{T})$ into $(Z, \mathscr{U})$ such that $f(X)$ is dense in Z.

THEOREM 2.1. *$(Y, \mathscr{V})$ is densely embedded in $(X, \mathscr{T})$.*

Proof. Let W be a subset of X containing exactly one element of each equivalence class. Let $g: W \to Y$ by $g(w) = D_w$. Clearly g is $1-1$ and onto. To verify that g is continuous, let $G \in \mathscr{V}$. Then $f^{-1}(G) \in \mathscr{T}$ and thus $f^{-1}(G) \cap W$ is open in W. Since $f^{-1}(G) = \cup\{D_x: D_x \in G\}$, it follows that $g^{-1}(G) = f^{-1}(G) \cap W$. To verify that g^{-1} is continuous, note that if $H \in \mathscr{T}$, then $H = \cup\{D_x: x \in H\}$. Hence $g(H \cap W) = f(H) \in \mathscr{V}$.

To see that $\overline{W} = X$, let $x \in X$. Then $x \in D_w$ for some $w \in W$. Thus w is in every neighborhood of x. Therefore $x \in \overline{W}$.

COROLLARY 2.1. *$(Y, \mathscr{V})$ is C-embedded in $(X, \mathscr{T})$.*

Proof. Let g be the homeomorphism in the previous proof. Given a real-valued, continuous map h on $(Y, \mathscr{V})$, then $h \circ f$ is a real-valued, continuous map on $(X, \mathscr{T})$ which extends $h \circ g$.

LEMMA 2.1. *If $g: X \to S$ is continuous and $(S, \mathscr{U})$ is a T_0-space, then $g(x) = g(y)$ whenever $y \in D_x$.*

Proof. Suppose $y \in D_x$ and $g(x) \neq g(y)$. Since $(S, \mathscr{U})$ is T_0, there is $G \in \mathscr{U}$ such that $G \cap \{g(x), g(y)\}$ is a singleton. Thus the open set $g^{-1}(G)$ contains exactly one of the points x, y which contradicts the equivalence relation determining D_x.

THEOREM 2.2. *$(Y, \mathscr{V})$ has property (*) iff $(X, \mathscr{T})$ does.*

(*) For every real-valued, continuous function g defined on the space, $g^{-1}(\{r: r \geqq 1\})$ or $g^{-1}(\{r: r \leqq 0\})$ is compact.

Proof. Let g be a real-valued, continuous function defined on X. As a result of Lemma 2.1, we may define a real-valued function h on Y by $h(D_x) = g(x)$ for each $D_x \in Y$. Hence $g = h \circ f$. It is easy to show that h is continuous and if C is a compact subset of Y, then $f^{-1}(C)$ is a compact subset of X. Therefore, if h has property (*), then g does also.

The proof of the converse is straightforward.

THEOREM 2.3. *If X is a set, Y is a partition of X and $\mathscr{V}$ is a T_0-topology on Y, then there is a unique topology $\mathscr{T}$ on X such that $(Y, \mathscr{V})$ is the T_0-identification space of $(X, \mathscr{T})$.*

Proof. Since Y is a collection of disjoint subsets of X which covers X, for each $x \in X$ there is exactly one $D_x \in Y$ such that $x \in D_x$. Let $f: X \rightarrow Y$ by $f(x) = D_x$. By Theorem 10.10 in [8] the family $\mathscr{T} = \{f^{-1}(B): B \in \mathscr{V}\}$ is the weakest topology on X such that f is continuous. We shall show that $(Y, \mathscr{V})$ is the T_0-identification of $(X, \mathscr{T})$.

Let $x, y \in X$. If $y \in D_x$ and $x \in f^{-1}(B)$ where $B \in \mathscr{V}$, then since $f^{-1}(B) = \cup\{D_x: D_x \in B\}$, it follows that $y \in f^{-1}(B)$, i.e., each member of D_x is in every open subset of X which contains x. On the other hand, if $y \notin D_x$, then $D_y \cap D_x = \varnothing$. Since $(Y, \mathscr{V})$ is T_0, there exists $B \in \mathscr{V}$ which contains D_y or D_x, but not both. Hence $f^{-1}(B)$ contains x or y, but not both. Therefore the members of Y are exactly the classes which are determined by the equivalence relation on X where $x \approx y$ iff $\overline{\{x\}} = \overline{\{y\}}$.

Let $\mathscr{U}$ be the quotient topology on Y determined by f. Since $\mathscr{U}$ is the strongest topology on Y such that f is continuous, $\mathscr{V} \subset \mathscr{U}$. If $G \in \mathscr{U}$, then $f^{-1}(G) \in \mathscr{T}$ and there is $B \in \mathscr{V}$ such that $f^{-1}(B) = f^{-1}(G)$. Since f is onto, $B = f(f^{-1}(B)) = f(f^{-1}(G)) = G$.

To see that $\mathscr{T}$ is unique, let $\mathscr{S}$ be a topology on X such that $(Y, \mathscr{V})$ is the T_0-identification of $(X, \mathscr{S})$. Since $\mathscr{T}$ is the weakest topology on X such that f is continuous, $\mathscr{T} \subset \mathscr{S}$. Suppose $S \in \mathscr{S} \backslash \mathscr{T}$. Since f is an open map, $f(S) \in \mathscr{V}$. So $f^{-1}(f(S)) \in \mathscr{T}$ and there is $t \in f^{-1}(f(S)) \backslash S$. Now $t \in D_s$ for some $s \in S$. Thus s is a member of a set in $\mathscr{S}$ not containing t, which contradicts the equivalence relation $\approx$.

THEOREM 2.4. *Let $(X, \mathscr{T})$ be a subspace of $(S, \mathscr{S})$ whose T_0-identification space is $(T, \mathscr{U})$. If $(Y, \mathscr{V})$ is C-embedded in $(T, \mathscr{U})$, then $(X, \mathscr{T})$ is C-embedded in $(S, \mathscr{S})$.*

Proof. Let g be a continuous, real-valued function on X. As a result of Lemma 2.1, we may define a real-valued function h on Y by $h(D_x) = g(x)$ for each $D_x \in Y$. Hence $g = h \circ f$ and h is continuous. By assumption h has a continuous extension k to T. Let $e: S \rightarrow T$ be the quotient map $e(s) = [s]$ where $[s]$ is the equivalence class containing s. Then $k \circ e$ is a continuous extension of g to S.

THEOREM 2.5. *Let $(Y, \mathscr{V})$ be a dense subspace of the T_0-space $(T, \mathscr{U})$. Then there is a topological space $(S, \mathscr{S})$ such that $(T, \mathscr{U})$*

is the T_0-identification of $(S, \mathscr{S})$ and $(X, \mathscr{T})$ is densely embedded in $(S, \mathscr{S})$. Furthermore, if $(X, \mathscr{T})$ is C-embedded in $(S, \mathscr{S})$, then $(Y, \mathscr{V})$ is C-embedded in $(T, \mathscr{U})$.

Proof. Let $S = X \cup (T \backslash Y)$, so without loss of generality we may assume $T \cap X = \varnothing$. For each open subset A of T form $A^* = \cup\{D_x: D_x \in A \cap Y\} \cup A \backslash Y$. Then $\{A^*: A \in \mathscr{U}\}$ is a topology on S. As usual, define $x \approx y$ for $x, y \in S$ iff $\overline{\{x\}}^S = \overline{\{y\}}^S$. Note that when x and y are distinct points in S, then $x \approx y$ iff $x, y \in X$ and $x \sim y$ in X. Thus $\approx$ determines the members of T, with the identification of $\{t\}$ with t whenever $t \in T \backslash Y$. It is easy to show that the quotient topology on T agrees with $\mathscr{U}$ and that $(X, \mathscr{T})$ is a dense subspace of $(S, \mathscr{S})$.

Let h be a real-valued, continuous function on Y. Then $h \circ f$ is a real-valued, continuous function on X, and thus has a continuous extension j to S. As a result of Lemma 2.1, we may define a real-valued function k on T by $k(D_x) = j(x)$ for $D_x \in Y$ and $k(t) = j(t)$ for $t \in T \backslash Y$. Let $e: S \to T$ be the quotient map. Then $j = k \circ e$, k is continuous and $k|_Y = h$.

Let $C^*(X)$ be the set of bounded, real-valued, continuous functions on X and let $C^*(X)$ have the topology of uniform convergence. By $A(X)$ we denote the subset of $C^*(X)$ consisting of those functions which are constant on the complement of some compact set in $(X, \mathscr{T})$.

THEOREM 2.6. *Let $(Y, \mathscr{V})$ be the T_0-identification space of $(X, \mathscr{T})$. Then $A(X)$ is dense in $C^*(X)$ iff $A(Y)$ is dense in $C^*(Y)$.*

Proof. ($\Leftarrow$) Let $g \in C^*(X)$ and $\varepsilon > 0$. As a result of Lemma 2.1, we may define a real-valued function h on Y by $h(D_x) = g(x)$ for each $D_x \in Y$. Then $g = h \circ f$ and $h \in C^*(Y)$. Since $A(Y)$ is dense in $C^*(Y)$, there is a continuous function k which is constant on the complement of a compact set C in Y and satisfies $|k(D_x) - h(D_x)| < \varepsilon$ for each $D_x \in Y$. Then $f^{-1}(C)$ is compact in X and $k \circ f$ is constant on the complement of $f^{-1}(C)$. Also $|k(f(x)) - g(x)| < \varepsilon$ for each $x \in X$.

($\Rightarrow$) Let $h \in C^*(Y)$ and $\varepsilon > 0$. Then $h \circ f \in C^*(X)$. Since $A(X)$ is dense in $C^*(X)$, there is a continuous function g which is constant on the complement of a compact set K in X and satisfies $|h(f(x)) - g(x)| < \varepsilon$ for each $x \in X$. As a result of Lemma 2.1, we may define a real-valued function k on Y by $k(D_x) = g(x)$ for each $D_x \in Y$. Then $g = k \circ f$ and k is constant on the complement of the compact set $f(K)$ in Y. Also $|h(D_x) - k(D_x)| < \varepsilon$ for each $D_x \in X$.

3. Uniform identification spaces. For the definitions of a uniform space and a proximity space, see [8]. Recall that if $U, V \subset X \times X$, then $U \circ V = \{(x, y): (x, t) \in U \text{ and } (t, y) \in V \text{ for some } t\}$ and $U(x) = \{y: (x, y) \in U\}$. If $(X, \mathscr{H})$ is a uniform space and Z is a set with $h: X \to Z$ a map onto Z, then the quotient uniformity induced on Z by h is $\{A \subset Z \times Z$: there is $H \in \mathscr{H}$ such that $(x, y) \in H$ implies $(h(x), h(y)) \in A\}$, which is the largest uniformity on Z such that h is uniformly continuous, see [9, p. 255]. It is easily verified that the quotient uniformity is $\{g(H): H \in \mathscr{H}\}$ where $g(H) = \{(h(a), h(b)): (a, b) \in H\}$.

Let $(X, \mathscr{H})$ be a uniform space. For $x, y \in X$, define $x \sim y$ iff $y \in H(x)$ for each $H \in \mathscr{H}$. Then $\sim$ is an equivalence relation on X. Throughout this section Y is the set of equivalence classes, D_x is the member of Y containing x, $f: X \to Y$ by $f(x) = D_x$ and $\mathscr{K}$ is the quotient uniformity on Y induced by f. Also, $(Y, \mathscr{K})$ is called *the uniform identification space* of $(X, \mathscr{H})$.

LEMMA 3.1. *$(Y, \mathscr{K})$ is a separated uniform space.*

Proof. Suppose D_x and D_y are distinct equivalence classes. Then there exists $H \in \mathscr{H}$ such that $H(x) \cap H(y) = \varnothing$. Since $D_y \subset H(y)$, $H(x) \cap D_y = \varnothing$. Choose $F \in \mathscr{H}$ such that $F \circ F \subset H$. If there exists $a \in D_x$ and $b \in D_y$ such that $(a, b) \in F$, then $(x, a) \in F$, since $x \in D_x \subset F(x)$. Hence $(x, b) \in F \circ F$, and thus $b \in H(x)$ which is a contradiction. We conclude that $(D_x \times D_y) \cap F = \varnothing$. Therefore $(D_x, D_y) \notin g(F) \in \mathscr{K}$.

COROLLARY 3.1. *If $(X, \mathscr{H})$ is a uniform space, then $\bigcap\{H: H \in \mathscr{H}\} = \bigcup\{D_x \times D_x: x \in X\}$.*

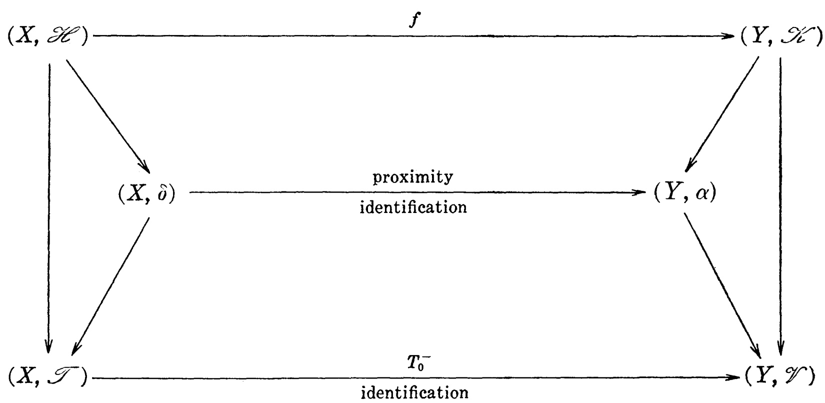
Proof. Using F in the proof of Lemma 3.1, it follows that $\bigcap\{H: H \in \mathscr{H}\} \subset \bigcup\{D_x \times D_x: x \in X\}$. The other containment is a result of $D_x \subset H(x)$ for each $H \in \mathscr{H}$.

THEOREM 3.1. *Let $(X, \mathscr{H})$ be a uniform space, let δ be the proximity on X induced by $\mathscr{H}$ and let α be the proximity on Y induced by $\mathscr{K}$. Then the following diagram commutes.*

$$\begin{array}{ccc} (X, \mathscr{H}) & \xrightarrow{\quad f \quad} & (Y, \mathscr{K}) \\ \downarrow & & \downarrow \\ (X, \delta) & \xrightarrow[\text{identification}]{\text{proximity}} & (Y, \alpha) \end{array}$$

Proof. In (X, δ), the equivalence classes are determined by the relation $x \approx y$ iff $x\delta y$ [9, p. 276]. Since $x\delta y$ iff $y \in H(x)$ for each $H \in \mathscr{H}$ iff $x \sim y$, it follows that the same members are determined for Y by either path. Since the proximity induced by a quotient uniformity is the quotient proximity of the induced proximity [9, p. 276], the diagram commutes.

COROLLARY 3.2. *In addition, let $\mathscr{T}$ be the topology on X induced by δ and let $\mathscr{Y}$ be the topology on Y induced by α. Then the following diagram commutes.*



Proof. The topology induced by a uniformity is the induced topology of the induced proximity [8, Theorem 21.15]. Starting with (X, δ), it is known that the paths are equivalent [9, p. 276].

LEMMA 3.2. *Let $(X, \mathscr{H})$ be a uniform space and let $H \in \mathscr{H}$. Then there is $E \in \mathscr{H}$ such that $E \subset H$ and $E = \cup\{D_x \times D_y : (x, y) \in E\}$.*

Proof. Find symmetric $G \in \mathscr{H}$ such that $G \circ G \subset H$. Note that $G \subset G \circ G$. Then find $F \in \mathscr{H}$ such that $F \circ F \subset G$. If $(r, s) \notin H$, then $H(r) \cap \{s\} = \emptyset$. If there exists $t \in G(r) \cap G(s)$, then (r, t) and (s, t) are in G. Since G is symmetric, $(r, s) \in G \circ G$, which is a contradiction. Thus $G(r) \cap G(s) = \emptyset$. As in the proof of Lemma 3.1 we can conclude that $(D_r \times D_s) \cap F = \emptyset$. Set $E = \cup\{D_x \times D_y : (x, y) \in F\}$. Since $(x, y) \in D_x \times D_y$, if follows that $F \subset E$ and hence, $E \in \mathscr{H}$. If $(a, b) \in E$, then $(D_a \times D_b) \cap F \neq \emptyset$, and from the above work $(a, b) \in H$. Clearly, $E = \cup\{D_x \times D_y : (x, y) \in E\}$.

THEOREM 3.2. *$(Y, \mathscr{K})$ is uniformly isomorphic to a uniform subspace of $(X, \mathscr{H})$.*

Proof. Let S be a subset of X consisting of exactly one point from each equivalence class in Y. The relative uniformity on S is $\{H \cap (S \times S): H \in \mathscr{H}\}$. Consider the map $d: S \to Y$ by $d(s) = D_s$. Clearly, d is $1 - 1$ and onto. To verify that d is uniformly continuous, let $K \in \mathscr{K}$. Then there is $H \in \mathscr{H}$ such that $f(H) = K$. If $(x, y) \in H \cap (S \times S)$, then $(d(x), d(y)) = (f(x), f(y)) \in f(H)$. To verify that d^{-1} is uniformly continuous, let $H \in \mathscr{H}$ and consider $E \cap (S \times S)$ where E is the entourage in $\mathscr{H}$ guaranteed by Lemma 3.2. If $(D_x, D_y) \in f(E)$ which is in $\mathscr{K}$, then $(d^{-1}(D_x), d^{-1}(D_y)) \in f^{-1}(f(E)) \cap (S \times S) = E \cap (S \times S) \subset H \cap (S \times S)$.

THEOREM 3.3. *Let* $(Y, \mathscr{K})$ *be the uniform identification space of* $(X, \mathscr{H})$. *Then* $\mathscr{F} = \{g^{-1}(K): K \in \mathscr{K}\}$ *is a base for* $\mathscr{H}$, *where* $g^{-1}(K) = \{(a, b) \in X \times X: (f(a), f(b)) \in K\}$.

Proof. By Theorem 20.21 in [8], $\mathscr{F}$ is a subbase for the weakest uniformity $\mathscr{G}$ on X such that f is uniformly continuous. Thus $\mathscr{G} \subset \mathscr{H}$. To verify that $\mathscr{F}$ is a base, note that $g^{-1}(K) = \cup\{D_x \times D_y: (D_x, D_y) \in K\}$ and $D_x \times D_y = D_r \times D_s$ or $(D_x \times D_y) \cap (D_r \times D_s) = \varnothing$, from which it follows that $g^{-1}(K) \cap g^{-1}(L) = g^{-1}(K \cap L)$. If $H \in \mathscr{H}$, then let $E \in \mathscr{H}$ be the entourage guaranteed by Lemma 3.2. Therefore $E = g^{-1}(g(E))$ and $g(E) \in \mathscr{K}$. Hence $\mathscr{H} \subset \mathscr{G}$.

THEOREM 3.4. *Let* X *be a set, let* Y *be a partition of* X *and let* $\mathscr{K}$ *be a separated uniformity on* Y. *Then there is a uniformity* $\mathscr{H}$ *on* X *such that* $(Y, \mathscr{K})$ *is the uniform identification space of* $(X, \mathscr{H})$.

Proof. Since Y is a collection of disjoint subsets of X which covers X, for each $x \in X$ there is exactly one member D_x of Y such that $x \in D_x$. Let $f: X \to Y$ by $f(x) = D_x$. By Theorem 20.21 in [8], the family $\mathscr{F} = \{g^{-1}(K): K \in \mathscr{K}\}$ is a subbase for the weakest uniformity $\mathscr{H}$ on X such that f is uniformly continuous. Here $g^{-1}(K) = \{(a, b) \in X \times X: (f(a), f(b)) \in K\}$. We shall show that $(Y, \mathscr{K})$ is the uniform identification space of $(X, \mathscr{H})$.

If $y \in D_x$ and $K \in \mathscr{K}$, then $(D_x, D_x) \in K$ and hence $y \in g^{-1}(K)(x)$. If $y \notin D_x$, then since $\mathscr{K}$ is separated, there is some $K_0 \in \mathscr{K}$ such that $(D_x, D_y) \notin K_0$. Thus $y \notin g^{-1}(K_0)(x)$. We conclude that the members of Y are exactly the classes which are determined by the equivalence relation on X where $x \sim y$ iff $y \in H(x)$ for each $H \in \mathscr{H}$.

Let $\mathscr{L}$ be the quotient uniformity on Y induced by f. Since f is uniformly continuous with respect to $\mathscr{K}$ and $\mathscr{L}$, $\mathscr{K} \subset \mathscr{L}$. On the other hand, if $L \in \mathscr{L}$, then there is $H \in \mathscr{H}$ such that $h(H) = L$, where $h(H) = \{(f(r), f(s)): (r, s) \in H\}$. Since $\mathscr{F}$ is a base for $\mathscr{H}$, there is $F \in \mathscr{F}$ such that $F \subset H$. Then there is $K \in \mathscr{K}$ such that

$g^{-1}(K) = F$. Thus $K \subset L$ and therefore $L \in \mathscr{K}$.

THEOREM 3.5. *Let* $(X, \mathscr{T})$ *be a topological space and let* $(Y, \mathscr{V})$ *be its* T_0*-identification space. Let* Θ *be the family of all uniformities on* X *compatible with* $\mathscr{T}$ *and let* Ω *be the family of all uniformities on* Y *compatible with* $\mathscr{V}$. *Then* Θ *and* Ω *are order isomorphic.*

Proof. By Corollary 3.2 we may define $h: \Theta \to \Omega$ by $h(\mathscr{H})$ is the identification uniformity of $\mathscr{H}$. As a result of Theorem 3.3, h is $1 - 1$. By Theorem 3.4 in conjunction with Theorem 2.3 and Corollary 3.2, h is onto. Since $h(\mathscr{H})$ is a quotient uniformity, it follows that h preserves order. Noting how $h^{-1}(\mathscr{K})$ is formed in the proof of Theorem 3.4 allows us to conclude that h^{-1} preserves order.

It is noted that Theorem 3.5 can also be proved from Theorem 2.1 in [7].

4. Unique compatible uniformity and proximity. Early in the study of uniform spaces it was observed that a compact, completely regular topological space admits exactly one compatible uniformity [8, Theorem 20.38]. Using normally separated sets, Doss [2] characterized Tychonoff spaces which have exactly one compatible uniformity. Later Gál [3] gave two additional characterizations of Tychonoff spaces with this phenomenon. Newns [6] has given a characterization based on the structure of the uniformity. Doss' work is extended to completely regular spaces in Theorem 4.1(d), Gál's work in Theorem 4.1(e) and (f), and Newns' work in Theorem 4.1(b).

In Corollary 2.2 of [4] it is shown that a Tychonoff space has a unique compatible proximity iff it has a unique compatible uniformity. Note that [4] requires a Hausdorff assumption since Corollary 2.2 is based upon the Stone-Čech and Smirnov compactifications. This result is also in [9, p. 277]. We prove in the next theorem that this result is valid without a Hausdorff assumption.

THEOREM 4.1. *Let* $(X, \mathscr{T})$ *be a completely regular topological space. Then the following are equivalent*:

(a) *There is a unique uniformity on* X *compatible with* $\mathscr{T}$.

(b) *There is a unique totally bounded uniformity on* X *compatible with* $\mathscr{T}$.

(c) *There is a unique proximity on* X *compatible with* $\mathscr{T}$.

(d) $(X, \mathscr{T})$ *has the property that for every real-valued, continuous function* f *defined on* X, $\{x: f(x) \geqq 1\}$ *or* $\{x: f(x) \leqq 0\}$ *is compact.*

(e) *$(X, \mathscr{T})$ is C-embedded in every completely regular space containing $(X, \mathscr{T})$ as a dense subspace.*

(f) *$A(X)$ is dense in $C^*(X)$.*

Proof. Let $(Y, \mathscr{Y})$ be the T_0-identification space of $(X, \mathscr{T})$.

(b) $\Leftrightarrow$ (c). The implications follow from Theorem 21.28 in [8].

(a) $\Leftrightarrow$ (c). Since $\mathscr{T}$ and $\mathscr{Y}$ are lattice isomorphic, it follows from [1] that the family of proximities on X compatible with $\mathscr{T}$ is isomorphic to the family of proximities on Y compatible with $\mathscr{Y}$. Hence (b) is equivalent to the existence of a unique proximity on Y compatible with $\mathscr{Y}$. From [4, Corollary 2.2] this is equivalent to the existence of a unique uniformity on Y compatible with $\mathscr{Y}$. By Theorem 3.5 this last statement is equivalent to (a).

(a) $\Leftrightarrow$ (d). Couple Theorems 3.5 and 2.2 with Doss' Theorem in [2].

(a) $\Leftrightarrow$ (e). Let X be a dense subspace of the completely regular space S whose T_0-identification space is T. Denote the equivalence classes of S by $[s]$ where $s \in S$. If $s \in X$, then we identify the equivalence class D_s of X with $[s]$ of S. Thus Y is densely embedded in T. It follows from Theorem 3.5 that (a) is equivalent to Y having a unique uniformity, which by Gál's Theorem (v) in [3] implies that Y is C-embedded in T, and by Theorem 2.4 X is C-embedded in S. On the other hand, (d) implies by Theorem 2.5 that Y is C-embedded in every Tychonoff space containing Y as a dense subspace, which by Gál's Theorem (v) in [3], is equivalent to Y having a unique uniformity.

(a) $\Leftrightarrow$ (f). Couple Theorems 3.5 and 2.6 with Gál's Theorem (iv) in [3].

Each of the following three statements is equivalent to Theorem 4.1(d). In the terminology of [2], of any two normally separable sets in X, at least one is compact. In the terminology of [5], if A and B are functionally distinguishable subsets of X, then A or B is compact. In the terminology of [9], of any two disjoint zero-sets in X, at least one is compact.

Comparing Theorem 4.1(d) with Theorem 7.20 in [5], it is noted that locally compact can be deleted from the statement of Theorem 7.20.

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