PACIFIC JOURNAL OF MATHEMATICS
Vol. 95, No. 2, 1981

FINITE SIGNED MEASURES ON FUNCTION SPACES

A. J. VAN HAAGEN

Some results for probability measures on function spaces are extended to finite signed measures (FSM's). In particular FSM's on the space of continuous functions and right-continuous functions with left-hand limits are patched together by a procedure of Stroock and Varadhan. Given an increasing sequence of stopping times the procedure is carried out repeatedly. A sequence of transition functions and, an extension result for the linear maps associated with these transition functions are obtained.

Introduction. In recent years some papers have appeared related to signed measures on function spaces (see [3] and [4]). This paper extends certain results for probability measures on function spaces to finite signed measures (FSM's) on such spaces. A more detailed discussion can be found in [8].

In part I we introduce conditional FSM's and consider the existence of a regular conditional distribution (RCD) of an FSM on a standard measurable space mimicking Chapter V of [5]. Further, the Jordan decomposition of an RCD is investigated. We then consider a sequence of transition functions and associate linear maps between Banach spaces of FSM's with these transition functions. An extension result for the linear maps is obtained.

In part II FSM's on $\Omega = C([0, \infty); S)$ and $\Omega = D([0, \infty); S)$ with S a separable metric space are patched together by the procedure used in [6] for probability measures on $C([0, \infty); R^d)$. In fact, if $\mathscr{M}^s$ is the σ-field on Ω generated by the coordinate projections $\{X_t,\ t \geqq s\}$ and τ an s-stopping time with respect to the σ-fields $\mathscr{M}_t^s = \sigma\{X_u,\ s \leqq u \leqq t\}$, then an FSM on $\mathscr{M}_\tau^s$ is patched together with a family $\{\mu_\omega\}_{\omega \in \Omega}$ of FSM's where μ_ω has domain $\mathscr{M}^{\tau(\omega)}$ if $\tau(\omega) < \infty$.

If S is a complete separable metric space, $(\Omega, \mathscr{M}^s)$ and $(\Omega, \mathscr{M}_\tau^s)$ are standard measurable spaces. In this case any FSM on $(\Omega, \mathscr{M}^s)$ with an RCD given $\mathscr{M}_\tau^s$ can be thought of as obtained by patching. If $\tau_0, \tau_1, \cdots$ is an increasing sequence of s-stopping times $\mathscr{M}_{\tau_0}^s$, $\mathscr{M}_{\tau_1}^s, \cdots$ is the corresponding sequence of σ-fields and, given families of FSM's $\{\mu_{n\omega}\}_\omega$ on $\mathscr{M}^{\tau_n(\omega)}$ for each n with the right properties the patching procedure can be applied repeatedly. We have in fact an associated sequence of transition functions and the results of part I apply.

Basic facts on FSM's are taken from [2].

Part I. RCD's of FSM's on Standard Measurable Spaces

1. Conditional FSM's. Let $(\Omega, \mathscr{F}, \mu)$ be an FSM space and Σ a sub σ-field of $\mathscr{F}$. If μ_Σ is the restriction of μ to Σ, then a μ_Σ-null set is not necessarily a μ-null set. However we can prove the following.

THEOREM 1.1. *If each μ_Σ-null set of Ω is also a μ-null set, and if B is any $\mathscr{F}$-measurable set, then there exists a Σ-measurable real function $\mu(B|\Sigma)$ such that for all $A \in \Sigma$*

$$\mu(A \cap B) = \int_A \mu(B|\Sigma) d\mu_\Sigma .$$

Any two such functions for given B must coincide μ-a.e. $\mu(B|\Sigma)$ is called the conditional FSM *of B given Σ.*

Proof. For $B \in \mathscr{F}$ we define an FSM λ on Σ by $\lambda(A) = \mu(A \cap B)$ for all $A \in \Sigma$. As λ is absolutely continuous with respect to μ_Σ it follows by the Radon-Nikodym theorem that there exists a Σ-measurable real function f on Ω such that

$$\mu(A \cap B) = \int_A f d\mu_\Sigma \qquad \text{for all } A .$$

We can take f to be $\mu(B|\Sigma)$.

THEOREM 1.2. *With the assumptions on μ of Theorem* 1.1 *let $\{B_n\}_1^\infty$ be a sequence of disjoint $\mathscr{F}$-measurable sets. Then*

$$\mu(\bigcup_n B_n|\Sigma) = \sum_n \mu(B_n|\Sigma) \qquad \mu\text{-a.e.}$$

Proof. Let $\mu = \mu^+ - \mu^-$ be the Jordan decomposition of μ and let $|\mu_\Sigma|$ be the total variation of μ_Σ. If $B \in \mathscr{F}$, there exist Σ-measurable real functions $h_1(B|\Sigma) \geqq 0$ and $h_2(B|\Sigma) \geqq 0$ such that for all $A \in \Sigma$

$$\mu^+(A \cap B) = \int_A h_1(B|\Sigma) d|\mu_\Sigma| ,$$

$$\mu^-(A \cap B) = \int_A h_2(B|\Sigma) d|\mu_\Sigma| .$$

If we put $h(B|\Sigma) = h_1(B|\Sigma) - h_2(B|\Sigma)$, then $|\mu(B|\Sigma)| = |h(B|\Sigma)|$ μ-a.e. It now follows easily that for all $A \in \Sigma$

$$\int_A \sum_n \mu(B_n|\Sigma) d\mu_\Sigma = \int_A \mu(\bigcup_n B_n|\Sigma) d\mu_\Sigma .$$

REMARK 1.3. Note that for all $B \in \mathscr{F}$, $|\mu(B|\Sigma)| \leqq h_1(\Omega|\Sigma) + h_2(\Omega|\Sigma)$ except on a μ-null set N (which depends on B in general).

DEFINITION 1.4. Let $(\Omega, \mathscr{F}, \mu)$ be an FSM space, Σ a sub σ-field of $\mathscr{F}$ and R the set of real numbers. The function $Q: \Omega \times \mathscr{F} \rightarrow R$ is called an RCD of μ given Σ if it has the properties:

(i) For each $B \in \mathscr{F}$, $\omega \rightarrow Q(\omega, B)$ is a Σ-measurable map from Ω into R.

(ii) For each $\omega \in \Omega$, $Q(\omega, \cdot)$ is an FSM on $\mathscr{F}$ with $Q(\omega, \Omega) = 1$.

(iii) For all $A \in \Sigma$ and $B \in \mathscr{F}$

$$\mu(A \cap B) = \int_A Q(\omega, B) d\mu_\Sigma(\omega) .$$

In §3 we give conditions under which Q exists.

LEMMA 1.5. *Let $(\Omega, \mathscr{F}, \mu)$ be a countably generated* FSM *space and Σ a sub σ-field of $\mathscr{F}$. If Q_1 and Q_2 are* RCD's *of μ given Σ, then $Q_1(\omega, B) = Q_2(\omega, B)$ for all $B \in \mathscr{F}$ except for ω in a μ-null set of Σ which is independent of B.*

Proof. The proof is the same as for regular conditional probability distributions (RCPD's).

2. Extension theorems for FSM's.

In this section we generalize theorems for probability measures appearing in Chapter V of [5] to theorems on FSM's.

DEFINITION 2.1. Let $\{\mathscr{B}_n\}_1^\infty$ be an increasing family of σ-fields on the space X. The family of FSM's $\{\mu_n\}_1^\infty$ is said to be consistent if μ_n is defined on $\mathscr{B}_n$ and for all $A \in \mathscr{B}_n$ and $m \geqq n$, $\mu_n(A) = \mu_m(A)$. The family is said to be uniformly bounded if $\sup_n |\mu_n|(X) < \infty$.

We will need the following simple lemma of which we omit the proof.

LEMMA 2.2. *If $\{\mu_n\}_1^\infty$ is a uniformly bounded consistent family of* FSM's *on the σ-fields $\{\mathscr{B}_n\}_1^\infty$, then there exists a unique finitely additive set function μ on the field $\bigcup_n \mathscr{B}_n$ with the properties:*

(i) *$\mu(A) = \mu_n(A)$ for all $A \in \mathscr{B}_n$ and $n = 1, 2, \cdots$.*

(ii) *Given $\varepsilon > 0$, there exists a positive integer n_0 such that $|\mu|(A) - |\mu_n|(A) < \varepsilon$ for all $A \in \mathscr{B}_n$ if $n \geqq n_0$.*

THEOREM 2.3 (*analogue of Theorem* 4.1 *on* p. 141 *of* [5]). *Let $(X, \mathscr{B})$ be a measurable space and for each $n = 1, 2, \cdots$ $\mathscr{B}_n$ is a* *sub σ-field of $\mathscr{B}$ such that*

(i) $\mathscr{B}_1 \subseteq \mathscr{B}_2 \subseteq \cdots$, *and* $\bigcup_n \mathscr{B}_n$ *generates* $\mathscr{B}$.

(ii) $(X, \mathscr{B}_n)$ *is a standard measurable space for* $n = 1, 2, \cdots$.

Then, in order that every uniformly bounded consistent sequence of FSM's *on* $\mathscr{B}_1, \mathscr{B}_2, \cdots$ *be extendable to an* FSM *on* $\mathscr{B}$, *it is necessary and sufficient that* $\bigcap_1^\infty A_n \neq \emptyset$ *for each decreasing sequence* $\{A_n\}_1^\infty$ *of subsets of* X *such that* A_n *is an atom of* $\mathscr{B}_n$ *for all* n. *If this is the case,* $(X, \mathscr{B})$ *is also standard, and if* $\{\mu_n\}_1^\infty$ *is such a sequence of* FSM's, *the extension is unique. Moreover,* $|\mu|(X) = \sup_n |\mu_n|(X)$.

Proof. The necessity is proved in [5]. In order to prove the sufficiency we first prove the analogue for FSM's of Theorem 3.1 on p. 138 of [5]. We use the notation of [5] and make the further assumption that $\sup_n |\mu_n|(Z_n) < \infty$.

Let μ be the finitely additive bounded set function on $\mathscr{F}$ analogous in the obvious way to μ on p. 139 of [5]. We will show that if $\{A_n\}_1^\infty$ is a decreasing sequence in $\mathscr{F}$ and if $|\mu|(A_n) \geqq \delta > 0$ for all n and some $\delta > 0$, then $\bigcap_n A_n \neq \emptyset$. Without loss of generality we may assume $A_n \in \widetilde{\mathscr{D}}_n$ for $n = 1, 2, \cdots$. For $n = 1, 2, \cdots$ and $\mathfrak{l} = 0, 1, \cdots$ we can write $A_n = \widetilde{\varphi}_{n+\mathfrak{l}}^{-1}(B_{n,\mathfrak{l}})$ where $B_{n,\mathfrak{l}} \in \mathscr{D}_{n+\mathfrak{l}}$. For each n and $\mathfrak{l}$ there exists a compact set $K_{n,\mathfrak{l}}^0 \subseteq B_{n,\mathfrak{l}}$ in $\mathscr{D}_{n+\mathfrak{l}}$ such that $|\mu_{n+\mathfrak{l}}|(B_{n,\mathfrak{l}} - K_{n,\mathfrak{l}}^0) \leqq \delta/4^n$. Put $\widetilde{K}_{n,\mathfrak{l}} = \widetilde{\varphi}_{n+\mathfrak{l}}^{-1}(K_{n,\mathfrak{l}}^0)$. By Theorem 2.6 on p. 136 of [5], Z is compact and $\widetilde{\varphi}_{n+\mathfrak{l}}$ is continuous. Hence, $\widetilde{K}_{n,\mathfrak{l}}$ is compact. Moreover, $\widetilde{K}_{n,\mathfrak{l}} \subseteq A_n$ for $\mathfrak{l} = 0, 1, \cdots$ and

$$(*) \qquad |\mu_{\widetilde{\mathscr{D}}_{n+\mathfrak{l}}}|(A_n - \widetilde{K}_{n,\mathfrak{l}}) \leqq \delta/4^n \qquad \text{for all } n \text{ and } \mathfrak{l}\,.$$

Now $\{\mu_{\mathscr{D}_j}\}_1^\infty$ is a uniformly bounded consistent family of FSM's. By Lemma 2.2 there exists for each n an $\mathfrak{l}(n) > 0$ such that $|\mu|(A_n - \widetilde{K}_{n,\mathfrak{l}(n)}) - |\mu_{\widetilde{\mathscr{D}}_{n+\mathfrak{l}(n)}}|(A_n - \widetilde{K}_{n,\mathfrak{l}(n)}) < \delta/4^{n+1}$. Together with $(*)$ we obtain $|\mu|(A_n - \widetilde{K}_{n,\mathfrak{l}(n)}) < 5\delta/4^{n+1}$.

Put $K_n = \bigcap_{j=1}^n \widetilde{K}_{j,\mathfrak{l}(j)}$ for $n = 1, 2, \cdots$. As in [5] it follows that $\bigcap_n A_n \neq \emptyset$. Thus μ is countably additive on $\mathscr{F}$. Therefore μ has a unique extension to an FSM on $\mathscr{D}$, which we also denote by μ, and $|\mu|(Z) = \sup_n |\mu_n|(Z_n)$.

We can now prove the analogue for FSM's of Theorem 3.2 on p. 139 of [5]. This proof and the proof of the theorem at hand go through in the same way as in [5]. For details see [8].

The next theorem is Kolmogorov's extension theorem for FSM's. Its proof is similar to Parthasarathy's proof for probability measures. With the notation of Theorem 5.1 on p. 144 of [5] we have the following.

THEOREM 2.4. *Let* $(X_\alpha, \mathscr{B}_\alpha)$, $\alpha \in I$, *be standard separable measurable spaces. If* $\{\mu_F: F \subseteqq I, F \text{ finite}\}$ *is a uniformly bounded consistent family of* FSM's, *then there exists a unique* FSM μ *on* $\mathscr{B}^I$ *such that* $\mu_F(A) = \mu(\pi_{IF}^{-1}(A))$ *for all* $A \in \mathscr{B}^F$ *and all finite* $F \subseteqq I$. *Moreover*, $|\mu|(X^I) = \sup_{F \subseteqq I\ F\ finite} |\mu_F|(X^F)$.

3. The existence of an RCD of an FSM.

THEOREM 3.1 (*analogue of Theorem* 7.1 *on* p. 145 *of* [5]). *Let* $(X, \mathscr{B})$ *be a standard measurable space and* Σ *a sub d-field of* $\mathscr{B}$. *Let* μ *be an* FSM *on* $\mathscr{B}$. *The condition that a* μ_Σ*-null set of* X *is a* μ*-null set is necessary and sufficient for the existence of an* RCD *of* μ *given* Σ.

Proof. The necessity follows immediately from the definition. If X is countable the sufficiency is a direct consequence of Theorems 1.1 and 1.2. Let X be uncountable. As is shown in [5] there exists an increasing sequence of finite σ-fields $\{\mathscr{B}_n\}_1^\infty$ such that $\bigcup_n \mathscr{B}_n$ generates $\mathscr{B}$ and any uniformly bounded consistent sequence of FSM's on $\{\mathscr{B}_n\}_1^\infty$ is extendable to an FSM on $\mathscr{B}$.

Following [5] it is easy to see that for $n = 1, 2, \cdots$ there exists a function $Q_n: X \times \mathscr{B}_n \to R$ such that

(i) for each $A \in \mathscr{B}_n$, $x \to Q_n(x, A)$ is Σ-measurable.
(ii) for each $x \in X$, $Q_n(x, \cdot)$ is an FSM on $\mathscr{B}_n$ and $Q_n(x, X) = 1$.
(iii) for all $A \in \Sigma$ and $B \in \mathscr{B}_n$

$$\mu(A \cap B) = \int_A Q_n(x, B) d\mu_\Sigma(x) .$$

By the argument in [5] and Remark 1.3 there exists a μ-null set $N \in \Sigma$ such that for each $x \in X - N$, $Q_1(x, \cdot), Q_2(x, \cdot), \cdots$ is a uniformly bounded consistent sequence of FSM's on $\mathscr{B}_1, \mathscr{B}_2, \cdots$. Thus for each $x \in X - N$ there exists a unique FSM, Q_x, on $\mathscr{B}$ such that $Q_x(A) = Q_n(x, A)$ for all $A \in \mathscr{B}_n$ and $n = 1, 2, \cdots$. Define the function Q on $X \times \mathscr{B}$ by

$$Q(x, B) = \begin{cases} Q_x(B) & \text{if } B \in \mathscr{B} \text{ and } x \in X - N , \\ P(B) & \text{if } B \in \mathscr{B} \text{ and } x \in N , \end{cases}$$

where P is any fixed probability measure on $\mathscr{B}$. Now $|Q|(x, X) = \sup_n |Q_n|(x, X)$ if $x \in X - N$ and, consequently, Remark 1.3 implies that $x \to |Q|(x, X)$ is μ_Σ-integrable. That Q is an RCD of μ given Σ follows now as in [5].

COROLLARY 3.2. *The map* $x \to |Q|(x, B)$, *from* X *into* R, *is* μ_Σ-*integrable for all* $B \in \mathscr{B}$.

Proof. Let $\mathscr{F}$ be a countable field generating $\mathscr{B}$. Then $|Q|(x, B) = \sup \sum_{i=1}^{n} |Q(x, B_i)|$, where the sup is taken over all finite sequences $\{B_i\}$ of disjoint sets in $\mathscr{F}$ such that $B_i \subseteqq B$. The assertion now follows.

COROLLARY 3.3. *If Σ is countably generated, then there exists a μ-null set $N \in \Sigma$ such that for all $A \in \Sigma$ and $B \in \mathscr{B}$, $Q(x, AB) = I_A(x)Q(x, B)$ if $x \in X - N$.*

Proof. Clearly, if $A \in \Sigma$ and $B \in \mathscr{B}$, $Q(x, AB) = I_A(x)Q(x, B)$ except for x in a μ-null set in Σ depending on A and B. Now both Σ and $\mathscr{B}$ are countably generated and the proof is completed by a standard argument.

The next theorem concerns the Jordan decomposition of an RCD.

THEOREM 3.4. *Let $(X, \mathscr{B})$ be a standard measurable space and Σ a countably generated sub σ-field of $\mathscr{B}$.*

Let μ be an FSM *on $\mathscr{B}$ with an* RCD, *Q, given Σ. If $Q(x, \cdot) = Q^+(x, \cdot) - Q^-(x, \cdot)$ is the Jordan decomposition of $Q(x, \cdot)$, then there exists probability measures P_1 and P_2 on $\mathscr{B}$ with* RCPD's *Q_1 and Q_2 given Σ, respectively, and there exists a μ-null set $N \in \Sigma$ such that for all $B \in \mathscr{B}$*

$$Q(x, B) = Q^+(x, X)Q_1(x, B) - Q^-(x, X)Q_2(x, B)$$

if $x \in X - N$.

Proof. From $Q^+(x, B) = 1/2\{|Q|(x, B) + Q(x, B)\}$ and $Q^-(x, B) = 1/2\{|Q|(x, B) - Q(x, B)\}$ it follows that the maps $x \to Q^+(x, B)$ and $x \to Q^-(x, B)$ are μ_Σ-integrable for each $B \in \mathscr{B}$. There exists a μ-null set $N_1 \in \Sigma$ such that $Q(x, AB) = I_A(x)Q(x, B)$ for all $A \in \Sigma$ and $B \in \mathscr{B}$ if $x \in X - N_1$. Hence, $Q^+(x, AB) = I_A(x)Q^+(x, B)$ and $Q^-(x, AB) = I_A(x)Q^-(x, B)$ if $x \in X - N_1$.

Let P be any probability measure on $(X, \mathscr{B})$ such that μ and P are absolutely continuous with respect to each other and let Q' be an RCPD of P given Σ. We set $F_1 = \{x: Q^+(x, X) = 0\}$ and $F_2 = \{x: Q^-(x, X) = 0\}$ and define, for all $B \in \mathscr{B}$,

$$Q_1(x, B) = \begin{cases} \dfrac{Q^+(x, B)}{Q^+(x, X)} & \text{if} \quad x \in X - F_1\,, \\ Q'(x, B) & \text{if} \quad x \in F_1\,, \end{cases}$$

$$Q_2(x, B) = \begin{cases} \dfrac{Q^-(x, B)}{Q^-(x, X)} & \text{if } x \in X - F_2, \\ Q'(x, B) & \text{if } x \in F_2. \end{cases}$$

Let probability measures P_1 and P_2 on $\mathscr{B}$ be defined by $P_1(B) = \int Q_1(x, B)dP(x)$ and $P_2(B) = \int Q_2(x, B)dP(x)$. Clearly, Q_1 and Q_2 are RCPD's given Σ of P_1 and P_2 respectively. Moreover, for all $A \in \Sigma$ and $B \in \mathscr{B}$,

$$\mu(A \cap B) = \int_A [Q^+(x, X)Q_1(x, B) - Q^-(x, X)Q_2(x, B)]d\mu_\Sigma(x) .$$

The proof is complete.

4. Sequences of transition functions.

DEFINITION 4.1. Let $\{\mathscr{B}_n\}_0^\infty$ be an increasing sequence of σ-fields on the space X. For all $n > m \geqq 0$ let the functions $f_{m,n}: X \times \mathscr{B}_n \to R$ have the properties:

(i) For each $B \in \mathscr{B}_n$, $x \to f_{m,n}(x, B)$ is a $\mathscr{B}_m$-measurable map from X into R.

(ii) For each $x \in X$, $f_{m,n}(x, \cdot)$ is an FSM on $\mathscr{B}_n$ and $f_{m,n}(x, X) = 1$.

(iii) $\sup_{x \in X} |f_{m,n}|(x, X) < \infty$.

(iv) For all $x \in X$, $A \in \mathscr{B}_m$ and $B \in \mathscr{B}_n$, $f_{m,n}(x, AB) = I_A(x)f_{m,n}(x, B)$.

$f_{m,n}$ will be called a transition function from $(X, \mathscr{B}_m)$ to $(X, \mathscr{B}_n)$. For $f_{n-1,n}$ we write f_n.

LEMMA 4.2. *Let $\{f_n\}_1^\infty$ be a sequence of transition functions with respect to the σ-fields $\{\mathscr{B}_n\}_0^\infty$. For $0 \leqq n \leqq m$ define the functions $f_{m,n}$ on $X \times \mathscr{B}_n$ by $f_{m,n}(x, B) = I_B(x)$. Then the functions $f_{m,n}: X \times \mathscr{B}_n \to R$ inductively defined for $n > m \geqq 0$ by*

$$f_{m,n}(x, B) = \int f_n(x', B)f_{m,n-1}(x, dx')$$

are transition functions and if $\sup_x |f_n|(x, X) = \alpha_n$, *then* $\sup_x |f_{m,n}|(x, X) \leqq \alpha_{m+1} \cdots \alpha_n$. *Moreover, for $B \in \mathscr{B}_n$ and $n > m \geqq 0$,* $f_{m,n}(x, B) = f_{m,n+1}(x, B)$ *and*

$$f_{m,n}(x, B) = \int f_{m+1,n}(x', B)f_{m+1}(x, dx') .$$

Proof. This is straight forward and will be omitted. We only observe that for $n > m \geqq 0$ and $B \in \mathscr{B}_n$,

$$f_{m,n}(x, B) = \int \Big[\cdots \Big[\int f_n(x_{n-1}, B)f_{n-1}(x_{n-2}, dx_{n-1}) \Big] \cdots \Big] f_{m+1}(x, dx_{m+1}) .$$

DEFINITION 4.3. For $n = 0, 1, \cdots$ let $M_n = M(X, \mathscr{B}_n)$ be the set of FSM's on $(X, \mathscr{B}_n)$. With the usual operations and the total variation norm M_n is a Banach space. Let $\{f_n\}_1^\infty$ be transition functions as in Lemma 4.2. For $n = 1, 2 \cdots$ define the linear maps $L_n: M_{n-1} \to M_n$ by $L_n(\mu)(B) = \int f_n(x, B) d\mu(x)$. L_n is injective and $\| L_n \| \leqq \alpha_n$. For $n > m \geqq 0$ let $L_{m,n} = L_n L_{n-1} \cdots L_{m+1}$. Then $L_{m,n}(\mu)(B) = \int f_{m,n}(x, B) d\mu(x)$ and $\| L_{m,n} \| \leqq \alpha_{m+1} \cdots \alpha_n$.

THEOREM 4.4. *Let $(X, \mathscr{B})$ be a measurable space and $\{\mathscr{B}_n\}_0^\infty$ an increasing sequence of sub σ-fields of $\mathscr{B}$ as in Theorem 2.3 and satisfying the extension condition. Let $f_1, f_2, \cdots$ be a sequence of transition functions with respect to $\{\mathscr{B}_n\}_0^\infty$ such that $\alpha = \prod_1^\infty \sup_x |f_n|(x, X) < \infty$. Then for each $m = 0, 1, \cdots$ there exists a unique transition function g_m from $(X, \mathscr{B}_m)$ to $(X, \mathscr{B})$ such that $g_m(x, B) = f_{m,n}(x, B)$ for all $x \in X$, $B \in \mathscr{B}_n$ and $n > m$. If M is the Banach space of* FSM's *on $(X, \mathscr{B})$, then for each $m = 0, 1, \cdots$ there exists a unique injective linear map $T_m: M_m \to M$ with norm $\| T_m \| \leqq \alpha$ such that $T_m(\mu)(B) = L_{m,n}(\mu)(B)$ for all $\mu \in M_m$, $B \in \mathscr{B}_n$ and $n > m$. Moreover, g_m is an* RCD *of $T_m(\mu)$ given $\mathscr{B}_m$ for all $\mu \in M_m$.*

Proof. Obviously, $f_{m,m+1}(x, \cdot)$, $f_{m,m+2}(x, \cdot), \cdots$ is a uniformly bounded consistent sequence of FSM's on $\mathscr{B}_{m+1}, \mathscr{B}_{m+2}, \cdots$ for each x. There exists a unique FSM $g_m(x, \cdot)$ on $\mathscr{B}$ such that $g_m(x, B) = f_{m,n}(x, B)$ for all $B \in \mathscr{B}_n$ and $n > m$. Moreover, $\sup_x |g_m|(x, X) \leqq \alpha$, and g_m is a transition function from $(X, \mathscr{B}_m)$ to $(X, \mathscr{B})$.

For all $\mu \in M_m$, $B \in \mathscr{B}_n$ and $n > m \geqq 0$

$$(*) \qquad L_{m,n}(\mu)(B) = \int g_m(X, B) d\mu(X) .$$

Thus $\{L_{m,n}(\mu)\}_{n=m+1}^\infty$ is a consistent sequence of FSM's such that $\sup_{n>m} |L_{m,n}(\mu)|(X) \leqq \alpha |\mu|(X)$. There exists a unique FSM on $\mathscr{B}$ which we will denote by $T_m(\mu)$ such that $T_m(\mu)(B) = L_{m,n}(\mu)(B)$ for all $B \in \mathscr{B}_n$, $n > m$. It is easily seen that $T_m: M_m \to M$ is a one-to-one linear map with $\| T_m \| \leqq \alpha$. The last assertion follows from (*) and the above.

COROLLARY 4.5. *For $n = 0, 1, \cdots$ let μ_n denote the restriction to $\mathscr{B}_n$ of an element μ of M, then the map $P_n: M \to M$ defined by $P_n(\mu) = T_n(\mu_n)$ is a continuous linear projection with null space $\{\mu \in M: \mu_n = 0\}$. Moreover, $P_n(\mu)(B) = \int g_n(x, B) d\mu(x)$ for all $\mu \in M$ and $B \in \mathscr{B}$, and $P_n P_m = P_n$ for all $m \geqq n$.*

Part II. FSM's on the function spaces $C([0, \infty); S)$ and $D([0, \infty); S)$

In §§5 and 6 the patching Theorem on p. 367 of [6] is generalized.

5. Extension of a τ-basic family of FSM's.

NOTATIONS AND DEFINITIONS 5.1, Let S be a metric space. $C([0, \infty); S)$ will denote the space of all continuous functions on $[0, \infty)$ with values in S. $D([0, \infty); S)$ denotes the space of all S-valued functions on $[0, \infty)$ that are right-continuous and have left-hand limits. The symbol Ω will be used both for $C([0, \infty); S)$ and $D([0, \infty); S)$. For each $t \in [0, \infty)$ we define the coordinate projection $X_t: \Omega \to S$ by $X_t(\omega) = \omega(t)$ for all $\omega \in \Omega$. Instead of X_t we also write $X(t)$. The Borel σ-field of S will be denoted by $\mathscr{B}(S)$ or $\mathscr{S}$. If $0 \leqq a \leqq t \leqq b < \infty$, then $\mathscr{M}_b^a = \sigma(X_t, a \leqq t \leqq b)$ is the σ-field on Ω generated by the maps X_t, $a \leqq t \leqq b$. If $b = \infty$ we write $\mathscr{M}^a$ for $\sigma(X_t, t \geqq a)$. $\mathscr{C}_b^a(\mathscr{C}^a)$ is the semifield of subsets of Ω of the form $\{X_{t_1} \in \Gamma_1, \cdots, X_{t_n} \in \Gamma_n\}$, where $t_1, \cdots, t_n$ are points in $[a, b]$ $([a, \infty))$, $\Gamma_1, \cdots, \Gamma_n$ Borel sets in S and n any positive integer. Clearly, $\mathscr{M}_b^a = \sigma(\mathscr{C}_b^a)$ and $\mathscr{M}^a = \sigma(\mathscr{C}^a)$. $\mathscr{C}^\infty$ and $\mathscr{M}^\infty$ will stand for $\{\varnothing, \Omega\}$.

If $s \geqq 0$, then an s-Markov time or s-stopping time is a map $\tau: \Omega \to [s, \infty]$ such that $\{\tau \leqq t\} \in \mathscr{M}_t^s$ for all $t \geqq s$. $\mathscr{M}_\tau^s = \{A \in \mathscr{M}^s: A \cap \{\tau \leqq t\} \in \mathscr{M}_t^s$ for all $t \geqq s\}$ is the σ-field of sets in $\mathscr{M}^s$ prior to τ. If $\omega \in \Omega$, the atom of $\mathscr{M}_\tau^s$ containing ω is given by

$$A_\omega = \begin{cases} \{\omega' \in \Omega: X(t, \omega') = X(t, \omega), s \leqq t \leqq \tau(\omega)\} & \text{if } \tau(\omega) < \infty , \\ \{\omega' \in \Omega: X(t, \omega) = X(t, \omega), t \geqq s\} & \text{if } \tau(\omega) = \infty . \end{cases}$$

DEFINITION 5.2. A family of FSM's $\{\mu_\omega\}_{\omega \in \Omega}$ will be called a τ-basic family if it has the following properties:

(i) For each $\omega \in \Omega$, μ_ω is an FSM on $\mathscr{M}^{\tau(\omega)}$ and the map

$$\omega \longrightarrow \begin{cases} \mu_\omega(A) & \text{if } \omega \in \{\tau \leqq t\} \text{ and } A \in \mathscr{M}^t , \\ 0 & \text{if } \omega \in \{\tau > \tau\} , \end{cases}$$

from Ω into R, is $\mathscr{M}_t^s$-measurable for all $t \geqq s$ and all $A \in \mathscr{M}^t$.

(ii) For each $\omega \in \tilde{\Omega} = \{\tau < \infty\}$ the complement of

$$\Omega_\omega = \{\omega' \in \Omega: X(\tau(\omega), \omega') = X(\tau(\omega), \omega)\}$$

is a μ_ω-null set of $\mathscr{M}^{\tau(\omega)}$.

(iii) For $\omega \in \Omega - \tilde{\Omega}$ and $A \in \mathscr{M}^\infty$, $\mu_\omega(A) = I_A(\omega)$.

For each $\omega \in \Omega$ the set function μ'_ω is defined on $(\Omega, \mathscr{C}^s)$ as follows:

$$\mu'_\omega(\{X(t_0) \in \Gamma_0, \cdots, X(t_n) \in \Gamma_n\})$$
$$= \begin{cases} I_{\{X(t_i) \in \Gamma_i; i=0,\cdots,\mathfrak{l}\}}(\omega)\mu_\omega(\{X(t_i) \in \Gamma_i;\ i = \mathfrak{l}+1, \cdots, n\}) \\ \qquad \text{if } \omega \in \{t_\mathfrak{l} \leqq \tau < t_{\mathfrak{l}+1}\} \text{ and } \mathfrak{l} = 0, 1, \cdots, n-1\,, \\ I_{\{X(t_i) \in \Gamma_i; i=0,\cdots,n\}}(\omega)\mu_\omega(\Omega) \\ \qquad \text{if } \omega \in \{\tau \geqq t_n\}\,, \end{cases}$$

with $0 \leqq s = t_0 < t_1 < \cdots < t_n$, $n \geqq 1$ and $\Gamma_0, \cdots, \Gamma_n \in \mathscr{S}$.

LEMMA 5.3. *For each $\omega \in \Omega$, μ'_ω is a countably additive set function on the semifield $\mathscr{C}^s$.*

Proof. Fix $\omega_0 \in \Omega$ and put $\tau_0 = \tau(\omega_0)$. Let $C = \{X(t_i) \in \Gamma_i; i = 0, \cdots, n\}$ with $s = t_0 < t_1 < \cdots < t_n$ be a set in $\mathscr{C}^s$.

Suppose $\tau_0 \in [t_\mathfrak{l}, t_{\mathfrak{l}+1})$ where $0 \leqq \mathfrak{l} \leqq n-1$. We can write $C = AB$ with $A = \{X(t_i) \in \Gamma_i; i = 0, \cdots, \mathfrak{l}\}$ and $B = \{X(t_i) \in \Gamma_i; i = \mathfrak{l}+1, \cdots, n\}$. Now $A \in \mathscr{C}^s_{\tau_0}$ and $B \in \mathscr{C}^{\tau_0+}$ where $\mathscr{C}^{\tau_0+}$ is the collection of sets of the form $\{X(u_1) \in \Delta_1, \cdots, X(u_n) \in \Delta_n\}$ with $\tau_0 < u_1 < \cdots < u_n$, $n \geqq 1$ and $\Delta_1, \cdots, \Delta_n$ Borel sets of S. Clearly, $\mu'_{\omega_0}(C) = I_A(\omega_0)\mu_{\omega_0}(B)$.

If $\tau_0 \in [t_n, \infty)$, $C = AB$ with $A = C \in \mathscr{C}^s_{\tau_0}$ and $B = \Omega \in \mathscr{C}^{\tau_0+}$. If $\tau_0 = \infty$, $C = AB$ with $A = C \in \mathscr{C}^s_{\tau_0} = \mathscr{C}^s_\infty = \mathscr{C}^s$ and $B = \Omega \in \mathscr{C}^{\tau_0+} = \mathscr{C}^\infty$.

In both cases $\mu'_{\omega_0}(C) = I_A(\omega_0)\mu_{\omega_0}(B)$. To show countable additivity of μ'_{ω_0} on $\mathscr{C}^s$, let $C = \bigcup_{k=1}^\infty A_kB_k = AB \in \mathscr{C}^s$, where $A_1B_1, A_2B_2, \cdots$ are disjoint members of $\mathscr{C}^s$ with $A_k \in \mathscr{C}^s_{\tau_0}$, $B_k \in \mathscr{C}^{\tau_0+}$, $A \in \mathscr{C}^s_{\tau_0}$ and $B \in \mathscr{C}^{\tau_0+}$. Clearly, $\mu'_{\omega_0}(\bigcup_k A_kB_k) = I_A(\omega_0)\mu_{\omega_0}(B)$.

In the case that $\omega_0 \in \Omega - \tilde{\Omega}$ countable additivity is obvious. Now let $\omega_0 \in \tilde{\Omega}$ and note that $|\mu_{\omega_0}|(\Omega_0^c) = 0$, where $\Omega_0 = \Omega_{\omega_0}$. If $\omega_0 \in A \cap \tilde{\Omega}$, $\mu'_{\omega_0}(\bigcup_k A_kB_k) = \mu_{\omega_0}(B)$ and $\sum_k \mu'_{\omega_0}(A_kB_k) = \sum_k I_{A_k}(\omega_0)\mu_{\omega_0}(B_k)$. Let $J_0 = \{k: \omega_0 \in A_k\}$. Thus if $J_0 \neq \varnothing$, we have to show $\mu_{\omega_0}(B) = \sum_{k \in J_0}\mu_{\omega_0}(B_k)$ and if $J_0 = \varnothing$, $\mu_{\omega_0}(B) = 0$.

Let $J_0 \neq \varnothing$. We claim $\Omega_0B_k \cap \Omega_0B_{k'} = \varnothing$ if $k, k' \in J_0$ and $k = k'$. Suppose $\omega \in \Omega_0B_k \cap \Omega_0B_{k'}$. There exists an ω_1 such that

$$\omega_1 = \begin{cases} \omega_0 & \text{on } [s, \tau_0]\,, \\ \omega & \text{on } [\tau_0, \infty)\,. \end{cases}$$

Now $\omega_0 \in A_kA_{k'}$ and it follows that $\omega_1 \in \Omega_0A_kA_{k'}B_kB_{k'}$, a contradiction. By a similar kind of argument it is seen that $\Omega_0B = \bigcup_{k \in J_0}\Omega_0B_k$. It follows that $\mu_{\omega_0}(B) = \sum_{k \in J_0}\mu_{\omega_0}(B_k)$. The remainder of the proof is along similar lines.

LEMMA 5.4. *For each $\omega \in \Omega$, μ'_ω has a unique extension to an* FSM *on $\mathscr{M}^s$. The total variation on Ω of this* FSM *does not exceed $|\mu_\omega|(\Omega)$.*

Proof. Fix $\omega_0 \in \Omega$ and put $\tau_0 = \tau(\omega_0)$. μ'_{ω_0} has a unique countably additive extension to the field $\mathscr{F}$ generated by $\mathscr{C}^s$. Denoting this by μ'_{ω_0}, we have

$$|\mu'_{\omega_0,\mathscr{F}}|(\Omega) = \sup \sum_{k=1}^{n} |\mu'_{\omega_0}(E_k)| ,$$

where the sup is taken over all finite sequences $\{E_k\}$ of disjoint sets in $\mathscr{C}^s$. We can write $E_k = A_k B_k$ with $A_k \in \mathscr{C}^s_{\tau_0}$ and $B_k \in \mathscr{C}^{\tau_0+}$ for $k = 1, \cdots, n$. It follows that $|\mu'_{\omega_0,\mathscr{F}}|(\Omega) \leqq |\mu_{\omega_0}|(\Omega)$. Thus μ'_{ω_0} has a unique extension to an FSM on $\mathscr{M}^s$.

REMARK 5.5. From now on μ'_ω will denote the extension to $\mathscr{M}^s$ and $\{\mu'_\omega\}_{\omega\in\Omega}$ will be called the family of FSM's associated with the τ-basic family $\{\mu_\omega\}_{\omega\in\Omega}$. In Lemma 5.9 we will see that $\mu'_{\omega,\mathscr{M}^{\tau(\omega)}} = \mu_\omega$ and it follows that $|\mu'_\omega|(\Omega) = |\mu_\omega|(\Omega)$.

LEMMA 5.6. *The map $\omega \to \mu'_\omega(B)$, from Ω into R, is $\mathscr{M}^s_\tau$-measurable for all $B \in \mathscr{M}^s$.*

Proof. It is easy to see that $\omega \to \mu'_\omega(C)$ is $\mathscr{M}^s_\tau$-measurable for each $C \in \mathscr{C}^s$. By the monotone class theorem the proof is completed.

LEMMA 5.7. *Let $\omega \in \Omega$. Then for all $A \in \mathscr{M}^s_\tau$ and $B \in \mathscr{M}^s$,*

$$\mu'_\omega(AB) = I_A(\omega)\mu'_\omega(B) .$$

Proof. Fix $\omega_0 \in \Omega$ and call $\tau(\omega_0) = \tau_0$. If $A = \{X(u) \in \Gamma\}$ with $s \leqq u \leqq \tau_0$ ($s \leqq u < \infty$ if $\tau_0 = \infty$) and $\Gamma \in \mathscr{S}$, then $\mu'_{\omega_0}(AB) = I_A(\omega_0)\mu'_{\omega_0}(B)$ for all $B \in \mathscr{M}^s$. By induction it follows that $\mu'_{\omega_0}(AB) = I_A(\omega_0)\mu'_{\omega_0}(B)$ for all $A \in \mathscr{C}^s_{\tau_0}$ and $B \in \mathscr{M}^s$. By a monotone class argument, $\mu'_{\omega_0}(AB) = I_A(\omega_0)\mu'_{\omega_0}(B)$ for all $A \in \mathscr{M}^s_{\tau_0}$ and $B \in \mathscr{M}^s$.

Now let $A \in \mathscr{M}^s_\tau$ and $B \in \mathscr{M}^s$. Since both $A \cap \{\tau = \tau_0\}$ and $\{\tau = \tau_0\}$ are in $\mathscr{M}^s_{\tau_0}$ it is easily seen that $\mu'_{\omega_0}(AB) = I_A(\omega_0)\mu'_{\omega_0}(B)$.

LEMMA 5.8. *For each $\omega \in \Omega$, $\mu'_\omega(B) = \mu_\omega(B)$ for all $B \in \mathscr{M}^{\tau(\omega)}$.*

Proof. The assertion is obvious if $\omega \in \Omega - \tilde{\Omega}$. Now let $\omega_0 \in \tilde{\Omega}$. If $B = \{X(t_i) \in \Gamma_i; i = 1, \cdots, m\}$ with $\tau_0 = \tau(\omega_0) < t_1 < \cdots < t_m$ and $\Gamma_1, \cdots, \Gamma_m \in \mathscr{S}$, then $\mu'_{\omega_0}(B) = \mu_{\omega_0}(B)$ and thus the assertion is true for all $B \in \sigma(\mathscr{C}^{\tau_0+})$. But $\sigma(\mathscr{C}^{\tau_0+}) = \mathscr{M}^{\tau_0}$ and the lemma is proved.

6. The patching theorem. In this section S will be a separable metric space. The σ-field $\mathscr{M}^s$ then is countably generated.

LEMMA 6.1. *Let $\{\mu_\omega\}_{\omega\in\Omega}$ be a τ-basic family of* FSM's *and $\{\mu'_\omega\}_{\omega\in\Omega}$ its associated family. For each $B \in \mathscr{M}^s$, the map $\omega \to |\mu'_\omega|(B)$, from*

Ω into R, is $\mathscr{M}_\tau^s$-measurable.

Proof. See proof of Corollary 3.2.

THEOREM 6.2 (*Patching theorem*). *Let μ be an* FSM *on $(\Omega, \mathscr{M}_\tau^s)$ and $\{\mu_\omega\}_{\omega\in\Omega}$ a τ-basic family of* FSM's. *Assume, moreover, that for all $\omega \in \Omega$, $\mu_\omega \neq 0$ and that the map $\omega \to |\mu_\omega|(\Omega)/\mu_\omega(\Omega)$, from Ω into the extended real line, is μ-integrable. If $\{\mu'_\omega\}_{\omega\in\Omega}$ is the family of* FSM's *associated with $\{\mu_\omega\}_{\omega\in\Omega}$, then the intgral*

$$\mu'(B) = \int \frac{\mu'_\omega(B)}{\mu'_\omega(\Omega)} d\mu(\omega) ,$$

with $B \in \mathscr{M}^s$, defines an FSM *μ' on $\mathscr{M}^s$. μ' has the properties:*

(i) *$\mu'(A) = \mu(A)$ for all $A \in \mathscr{M}_\tau^s$* i.e., *$\mu = \mu'_\tau$, the restriction of μ' to $\mathscr{M}_\tau^s$.*

(ii) $$\mu'(A \cap B) = \int_A \frac{\mu'_\omega(B)}{\mu'_\omega(\Omega)} d\mu'_\tau(\omega)$$

for all $A \in \mathscr{M}_\tau^s$ and $B \in \mathscr{M}^s$.

Proof. If $\{B_k\}_1^\infty$ is any sequence of disjoint sets in $\mathscr{M}^s$, then

$$\sum_1^\infty \int \left| \frac{\mu'_\omega(B_k)}{\mu'_\omega(\Omega)} \right| d|\mu|(\omega) \leqq \int \frac{|\mu_\omega|(\Omega)}{|\mu_\omega(\Omega)|} d|\mu|(\omega) .$$

By the dominated convergence theorem μ' is countably additive and thus an FSM on $\mathscr{M}^s$. By Lemma 5.7 $\mu'(A) = \mu(A)$ for $A \in \mathscr{M}_\tau^s$ and also (ii) follows.

THEOREM 6.3. *Let μ and μ' be as in Theorem* 6.2. *There exists an* RCD, *Q', of μ' given $\mathscr{M}_\tau^s$. Q' is unique in the sense of Lemma* 1.5. *Furthermore, if $N = \{\omega \in \Omega: \mu_\omega(\Omega) = 0\}$, then for all $\omega \in \Omega - N$:*

(i) *$Q'(\omega, B) = \mu_\omega(B)/\mu_\omega(\Omega)$ for all $B \in \mathscr{M}^{\tau(\omega)}$.*

(ii) *$Q'(\omega, A_\omega) = 1$ and $|Q'|(\omega, A_\omega^c) = 0$, where A_ω is the atom of $\mathscr{M}_\tau^s$ containing ω.*

Proof. Define a function $Q': \Omega \times \mathscr{M}^s \to R$ by

$$Q'(\omega, B) = \begin{cases} \dfrac{\mu'_\omega(B)}{\mu'_\omega(\Omega)} & \text{if } \omega \notin N \text{ and } B \in \mathscr{M}^s , \\ P(B) & \text{if } \omega \in N \text{ and } B \in \mathscr{M}^s , \end{cases}$$

where P is a fixed but arbitrary probability measure on $\mathscr{M}^s$. Clearly, Q' is an RCD of μ' given $\mathscr{M}_\tau^s$. As $\mathscr{M}^s$ is countably generated the second assertion follows. By Lemma 5.8, $Q'(\omega, B) =$

$\mu_\omega(B)/\mu_\omega(\Omega)$ for all $B \in \mathscr{M}^{\tau(\omega)}$ if $\omega \notin N$. The last part is an immediate consequence of Lemma 5.7.

THEOREM 6.4. *The* FSM μ' *defined in Theorem* 6.2 *in terms of* $\{\mu_\omega\}_{\omega \in \Omega}$ *and* μ *is the only* FSM *on* $\mathscr{M}^s$ *such that*

(i) $\mu'(A) = \mu(A)$ *for all* $A \in \mathscr{M}_\tau^s$,

(ii) μ' *has an* RCD, Q', *given* $\mathscr{M}_\tau^s$,

(iii) $Q'(\omega, B) = \mu_\omega(B)/\mu_\omega(\Omega)$ *for all* $B \in \mathscr{M}^{\tau(\omega)}$ *except for* ω *in a* μ'*-null set of* $\mathscr{M}_\tau^s$.

Proof. Suppose $\bar{\mu}$ is another FSM on $\mathscr{M}^s$ satisfying (i), (ii) and (iii), and let $\bar{Q}$ be its RCD given $\mathscr{M}_\tau^s$. Let $B = \{X(t_i) \in \Gamma_i; i = 0, \cdots, n\}$ with $s = t_0 < t_1 < \cdots < t_n$ and $\Gamma_0, \cdots, \Gamma_n \in \mathscr{S}$. Put $A_\iota = \{t_\iota \leqq \tau < t_{\iota+1}\}$ for $\iota = 0, 1, \cdots, n-1$ and $A_n = \{\tau \geqq t_n\}$. Clearly, $A_\iota \in \mathscr{M}_\tau^s$ for $\iota = 0, 1, \cdots, n$. Now it follows from Lemmas 5.7 and 5.8, and (i) and (iii) that

$$\begin{aligned}\mu'(B) &= \sum_{\iota=0}^{n} \int_{A_\iota} \frac{\mu'_\omega(B)}{\mu'_\omega(\Omega)} d\mu'_\tau(\omega) \\ &= \sum_{\iota=0}^{n} \bar{\mu}(A_\iota \cap B) = \bar{\mu}(B) .\end{aligned}$$

Thus μ' and $\bar{\mu}$ agree on $\mathscr{C}^s$ and hence on $\mathscr{M}^s$.

7. The Jordan decomposition of μ'_ω.

Let $\{\mu_\omega\}_{\omega \in \Omega}$ be a τ-basic family of FSM's and let $\mu_\omega = \mu_\omega^+ - \mu_\omega^-$ be the Jordan decomposition of μ_ω. For each $\omega \in \tilde{\Omega}$, $\mu_\omega^+(\Omega_\omega^c) = \mu_\omega^-(\Omega_\omega^c) = 0$ and we define μ'_ω, $\mu_\omega^{+\prime}$ and $\mu_\omega^{-\prime}$ on $(\Omega, \mathscr{C}^s)$ as in 5.2 and extend to $(\Omega, \mathscr{M}^s)$. For $\omega \in \Omega - \tilde{\Omega}$ put $\mu'_\omega(A) = \mu_\omega^{+\prime}(A) = I_A(\omega)$ and $\mu_\omega^{-\prime}(A) = 0$ for all $A \in \mathscr{M}^s$. We have the following.

LEMMA 7.1. *If* $\mu'_\omega = \mu_\omega^{\prime +} - \mu_\omega^{\prime -}$ *is the Jordan decomposition of* μ'_ω, *then for all* $\omega \in \Omega$ *and* $A \in \mathscr{M}^s$

$$\mu_\omega^{\prime +}(A) = \mu_\omega^{+\prime}(A) \quad \text{and} \quad \mu_\omega^{\prime -}(A) = \mu_\omega^{-\prime}(A) .$$

Proof. There exists for each $\omega \in \Omega$ a set $D_\omega \in \mathscr{M}^{\tau(\omega)}$ such that $\mu_\omega^+(A) = \mu_\omega(A \cap D_\omega)$ and $\mu_\omega^-(A) = -\mu_\omega(A \cap D_\omega^c)$ for all $A \in \mathscr{M}^{\tau(\omega)}$. Let $A \in \mathscr{C}^s$. It is not hard to see that $\mu'_\omega(A \cap D_\omega) = \mu_\omega^{+\prime}(A)$ and $\mu'_\omega(A \cap D_\omega^c) = -\mu_\omega^{-\prime}(A)$. It follows that $\mu'_\omega(A \cap D_\omega) = \mu_\omega^{+\prime}(A)$ and $\mu'_\omega(A \cap D_\omega^c) = -\mu_\omega^{-\prime}(A)$ for all $A \in \mathscr{M}^s$. Hence $\mu_\omega^{\prime +} = \mu_\omega^{+\prime}$ and $\mu_\omega^{\prime -} = \mu_\omega^{-\prime}$.

LEMMA 7.2. *Let* $\{\mu'_\omega\}_{\omega \in \Omega}$ *be the family of* FSM's *on* $(\Omega, \mathscr{M}^s)$ *associated with* $\{\mu_\omega\}_{\omega \in \Omega}$. *If* S *is a separable metric space, then the maps* $\omega \to \mu_\omega^{\prime +}(B)$ *and* $\omega \to \mu_\omega^{\prime -}(B)$, *from* Ω *into* R, *are* $\mathscr{M}_\tau^s$*-measura-*

ble for each $B \in \mathscr{M}^s$.

Proof. For all $B \in \mathscr{M}^s$ and $\omega \in \Omega$ we can write

$$\mu_\omega'^{+}(B) = \frac{1}{2}\{|\mu'_\omega|(B) + \mu'_\omega(B)\} ,$$

$$\mu_\omega'^{-}(B) = \frac{1}{2}\{|\mu'_\omega|(B) - \mu'_\omega(B)\} .$$

Now apply Lemmas 5.6 and 6.1.

REMARK 7.3. It is now obvious that under the conditions of Theorem 6.2 we can write for all $A \in \mathscr{M}_\tau^s$ and $B \in \mathscr{M}^s$:

$$\int_A \frac{\mu'_\omega(B)}{\mu'_\omega(\Omega)} d\mu(\omega) = \int_A \frac{\mu_\omega^{+\prime}(B)}{\mu'_\omega(\Omega)} d\mu(\omega) - \int_A \frac{\mu_\omega^{-\prime}(B)}{\mu'_\omega(\Omega)} d\mu(\omega) .$$

8. The reverse of patching. From now on S will be a complete separable metric space. Let $0 \leqq a \leqq b < \infty$ and let $C_b^a = C([a, b]; S)$ be the space of continuous functions on $[a, b]$ with values in S. It is well known that C_b^a with the uniform metric is a complete separable metric space and that the Borel σ-field on C_b^a coincides with the σ-field generated by the coordinate projections, e.g., see [9]. Let $D_b^a = D([a, b]; S)$ be the space of functions on $[a, b]$ that are right-continuous and have left-hand limits, with values in S. If $S = R$, it is shown in [1] that D_b^a with the Skorohod topology is a separable completely metrizable space. Its Borel σ-field coincides with the σ-field generated by the coordinate projections. If S is any complete separable metric space the same can be shown, see for instance [8]. That $(\Omega, \mathscr{M}^s)$ is standard is then an easy application of Theorem 2.3.

If τ is an s-Markov time as in 5.1, it is shown in [6] that $(\Omega, \mathscr{M}_\tau^s)$ is standard in the case that $\Omega = C([0, \infty]; R^d)$ (see also [7]). Along similar lines it can be shown that $(\Omega, \mathscr{M}_\tau^s)$ is standard if $\Omega = C([0, \infty); S)$ or $\Omega = D\ ([0, \infty); S)$ with S any complete separable metric space. In fact $\mathscr{M}_\tau^s$ is generated by the collection of sets of the form $\{X(t_1 \wedge \tau) \in \Gamma_1, \cdots, X(t_n \wedge \tau) \in \Gamma_n\}$ with $t_1, \cdots, t_n$ points in $[s, \infty)$, $\Gamma_1, \cdots, \Gamma_n \in \mathscr{S}$ and $n \geqq 1$. Also $\Omega(\tau) = \{\omega\colon X(t, \omega) = X(t \wedge \tau(\omega), \omega)$ for all $t \geqq 0\}$ is a set in $\mathscr{M}^s$ (compare p. 395 of [6]). If $\Psi_\tau\colon \Omega \to \Omega$ is defined by $\Psi_\tau(\omega)(t) = \omega(t \wedge \tau(\omega))$ for all $\omega \in \Omega$ and $t \geqq 0$, then $\Psi_\tau^{-1}(\mathscr{M}^s \cap \Omega(\tau)) = \mathscr{M}_\tau^s$. For details see [8].

We have the following extension of Theorem 1.3.4 on p. 34 of [7].

THEOREM 8.1. *Let* μ *be an* FSM *on* $(\Omega, \mathscr{M}^s)$. *The condition* *that a $\mu_{\mathscr{M}_\tau^s}$-null set of Ω is a μ-null set, is necessary and sufficient for the existence of an* RCD *of μ given $\mathscr{M}_\tau^s$. If the condition is satisfied there exists an* RCD, Q, *of μ given $\mathscr{M}_\tau^s$ such that*

$$Q(\omega, AB) = I_A(\omega)Q(\omega, B)$$

for all $\omega \in \Omega$, $A \in \mathscr{M}_\tau^s$ and $B \in \mathscr{M}^s$.

Proof. The first assertion follows from Theorem 3.1. Assume the condition is satisfied and let Q' be an RCD of μ given $\mathscr{M}_\tau^s$. By Corollary 3.3, there exists a μ-null set $N \in \mathscr{M}_\tau^s$ such that for all $A \in \mathscr{M}_\tau^s$ and $B \in \mathscr{M}^s$, $Q'(\omega, AB) = I_A(\omega)Q'(\omega, B)$ if $\omega \notin N$. Define Q on $\Omega \times \mathscr{M}^s$ by

$$Q(\omega, B) = \begin{cases} Q'(\omega, B) & \text{for } B \in \mathscr{M}^s \text{ and } \omega \notin N\,, \\ I_B(\Psi_\tau(\omega)) & \text{for } B \in \mathscr{M}^s \text{ and } \omega \in N\,. \end{cases}$$

Clearly, Q has the required properties.

THEOREM 8.2. *Let μ' be an* FSM *on $(\Omega, \mathscr{M}^s)$ and let μ be its restriction to $\mathscr{M}_\tau^s$. Assume that a μ-null set of Ω is also a μ'-null set. Then there exists a τ-basic family of* FSM's $\{\mu_\omega\}_{\omega\in\Omega}$ *with the properties* (i) $\mu_\omega(\Omega) = 1$ *for all $\omega \in \Omega$,* (ii) *the map $\omega \to |\mu_\omega|(\Omega)$ is μ-integrable and* (iii) *μ' coincides with the* FSM *obtained by patching μ and $\{\mu_\omega\}_{\omega\in\Omega}$ together as in Theorem* 6.2.

Proof. Let Q' be an RCD of μ' given $\mathscr{M}_\tau^s$ such that $Q'(\omega, AB) = I_A(\omega)Q'(\omega, B)$ for all $\omega \in \Omega$, $A \in \mathscr{M}_\tau^s$ and $B \in \mathscr{M}^s$. Define the family of FSM's $\{\mu_\omega\}_{\omega\in\Omega}$ as follows: For each $\omega \in \Omega$, $\mu_\omega(A) = Q'(\omega, A)$ for all $A \in \mathscr{M}^{\tau(\omega)}$. $\{\mu_\omega\}_{\omega\in\Omega}$ is a τ-basic family. Let $\{\mu'_\omega\}_{\omega\in\Omega}$ be the family of associated FSM's on $(\Omega, \mathscr{M}^s)$. It is easily seen that $\mu'_\omega(B) = Q'(\omega, B)$ for all $\omega \in \Omega$ and $B \in \mathscr{M}^s$. It follows that $\{\mu_\omega\}_{\omega\in\Omega}$ has properties (ii) and (iii).

9. Patching countably many times. Let $\tau_0 \leqq \tau_1 \leqq \tau_2 \leqq \cdots$ be a sequence of s-Markov times, from Ω into $[s, \infty]$, with respect to the family of σ-fields $\{\mathscr{M}_t^s, t \geqq s\}$. For each $n = 0, 1, 2, \cdots$ let $\{\mu_{n\omega}\}_{\omega\in\Omega}$ be a τ_n-basic family of FSM's and $\{\mu'_{n\omega}\}_{\omega\in\Omega}$ its associated family of FSM's on $(\Omega, \mathscr{M}^s)$. Assume, moreover, that

$$\alpha = \prod_0^\infty \sup_{\omega\in\Omega} \frac{|\mu_{n\omega}|(\Omega)}{|\mu_{n\omega}(\Omega)|} < \infty\ .$$

For $n = 0, 1, 2, \cdots$ define the transition functions f_{n+1}, from $(\Omega, \mathscr{M}_{\tau_n}^s)$ to $(\Omega, \mathscr{M}_{\tau_{n+1}}^s)$, by $f_{n+1}(\omega, B) = \mu'_{n\omega}(B)/\mu'_{n\omega}(\Omega)$. Let $M_n = M(\Omega, \mathscr{M}_{\tau_n}^s)$ be the Banach space of all FSM's on $(\Omega, \mathscr{M}_{\tau_n}^s)$. If $\mathscr{M}^* =$

$\sigma(\bigcup_0^\infty \mathscr{M}_{\tau_n}^s)$, let M^* be the Banach space of FSM's on $(\Omega, \mathscr{M}^*)$. With $f_{m,n}$ defined as in 4.2 and $L_{m,n}$ as in 4.3 we can apply Theorem 4.4.

ACKNOWLEDGMENTS. It is a pleasure to thank R. B. Kirk and P. Feinsilver for helpful conversations.

REFERENCES

1. P. Billingsley, *Convergence of Probability Measures*, John Wiley, New York, 1968.
2. N. Dunford and J. T. Schwartz, *Linear Operators*, John Wiley (Interscience), New York, Part I.
3. K. J. Hochberg, *A signed measure on path space related to Wiener measure*, Ann. Probability, **6** (1978), 433–458.
4. V. Yu. Krylov, *Some properties of the distribution corresponding to the equation* $\partial u/\partial t = (-1)^{q+1}\partial^{2q}u/\partial x^{2q}$, Soviet Math. Dokl., **1** (960), 760–763.
5. K. R. Parthasarathy, *Probability Measures on Metric Spaces*, Academic Press, New York, 1967.
6. D. W. Strook and S. R. S. Varadhan, *Diffusion processes with continuous coefficients, I*, Comm. Pure Appl. Math., XXII (1969), 345–400.
7. ———, *Multidimensional Diffusion Processes*, Springer Verlag, Berlin-Heidelberg-New York, 1979.
8. A. J. Van Haagen, *Finite signed measures on function spaces*, dissertation for the Ph. D degree, Southern Illinois University at Carbondale, (1979).
9. W. Whitt, *Weak Convergence of Probability Measures on the Function Space* $C[0, \infty)$, Ann. Math. Stat., **41** (1970), 939–944.

Received April 23, 1980.

19 ZEEBURGERPAD
1018 AH AMSTERDAM, THE NETHERLANDS

Current address: Southern Illinois University
Carbondale, IL 62901