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RATIOS OF INTERPOLATING BLASCHKE PRODUCTS

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Every unimodular function on the unit circle can be uniformly approximated by ratios of interpolating Blaschke products. As a consequence, we show that points of the maximal ideal space of H^∞ can be separated by interpolating Blaschke products.

1. Introduction. Let Δ denote the open unit disc in $\boldsymbol{C}$ and let $H^\infty(\Delta)$ be the Banach algebra of functions bounded and analytic on Δ. A sequence of points $\{z_j\}$ in Δ is called an interpolating sequence if for every bounded sequence $\{\alpha_j\}$ of complex numbers there is a function $F \in H^\infty(\Delta)$ such that

$$F(z_j) = \alpha_j\ , \qquad j = 1, 2, \cdots .$$

Lennart Carleson [1] has shown that $\{z_j\}$ is an interpolating sequence if and only if

$$\inf_j \prod_{\substack{k \\ k \neq j}} \rho(z_j, z_k) > 0\ .$$

Here ρ denotes the pseudo hyperbolic metric; $\rho(w, z) = |(w-z)/(1-\bar{w}z)|$ for $w, z \in \Delta$.

For an arc I on the unit circle $\boldsymbol{T}$ let $|I|$ denote the length of I and let $S(I)$ denote the shadow region of I, $S(I) = \{z \in \Delta: (z/|z|) \in I, 1 - |I| < |z| < 1\}$. A positive measure μ on Δ is called a Carleson measure if

$$\sup_I \frac{1}{|I|}\mu(S(I)) = \|\mu\|_C < \infty\ ,$$

where the above supremum is taken over all arcs I of $\boldsymbol{T}$. There is also a characterization of interpolating sequences in terms of Carleson measures. A sequence $\{z_j\}$ is an interpolating sequence if and only if

$$\text{(i)} \qquad \inf_{\substack{j \neq k \\ j, k}} \rho(z_j, z_k) > 0$$

and

$$\text{(ii)} \qquad \sum (1 - |z_j|)\delta_{z_j} \quad \text{is a Carleson measure}\ ,$$

where δ_z denotes the Dirac δ measure at z.

A Blaschke product with simple zeros lying on an interpolating sequence is called an interpolating Blaschke product. The purpose of this paper is to study ratios of interpolating Blaschke products and then use the information gathered to prove a theorem about the maximal ideal space of H^∞. A complex valued function u on $\boldsymbol{T}$ is called unimodular if $|u(e^{i\theta})| = 1$ for almost all θ. Our first result asserts that every unimodular function on $\boldsymbol{T}$ can be uniformly approximated by ratios of interpolating Blaschke products. (Recall that every Blaschke product has nontangential boundary values satisfying $|B(e^{i\theta})| = 1$ almost everywhere.)

THEOREM 1. *If u is a unimodular function on $\boldsymbol{T}$ and $\varepsilon > 0$, there are interpolating Blaschke products B_1 and B_2 such that*

$$\left\|u - \frac{B_1}{B_2}\right\|_\infty < \varepsilon .$$

Theorem 1 is a refined version of a theorem of Douglas and Rudin [5]. They proved Theorem 1 with B_1 and B_2 (not necessarily interpolating) Blaschke products. Theorem 1 will be proved in §2.

Our next result answers a question of John Garnett and Donald Marshall.

THEOREM 2. *Interpolating Blaschke products separate the points of the maximal ideal space of $H^\infty(\Delta)$.*

Theorem 2 follows rather easily from Theorem 1 and known results. The strongest result previously known is Ziskind's theorem [13]: If m_1 and m_2 are homomorphisms in the maximal ideal space of $H^\infty(\Delta)$ with m_1 lying on the Silov boundary and m_2 lying off the Silov boundary, there is an interpolating Blaschke B such that $m_1(B) \neq m_2(B)$. Theorem 2 will be proved in §3.

Section 4 will be devoted to some comments and open questions.

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2. Proof of Theorem 1. To prove Theorem 1 it is sufficient to show that for any unimodular function u there are interpolating Blaschke products B_1 and B_2 such that

$$\left\|\arg\left(\frac{UB_2}{B_1}\right)\right\|_\infty < \varepsilon$$

on $\boldsymbol{T}$. Here we define the function arg by

$$\arg\left(e^{i(\theta+2\pi k)}\right) = \theta , \qquad -\pi < \theta \leqq \pi , \qquad k \in \boldsymbol{Z} .$$

Our proof of Theorem 1 uses an approximation argument due to A. M. Davie which appears in [4]. The idea of this approximation argument is quite simple. For the rest of this section $\varepsilon > 0$ will denote a small positive constant; $\varepsilon \leqq 1/10$ will do. Suppose α is a real number, $\varepsilon/2 < \alpha \leqq \pi$, and set $\beta = 4\pi\varepsilon^3/\alpha$. Let $r = 1 - \eta$, where $0 < \eta < 1$ is a small number, and let A_1 be the Blaschke product with simple zeros at the points

$$\left\{re^{i\theta_j}\colon \theta_j = j\beta\eta - \varepsilon^3\eta,\ 0 \leqq j \leqq \left[\frac{2\pi}{\beta\eta}\right] - 1\right\}.$$

Also let A_2 be the Blaschke product with simple zeros at the points

$$\left\{re^{i\theta_k}\colon \theta_k = k\beta\eta - \varepsilon^3\eta,\ 0 \leqq k \leqq \left[\frac{2\pi}{\beta\eta}\right] - 1\right\}.$$

Then one can check by hand that

$$\left|\alpha - \arg\frac{A_1(x)}{A_2(x)}\right| < \frac{\varepsilon}{2}, \qquad x \in \boldsymbol{T},$$

if A_1 and A_2 are first multiplied by suitable unimodular constants. Suppose now that I is a subarc of $\boldsymbol{T}$ and η is very small with respect to $|I|$. Let B_1 and B_2 be the Blaschke products having simple zeros at $\{z\colon A_1(z) = 0,\ z/|z| \in I\}$ and $\{z\colon A_2(z) = 0,\ z/|z| \in I\}$ respectively. Then $\arg B_1(x)/B_2(x) \approx \alpha$ for most x which lie in I and $\arg B_1(x)/B_2(x) \approx 0$ for most x which lie off I. More precisely, the following lemma holds.

LEMMA 2.1. *Suppose I is a subarc of $\boldsymbol{T}$ and α is a real number, $|\alpha| \leqq \pi$. Then if $\varepsilon, \delta, \eta > 0$ are three positive numbers, there are finite Blaschke products A_1 and A_2 with simple zeros such that*

$$\left|\left\{x \in I\colon \left|\alpha - \arg\frac{A_1(x)}{A_2(x)}\right| \geqq \frac{\varepsilon}{2}\right\}\right| < \eta \tag{2.1}$$

$$\left|\left\{x \notin I\colon \left|\arg\frac{A_1(x)}{A_2(x)}\right| \geqq \delta\right\}\right| < \eta \tag{2.2}$$

$$\frac{z}{|z|} \in I \text{ if } z \text{ is a zero of } A_1 \text{ or } A_2 \tag{2.3}$$

$$\rho(z_1, z_2) \geqq \varepsilon^3 \text{ if } z_1 \text{ and } z_2 \text{ are two zeros } A_1\ (A_2). \tag{2.4}$$

Furthermore, there is a positive constant $\gamma_0 = \gamma_0(\varepsilon, \delta, \eta, I) > 0$ such that whenever $0 < \gamma < \gamma_0$ we can choose A_1 and A_2 as above so that $1 - |z| = \gamma$ for all zeros z of A_1 and A_2.

Proof. This is essentially proved in Lemma 12.3 of [4]. Only (2.2) requires verification. Let C_1 and C_2 be the two Blaschke factors given by $C_1(z) = c_1 \cdot (z - z)_1/(1 - \bar{z})_1 z$, $C_2(z) = c_2 \cdot (z - z)_2/(1 - \bar{z})_2 z$, where

$$z_1 = (1 - \gamma)e^{i(\theta_0 - \varepsilon^3\gamma)}\,, \qquad z_2 = (1 - \gamma)e^{i(\theta_0 + \varepsilon^3\gamma)}\,,$$

where $|c_1| = |c_2| = 1$ are chosen so that $C_1(e^{i(\theta_0+\pi)}) = C_2(e^{i(\theta_0+\pi)})$. Then

$$\left|\arg \frac{C_2(x)}{C_2(x)}\right| \leqq 10\varepsilon^3\Big(1 + \frac{|x - e^{i\theta_0}|}{\gamma^2}\Big)^{-1}\,. \tag{2.5}$$

Since the ratio A_1/A_2 given by the construction in Lemma 12.3 of [6] is a product of factors like C_1/C_2 or C_2/C_1, inequality (2.2) follows from (2.3), (2.4), and (2.5), if all zeros z of A_1 and A_2 have modulus $|z| = 1 - \gamma$ where γ is sufficiently small.

Repeating the argument of Lemma 2.1 N times, we obtain the following lemma.

LEMMA 2.2. *Suppose $I_1, I_2, \cdots, I_N$ are a finite number of disjoint subarcs of T and $\alpha_1, \cdots, \alpha_N$ are N real numbers, $|\alpha_j| \leqq \pi$, $1 \leqq j \leqq N$. Then if $\varepsilon, \delta, \eta > 0$ are three positive numbers, there are Blaschke products A_1 and A_2 with simple zeros such that*

$$\left|\left\{x \in I_j\colon \left|\alpha_j - \arg\frac{A_1(x)}{A_2(x)}\right| \geqq \frac{\varepsilon}{2}\right\}\right| < \frac{\eta}{4N}\,, \qquad 1 \leqq j \leqq N\,. \tag{2.6}$$

$$\left|\left\{x \notin \bigcup_{j=1}^{N} I_j\colon \left|\arg\frac{A_1(x)}{A_2(x)}\right| \geqq \delta\right\}\right| < \frac{\eta}{4}\,, \tag{2.7}$$

$$\frac{z}{|z|} \in \bigcup_{j=1}^{N} I_j \text{ if } z \text{ is a zero of } A_1 \text{ or } A_2\,. \tag{2.8}$$

$$\rho(z_1, z_2) \geqq \varepsilon^3 \text{ if } z_1 \text{ and } z_2 \text{ are two zeros of } A_1(A_2). \tag{2.9}$$

Furthermore, there is a positive constant $\gamma_0 = \gamma_0(\varepsilon, \delta, \eta, I_1, \cdots, I_N) > 0$ such that whenever $0 < \gamma < \gamma_0$ we can choose A_1 and A_2 as above so that $1 - |z| = \gamma$ for all zeros z of A_1 and A_2.

We now construct the Blaschke products B_1 and B_2 of Theorem 1. Let u be the unimodular function in the statement of the theorem and at stage one put $v_1 = \arg u$. Then $\|v_1\|_\infty \leqq \pi$. Let $\eta_{-1} = 2\pi$, $\eta_0 = 1$, and let $E_1 = \{x\colon |v_1(x)| \geqq \varepsilon\}$. Find $\delta_1 > 0$ such that

$$|\{x\colon \varepsilon - \delta_1 \leqq |v_1(x)| < \varepsilon\}| < \frac{\eta_0}{4} = \frac{1}{4}\,.$$

Find also a finite collection of disjoint subarcs of T, $I_1^1, I_2^1, \cdots, I_N^1$, and N_1 real numbers $\alpha_1, \alpha_2, \cdots, \alpha_{N_1}$, $|\alpha_j| \leqq \pi$, $1 \leqq j \leqq N_1$, such that

$$\left|\left\{x: \left|v_1(x)\chi_{E_1}(x) - \sum_{j=1}^{N_1} \alpha_j \chi_{X_j^1}(x)\right| \geqq \frac{\varepsilon}{4}\right\}\right| < \frac{\eta_0}{4} = \frac{1}{4} .$$

Applying Lemma 2.2 with $\delta = \delta_1$ and $\eta = \eta_0 = 1$, we can find finite Blaschke products $A_{1,1}$ and $A_{2,1}$ with simple zeros such that

$$\left|\left\{x: \left|v_1(x) - \arg \frac{A_{1,1}(x)}{A_{2,1}(x)}\right| \geqq \varepsilon\right\}\right| < \eta_0 = 1 .$$

If z is a zero of $A_{1,1}$ or $A_{2,1}$ then $z/|z| \in \bigcup_{j=1}^{N_1} I_j'$. If z_1 and z_2 are two zeros of A_1 (A_2) then

$$\rho(z_1, z_2) \geqq \varepsilon^3 .$$

Furthermore we may choose $A_{1,1}$ and $A_{2,1}$ so that $1 - |z| = \eta_1$ whenever z is a zero of $A_{1,1}$ or $A_{2,1}$, and we may choose $\eta_1 \leqq \eta_0/4 = 1/4$. The Blaschke product $A_{1,1}$, satisfies

$$\sum_{A_{1,1}(z)=0} (1 - |z|) \leqq \eta_{-1}\varepsilon^{-3} = 2\pi\varepsilon^{-3} .$$

$A_{2,1}$ satisfies a similar inequality.

At stage two set

$$v_2(x) = \arg \left\{u(x)\frac{A_{2,1}(x)}{A_{1,1}(x)}\right\}$$

and let $E_2 = \{x: |v_2(x)| \geqq \varepsilon\}$. Find $\delta_2 > 0$ such that

$$|\{x: \varepsilon - \delta_2 \leqq |v_2(x)| < \varepsilon\}| < \frac{\eta_1}{4} .$$

Find also a finite collection of disjoint subarcs of T, $I_1^2, I_2^2, \cdots, I_{N_2}^2$, and N_2 real numbers $\alpha_1, \alpha_2, \cdots, \alpha_{N_2}$, $|\alpha_j| \leqq \pi$, $1 \leqq j \leqq N_2$, such that

$$\left|\left\{x: \left|v_2(x)\chi_{E_2}(x) - \sum_{j=1}^{N_2} \alpha_j \chi_{I_j^2}(x)\right| \geqq \frac{\varepsilon}{4}\right\}\right| < \frac{\eta_1}{4} .$$

Since $|\{x: |v_2(x)| \geqq \varepsilon\}| < \eta_0 = 1$, we can pick our intervals so that

$$\sum_{j=1}^{N_2} |I_j^2| < \eta_0 = 1 .$$

Applying Lemma 2.2 with $\delta = \delta_2$ and $\eta = \eta_1$, we can find Blaschke products $A_{1,2}$ and $A_{2,2}$ with simple zeros such that

$$\left|\left\{x: \left|v_2(x) - \arg \frac{A_{1,2}(x)}{A_{2,2}(x)}\right| \geqq \varepsilon\right\}\right| < \eta_1 .$$

If z is a zero of $A_{1,2}$ or $A_{2,2}$ then $z/|z| \in \bigcup_{j=1}^{N_2} I_j^2$. If z_1 and z_2 are two zeros of $A_{1,2}$ $(A_{2,2})$ then

$$\rho(z_1, z_2) \geqq \varepsilon^3 .$$

Therefore $A_{1,2}$ satisfies

$$\sum_{A_{1,2}(z)=0} (1 - |z|) \leqq \eta_0 \varepsilon^{-3} = \varepsilon^{-3} ,$$

and $A_{2,2}$ satisfies a similar inequality. We may choose $A_{1,2}$ and $A_{2,2}$ so that $1 - |z| = \eta_2$ whenever z is a zero of $A_{1,2}$ or $A_{2,2}$, and we may choose $\eta_2 \leqq \eta_1/4$.

Suppose by induction that we have found Blaschke products with simple zeros $A_{1,1}, A_{1,2}, \cdots, A_{1,n-1}$ and $A_{2,1}, A_{2,2}, \cdots, A_{2,n-1}$ having the following properties. If z is a zero of $A_{1,k}$ or $A_{2,k}$ then

$$1 - |z| = \eta_k \leqq \frac{\eta_{k-1}}{4} , \qquad 1 \leqq k \leqq n - 1 . \tag{2.10}$$

If z_1 and z_2 are two zeros of $A_{1,k}$ $(A_{2,k})$ then

$$\rho(z_1, z_2) \geqq \varepsilon^3 . \tag{2.11}$$

The zeros of $A_{1,k}$ satisfy

$$\sum_{A_{1,k}(z)=0} (1 - |z|) < \eta_{k-2} \varepsilon^{-3} , \qquad 1 \leqq k \leqq n - 1 , \tag{2.12}$$

and $A_{2,k}$ satisfies a similar inequality. If $B_{1,k} = \prod_{j=1}^{k} A_{1,j}$, $B_{2,k} = \prod_{j=1}^{k} A_{2,j}$ and $v_{k+1} = \arg \{u \cdot (B_{2,k}/B_{1,k})\}$ then

$$|\{x: |v_{k+1}(x)| \geqq \varepsilon\}| < \eta_{k-1} , \qquad 1 \leqq k \leqq n - 1 . \tag{2.13}$$

At stage n let $E_n = \{x: |v_n(x)| \geqq \varepsilon\}$. Find $\delta_n > 0$ such that

$$|\{x: \varepsilon - \delta_n \leqq |v_n(x)| < \varepsilon\}| < \frac{\eta_{n-1}}{4} .$$

Find also a finite collection of disjoint subarcs of T, $I_1^n, I_2^n, \cdots, I_{N_n}^n$, and N_n real numbers $\alpha_1, \alpha_2, \cdots, \alpha_{N_n}$, $|\alpha_j| \leqq \pi$, $1 \leqq j \leqq N_n$, such that

$$\left| \left\{x: \left| v_n(x) \chi_{E_n}(x) - \sum_{j=1}^{N_n} \alpha_j \chi_{I_j^n}(x) \right| \geqq \frac{\varepsilon}{4} \right\} \right| < \frac{\eta_{n-1}}{4} .$$

Since $|E_n| < \eta_{n-2}$ by the induction hypothesis, we can pick our intervals so that

$$\sum_{j=1}^{N_2} |I_j^n| < \eta_{n-2} .$$

Applying Lemma 2.2 with $\delta = \delta_n$ and $\eta = \eta_{n-1}$, we can find Blaschke products $A_{1,n}$ and $A_{2,n}$ with simple zeros such that

$$\left|\left\{x\colon \left|v_n(x) - \arg\frac{A_{1,n}(x)}{A_{2,n}(x)}\right| \geqq \varepsilon\right\}\right| < \eta_{n-1} .$$

If x is a zero of $A_{1,n}$ or $A_{2,n}$ then $z/|z| \in \bigcup_{j=1}^{N_n} I_j^n$. If z_1 and z_2 are two zeros of $A_{1,n}$ ($A_{2,n}$) then

$$\rho(z_1, z_2) \geqq \varepsilon^3 .$$

Therefore $A_{1,n}$ satisfies

$$\sum_{A_{1,n}(z)=0} (1 - |z|) \leqq \eta_{n-2}\varepsilon^{-3} ,$$

and $A_{2,n}$ satisfies a similar inequality. We may choose $A_{1,n}$ and $A_{2,n}$ so that $1 - |z| = \eta_n$ whenever z is a zero of $A_{1,n}$ or $A_{2,n}$, and we may choose $\eta_n \leqq \eta_{n-1}/4$.

Working formally for a moment, put $B_1 = \prod_{k=1}^{\infty} A_{1,k}$, $B_2 = \prod_{k=1}^{\infty} A_{2,k}$. Let $\{z_{1,j}\} = \{z\colon A_{1,k}(z) = 0$ for some $k \geqq 1\}$ and let $\{z_{2,j}\} = \{z\colon A_{2,k}(z) = 0$ for some $k \geqq 1\}$. Notice that by (2.10) every point in $\{z_{1,j}\}$ or $\{z_{2,j}\}$ appears with multiplicity one. Suppose we know that $\{z_{1,j}\}$ and $\{z_{2,j}\}$ are interpolating sequences. Then $\{z_{1,j}\}$ and $\{z_{2,j}\}$ are Blaschke sequences and the partial products in the definition of B_1 and B_2 converge to interpolating Blaschke products. Inequality (2.13) then shows that

$$\left\|\arg u - \arg\frac{B_1}{B_2}\right\|_\infty \leqq \varepsilon .$$

We now verify that $\{z_{1,j}\}$ is an interpolating sequence. The proof for $\{z_{2,j}\}$ is exactly the same. If $k \neq j$, inequalities (2.10) and (2.11) show

$$\rho(z_{1,j}, z_{1,k}) \geqq \varepsilon^3 .$$

To show that $\mu = \sum (1 - |z_{1,j}|)\delta_{z_{1,j}}$ is a Carleson measure, it is only necessary to verify the Carleson condition for arcs I of length $2\eta_n$, $n \geqq 1$. So suppose $|I| = 2\eta_n$. Then by (2.11),

$$\sum_{\substack{z_{1,j} \in \tilde{I} \\ (1-|z_{1,j}|)=\eta_n}} (1 - |z_{1,j}|) \leqq 2\varepsilon^{-3}|I|$$

and

$$\sum_{\substack{z_{1,j} \in \tilde{I} \\ (1-|z_{1,j}|)=\eta_{n+1}}} (1 - |z_{1,j}|) \leqq 2\varepsilon^{-3}|I| .$$

If $k \geqq 2$, then by (2.10) and (2.12),

$$\sum_{\substack{z_{1,j} \in \tilde{I} \\ (1-|z_{1,j}|)=\eta_{n+k}}} (1 - |z_{1,j}|) \leqq \eta_{n+k-2}\varepsilon^{-3}$$

$$\leq \eta_n \varepsilon^{-3} 4^{2-k}$$
$$= 8\varepsilon^{-3} 4^{-k} |I| .$$

Summing on $k \geqq 2$ we obtain

$$\sum_{z_{1,j} \in \tilde{I}} (1 - |z_{1,j}|) \leqq 5\varepsilon^{-3} |I| ,$$

so $\sum (1 - |z_{1,j}|)\delta_{z_{1,j}}$ is a Carleson measure. The proof of Theorem 1 is complete.

3. Proof of Theorem 2. The maximal ideal space of H^∞, $\mathscr{M}_{H^\infty}$, is divided into three disjoint parts, which we denote by $ш$, G, and H. $ш$ is the Šilov boundary of H^∞. See [5] for properties of $ш$. G is the set of all homomorphisms which lie in the closure (in the topology of $\mathscr{M}_{H^\infty}$) of an interpolating sequence in the disk. H is the set of homomorphisms which lie in neither $ш$ nor G. To study H we will have need of L. Carleson's corona theorem [2]:

If $m \in \mathscr{M}_{H^\infty}$ then there is a net $\{z_\alpha\}$ of points lying in the unit disk which converge to m in the topology of $\mathscr{M}_{H^\infty}$.

Let $m_1 \neq m_2$ be two homomorphisms in $\mathscr{M}_{H^\infty}$. To separate m_1 and m_2 by an interpolating Blaschke product there are four cases we must treat.

Case I. Either $m_1 \in G$ or $m_2 \in G$. Let us suppose $m_1 \in G$. Then there is an interpolating sequence $\{z_j\}$ such that m_1 is in the closure of the points $\{z_j\}$, but m_2 is not. If B is the interpolating Blaschke product with simple zeros at the points $\{z_j\}$, then $m_1(B) = 0$ and $m_2(B) \neq 0$.

Case II. $m_1 \in ш$ and $m_2 \notin ш$. This case has been treated previously by Ziskind [13]. Theorem 1 can also be used to treat this case. There is a Blaschke product B such that $m_1(B) = 1$ and $m_2(B) = 0$. (See e.g., [7].) Let B_1 and B_2 be two interpolating Blaschke products such that

$$\left\| B - \frac{B_1}{B_2} \right\|_\infty = \| BB_2 - B_1 \|_\infty < \frac{1}{2} .$$

Then $|m_2(B_1)| = |m_2(B_1 - BB_2)| < 1/2$. On the other hand, $|m_1(B_1)| = 1$ because $m_1 \in ш$.

Case III. $m_1, m_2 \in H$. By theorem of Hoffman [7], every point of G is a one-point Gleason part. Thus for every $\varepsilon > 0$ we can find $f_\varepsilon \in H^\infty$, $\|f_\varepsilon\|_\infty = 1$, such that $m_1(f_\varepsilon) = 0$ and $|m_2(f_\varepsilon)| > 1 - \varepsilon$. The unit ball of H^∞ is the norm closed convex hull of the Blaschke products [10], so we can find an inner function u_ε such that $m_1(u_\varepsilon) = 0$ and $|m_2(u)| > 1 - \varepsilon$. We now invoke a theorem of Ziskind [13].

THEOREM. (*Ziskind*) *Let u be an inner function. There are universal constants $0 < \varepsilon_0 < 1$, c_1, and c_2, and there is an interpolating Blaschke product B such that*

(3.1) *if* $|u(z)| \leqq \dfrac{1}{10}$ *then* $|B(z)| \leqq \dfrac{1}{10}$.

(3.2) *if* $B(z) = 0$ *then* $|u(z)| \leqq 1 - \varepsilon_0$.

(3.3) $\rho(z_1, z_2) \geqq c_1$ *whenever* z_1 *and* z_2 *are two zeros of* B.

(3.4) $\sum_{\substack{B(z_j)=0 \\ z_j \in \tilde{I}}} (1 - |z_j|) \leqq c_2 |I|$ *for all subarcs* I *of* $\boldsymbol{T}$.

For each $\varepsilon > 0$ find B_ε as in Ziskind's theorem, corresponding to the inner function u_ε. For a set $E \subset \Delta$ define $E^* \subset \boldsymbol{T}$ by

$E^* = \{e^{i\theta}\colon re^{i\theta_0} \in E$ for some r and θ_0, $|\theta - \theta_0| \leqq 1 - r\}$. E^* is the nontangential projection of E onto $\boldsymbol{T}$.

Now fix $\varepsilon > 0$. There are nets $\{w_\alpha\}$ and $\{z_\beta\}$ of points of Δ such that

$$w_\alpha \longrightarrow m_1$$

and

$$z_\beta \longrightarrow m_2 .$$

Inequality (3.1) shows $|m_1(B_\varepsilon)| \leqq 1/10$. By taking a subnet we can assume that $|u_\varepsilon(z_\beta)| > 1 - \varepsilon$ for all β. Let τ_β be a Mobius transformation which sends 0 to z_β, and let $v_{\varepsilon,\beta} = u_\varepsilon \circ \tau_\beta$. Then $|v_{\varepsilon,\beta}(0)| > 1 - \varepsilon$. Let $E_{\varepsilon,\beta} = \{z\colon |v_{\varepsilon,\beta}(z)| \leqq 1 - \varepsilon_0\}$, where ε_0 is the constant in Ziskind's theorem. Then

$$|E^*_{\varepsilon,\beta}| \leqq c_3(\varepsilon)$$

where

$$c_3(\varepsilon) \xrightarrow[\varepsilon \to 0]{} 0 .$$

This is essentially proved in §4 of [11]. Thus by (3.2)-(3.4),

$$|B_\varepsilon \circ \tau_\beta(0)| = |B_\varepsilon(z_\beta)| \geqq 1 - c_4(\varepsilon)$$

where

$$c_4(\varepsilon) \xrightarrow[\varepsilon \to 0]{} 0 .$$

Picking ε so small that $c_4(\varepsilon) \leqq 1/2$ we see that $|m_2(B)| \geqq 1/2$.

Case IV. $m_1, m_2 \in ш$. Let u be an inner function such that $m_1(u) = 1$, $m_2(u) = -1$. By Theorem 1 there are interpolating Blaschke products B_1 and B_2 so that $\|u - B_1/B_2\|_\infty \leqq 1/4$. Then

$$|m_1(B_2u) - m_1(B_1)| = |m_1(B_2) - m_1(B_1)| \leqq \frac{1}{4}$$

and

$$|m_2(B_2u) - m_2(B_1)| = |m_2(B_2) + m_2(B_1)| \leqq \frac{1}{4}.$$

If $m_1(B_1) = m_2(B_1)$ the above inequalities show $m_1(B_2) \neq m_2(B_2)$. (Since $m_1, m_2 \in ш$, $|m_1(v)| = |m_2(v)| = 1$ for all inner functions v.)

REMARKS 4. We conclude with some remarks and questions. It is possible that Theorem 1 could be derived from the Douglas-Rudin theorem but out attempts at this have been unsuccessful. In particular, a positive answer to the following question would suffice.

Question 1. If B_0 is a Blaschke product and $\varepsilon > 0$, is there an interpolating Blaschke product B_1 with $\|B_0 - B_1\|_\infty < \varepsilon$? One might hope that for a given B_0 there is a complex number α, $|\alpha| < \varepsilon$, such that $B_0 + \alpha/1 + \bar{\alpha}B_0$ is an interpolating Blaschke product. Unfortunately, this is not the case. Kahane [9] and Piranian [12] have shown that there is a Blaschke product B such that

$$|B'(z)| = o(1 - |z|),$$

while B has infinitely many zeros. On the other hand, if B is an interpolating Blaschke product with infinitely many zeros,

$$|B'(z)| = O(1 - |z|).$$

Theorem 2 shows that the uniform algebra generated by interpolating Blaschke products (call it J) is large in some sense. A natural question is just how large is J?

Question 2. Does $J = H^\infty$?

We do not even know if J and H^∞ have the same maximal ideal space. Note that by Marshall's theorem [10], an affirmative answer to Question 1 would give an affirmative answer to Question 2. By using Theorem 1 and the machinery in [10] one can reduce Question 2 to a problem concerning interpolation of bounded sequences by interpolating Blaschke products. An indication that J might be equal to H^∞ comes from the Chang-Marshall theorem [3], [11].

Finally, we note that BMO can be related to ratios of interpolating Blaschke products through the results of [8]. Perhaps arguments using BMO could be used to answer Questions 1 and 2.

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