PACIFIC JOURNAL OF MATHEMATICS
Vol. 95, No. 2, 1981

DENSITIES AND SUMMABILITY

A. R. FREEDMAN AND J. J. SEMBER

The ordinary asymptotic density of a set A of positive integers is $\nu(A)=\lim_{n\to\infty}A(n)/n$, where $A(n)$ is the cardinality of the set $A\cap\{1,2,\cdots,n\}$. It is known that the space of bounded strongly Cesàro summable sequences are just those bounded sequences that converge (in the ordinary sense) after the removal of a suitable collection of terms, the indices of which form a set A for which $\nu(A)=0$. In this paper we introduce a general concept of density and then examine the relationship, suggested by the above observation, between these densities and the strong convengence fields of various summability methods. These include all nonnegative regular matrix methods as well as the famous nonmatrix method called almost convergence.

The characterization of the bounded strongly Cesàro summable sequences mentioned above is significant in ergodic theory, where it relates to the study of weakly mixing transformations ([4], p. 38; [7], pp. 40–41).

The concept of a lower asymptotic density is presented axiomatically in § 2. Certain essential properties of these densities are proved and the "natural density" associated with the lower density is defined. The natural density has some of the properties of a measure but, in particular, is not a countably additive function. Of interest, therefore, are certain additivity properties (we call them (AP) and (APO)), valid for some natural densities, that are approximations to countable additivity.

Section 3 contains examples of densities. Of particular interest are those generated by nonnegative regular matrices, and another called uniform density.

In § 4 we investigate sequence spaces associated with a density. One such space is the space ω_δ of "nearly convergent" sequences (Definition 4.2) and another is the strong summability field $|c_s|$ of a summability method S that is "related" to the density in the sense of Definition 4.9. Whether or not the (APO) property holds for the density turns out to be crucial in the comparison of the sequence spaces ω_δ and $|c_s|$.

We use the following notation: The set of positive integers will be denoted by I. For A, $B\subseteq I$, we write $A\sim B$ (A is asymptotically equal to B) if the symmetric difference $A\Delta B$ is finite. For two sets A and B, the set-theoretic difference is denoted by $A\backslash B=\{x\colon x\in A,\ x\notin B\}$. Let $\varnothing$ denote the empty set. Sequences of real numbers will alternately be denoted by x, (x_i), or $(x_1, x_2, \cdots)$. The coordinatewise product of two sequences x, y will be denoted by $x \cdot y = (x_i y_i)$. For a sequence x, we let $|x| = (|x_i|)$, and if x is a sequence and l is some real number, we write $x - l = (x_i - l)$. We let ω denote the linear space of all real-valued sequences, and m, c, and c_0 will, as usual, denote the subspaces of ω consisting of the bounded sequences, the convergent sequences, and the sequences convergent to zero. Finally, if $M = (a_{nk})$ is an infinite matrix and $x = (x_i)$ is any sequence, the product $M \cdot x$ will denote the sequence (y_i), if it exists, where $y_i = \sum_{j=1}^{\infty} a_{ij} x_j$.

2. Densities and the additivity property. A function δ, defined for all sets of natural numbers and taking values in the closed interval $[0,1]$, will be called a *lower asymptotic density* (or just a *density*) if the following four axioms hold:

(D.1) if $A \sim B$, then $\delta(A) = \delta(B)$;
(D.2) if $A \cap B = \varnothing$, then $\delta(A) + \delta(B) \leqq \delta(A \cup B)$;
(D.3) for all A, B, $\delta(A) + \delta(B) \leqq 1 + \delta(A \cap B)$;
(D.4) $\delta(I) = 1$.

If δ is any density, we define $\bar{\delta}$, the *upper density associated with* δ, by

$$\bar{\delta}(A) = 1 - \delta(I \backslash A)$$

for any set of natural numbers A.

The first proposition lists the essential properties of δ and $\bar{\delta}$. The proofs are elementary and left to the reader.

PROPOSITION 2.1. *Let δ be a lower asymptotic density and $\bar{\delta}$ its associated upper density. For sets A, B of natural numbers, we have*

(i) $A \subseteqq B \Rightarrow \delta(A) \leqq \delta(B)$;
(ii) $A \subseteqq B \Rightarrow \bar{\delta}(A) \leqq \bar{\delta}(B)$;
(iii) *for all* A, B, $\bar{\delta}(A) + \bar{\delta}(B) \geqq \bar{\delta}(A \cup B)$;
(iv) $\delta(\varnothing) = \bar{\delta}(\varnothing) = 0$;
(v) $\bar{\delta}(I) = 1$;
(vi) $A \sim B \Rightarrow \bar{\delta}(A) = \bar{\delta}(B)$;
(vii) $\delta(A) \leqq \bar{\delta}(A)$.

We will say that a set $A \subseteqq I$ has *natural density with respect to* δ in case $\delta(A) = \bar{\delta}(A)$. We define

$$\eta_\delta = \{A : \delta(A) = \bar{\delta}(A)\} .$$

For $A \in \eta_\delta$, let $\nu_\delta(A) = \delta(A)$ (the *natural density of* A). In this paper we are mainly interested in sets A with natural density zero. Note that $A \in \eta_\delta$ and $\nu_\delta(A) = 0$ if and only if $\bar{\delta}(A) = 0$. Let

$$\eta_\delta^0 = \{A: \bar{\delta}(A) = 0\} .$$

The basic facts concerning ν_δ, η_δ, and η_δ^0 are contained in Propositions 2.2 and 2.3. We again omit the proofs.

PROPOSITION 2.2.
(i) *If* $A \sim I$, *then* $A \in \eta_\delta$ *and* $\nu_\delta(A) = 1$.
(ii) *If* $A \sim \varnothing$ (i.e., *if* A *is finite*), *then* $A \in \eta_\delta^0$.

PROPOSITION 2.3.
(i) ν_δ *is finitely additive*, i.e., *if* A, $B \in \eta_\delta$ *and* $A \cap B = \varnothing$, *then* $A \cup B \in \eta_\delta$ *and*

$$\nu_\delta(A \cup B) = \nu_\delta(A) + \nu_\delta(B) .$$

(ii) *If* $A_1, A_2, \cdots, A_n \in \eta_\delta^0$, *then* $\bigcup_{i=1}^n A_i \in \eta_\delta^0$.
(iii) *If* $A \in \eta_\delta$, *then* $(I/A) \in \eta_\delta$ *and* $\nu_\delta(I \backslash A) = 1 - \nu_\delta(A)$.
(iv) *If* $A \in \eta_\delta$ *and* $A \sim B$, *then* $B \in \eta_\delta$ *and* $\nu_\delta(A) = \nu_\delta(B)$.

A simple example shows that ν_δ is never countably additive: Taking $A_i = \{i\}$, $i = 1, 2, \cdots$, we have $A_i \in \eta_\delta$, $i = 1, 2, \cdots$ and $A_i \cap A_j = \varnothing$ $(i \neq j)$, but $\bigcup_{i=1}^\infty A_i = I$ and $\nu_\delta(I) = 1 \neq 0 = \sum_{i=1}^\infty \nu_\delta(A_i)$. However, for some densities, a similar property holds which we shall call the additivity property. This property of densities has been studied in other settings by Buck [1] and Freedman [2]. From its statement it is apparent that it is an approximation to countable additivity for ν_δ.

ADDITIVITY PROPERTY (AP). If $A_i \in \eta_\delta$, $i = 1, 2, \cdots$, and if $A_i \cap A_j = \varnothing$ $(i \neq j)$, then there exist sets B_i, $i = 1, 2, \cdots$, such that $B_i \sim A_i$, $i = 1, 2, \cdots$, $\bigcup_{i=1}^\infty B_i \in \eta_\delta$ and $\nu_\delta(\bigcup_{i=1}^\infty B_i) = \sum_{i=1}^\infty \nu_\delta(B_i)$.

In this paper we shall only need to consider a weaker property, namely, the additivity property for sets of zero natural density.

ADDITIVITY PROPERTY FOR SETS OF ZERO NATURAL DENSITY (APO). If $A_i \in \eta_\delta^0$, $i = 1, 2, \cdots$, and if $A_i \cap A_j = \varnothing$ $(i \neq j)$, then there exist sets B_i, $i = 1, 2, \cdots$ such that $B_i \sim A_i$, $i = 1, 2, \cdots$ and $\bigcup_{i=1}^\infty B_i \in \eta_\delta^0$.

If the condition that the sets A_i are disjoint is removed from (APO), we get an apparently stronger property (APO′). However, we can prove

PROPOSITION 2.4. *The properties* (APO) *and* (APO′) *are equivalent.*

Proof. Obviously (APO′) implies (APO). Now suppose (APO) holds and let (A_i) be any sequence of sets in η_δ^0. Define a pairwise disjoint sequence (A_i') in the usual way:

$$A_1' = A_1,\ A_{i+1}' = A_{i+1} \setminus \bigcup_{j=1}^{i} A_j \ .$$

Noting that $A_i' \in \eta_\delta^0$ we know, by (APO), that there exist sets $B_i' \sim A_i'$ such that $\bigcup_{i=1}^\infty B_i' \in \eta_\delta^0$. Letting $B_i = \bigcup_{j=1}^i B_j'$, it is easily seen that $B_i \sim A_i$ and $\bigcup_{i=1}^\infty B_i = \bigcup_{i=1}^\infty B_i'$.

The fact that (AP) implies (APO) is obvious. Whether or not the reverse implication holds is an open question.

3. Examples of densities and the (APO).

The term "asymptotic density" is often used for the function

$$d(A) = \liminf_{n\to\infty} \frac{A(n)}{n} \,,$$

where $A(n)$ is the number of elements in $A \cap \{1, 2, \cdots, n\}$. If χ_A denotes the characteristic sequence of A (thus χ_A is a sequence of 0's and 1's), and if C_1 denotes the Cesàro matrix

$$C_1 = \begin{pmatrix} 1 & 0 & 0 \cdots & & \\ \frac{1}{2} & \frac{1}{2} & 0 \cdots & & \\ \vdots & & & & \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 \cdots \\ \vdots & \vdots & & & \end{pmatrix}$$

then $A(n)/n$ is the nth term of the sequence $C_1 \cdot \chi_A$. Thus

$$d(A) = \liminf_{n\to\infty} (C_1 \cdot \chi_A)_n \ .$$

This function satisfies axioms (D.1)-(D.4).

This example suggests that there may be a general way to produce a density from a summability method. We now show that, for any nonnegative regular matrix, there is a natural way to do this.

PROPOSITION 3.1. *Let M be a nonnegative regular matrix and let δ_M be defined by*

$$\delta_M(A) = \liminf_{n\to\infty} (M \cdot \chi_A)_n \ .$$

Then δ_M *is a density* (i.e., *satisfies* (D.1)-(D.4)) *and, furthermore,*

$$\bar{\delta}_M(A) = \limsup_{n\to\infty} (M\cdot \chi_A)_n .$$

Proof. For convenience we write δ for δ_M.

(D.1). If $A \sim B$ there exists a positive integer N such that $\chi_A(j) = \chi_B(j)$ except, possibly, for $1 \leqq j \leqq N$. Thus

$$\begin{aligned}|(M\cdot\chi_A)_n - (M\cdot\chi_B)_n| &= \left|\sum_{j=1}^{\infty} a_{nj}\chi_A(j) - \sum_{j=1}^{\infty} a_{nj}\chi_B(j)\right| \\ &\leqq \sum_{j=1}^{N} a_{nj}|\chi_A(j) - \chi_B(j)| \leqq \sum_{j=1}^{N} a_{nj} \longrightarrow 0(n\to\infty) .\end{aligned}$$

It follows that $\liminf_{n\to\infty}(M\cdot\chi_A)_n = \liminf_n(M\cdot\chi_B)_n$ and so $\delta(A) = \delta(B)$.

(D.2) If $A \cap B = \varnothing$, then $\chi_{A\cup B} = \chi_A + \chi_B$. Hence

$$\begin{aligned}\delta(A\cup B) &= \liminf_{n\to\infty} (M\cdot\chi_{A\cup B})_n = \liminf_{n\to\infty} (M\cdot\chi_A + M\cdot\chi_B)_n \\ &\geqq \liminf_{n\to\infty} (M\cdot\chi_A)_n + \liminf_{n\to\infty} (M\cdot\chi_B) \\ &= \delta(A) + \delta(B) .\end{aligned}$$

(D.4) $\delta(I) = \liminf_{n\to\infty}(M\cdot\chi_I)_n = \liminf_{n\to\infty} \sum_{j=1}^{\infty} a_{nj} = 1$.

Before proving (D.3) we show that the formula for $\bar{\delta}(A)$ holds. First note that $\chi_{I\setminus A} = 1 - \chi_A$. Then, letting $\bar{1} = (1, 1, 1, \cdots)$,

$$\begin{aligned}\bar{\delta}(A) &= 1 - (\delta I\setminus A) = 1 - \liminf_{n\to\infty} (M\cdot\chi_{I\setminus A})_n \\ &= \limsup_{n\to\infty} (1 - M\cdot\chi_{I\setminus A})_n = \limsup_{n\to\infty} (1 - M\cdot\bar{1} + M\cdot\chi_A)_n \\ &= \limsup_{n\to\infty} (M\cdot\chi_A)_n .\end{aligned}$$

The last equality holds since $\lim_{n\to\infty}(1 - M\cdot\bar{1})_n = 0$.

(D.3) Since $\chi_{A\cap B} = \chi_A + \chi_B - \chi_{A\cup B}$, we can write

$$\begin{aligned}1 + \delta(A\cap B) &= 1 + \liminf_{n\to\infty} (M\cdot\chi_{A\cap B})_n \\ &\geqq 1 + \liminf_{n\to\infty} (M\cdot\chi_A)_n + \liminf_{n\to\infty} (M\cdot\chi_B)_n \\ &\qquad - \limsup_{n\to\infty}(M\cdot\chi_{A\cup B})_n \\ &= 1 + \delta(A) + \delta(B) - \bar{\delta}(A\cup B) = \delta(A) + \delta(B) \\ &\qquad + \delta(I\setminus(A\cup B)) \geqq \delta(A) + \delta(B) .\end{aligned}$$

One simple, but interesting, example is the density δ_J obtained from the identity matrix J. In this case $\delta_J(A) = 0$ if $I\setminus A$ is infinite, and otherwise $\delta_J(A) = 1$. Also, $\eta^0_{\delta_J}$ consists of just the finite subsets of the natural numbers and η_{δ_J} consists of the finite sets together with the sets A for which $A \sim I$.

We now show that the (APO) holds for any density obtained from a nonnegative regular matrix as in Proposition 3.1. In fact, the (AP) holds, but will not be proved here.

PROPOSITION 3.2. *Let M be a nonnegative regular matrix and let $\delta = \delta_M$ be as in Proposition 3.1. Then if (A_i) is a disjoint sequence of sets in η_δ^0, there exists a sequence (B_i) of sets such that $B_i \sim A_i$, $i = 1, 2, \cdots$, and $\bigcup_{i=1}^\infty B_i \in \eta_\delta^0$.*

Proof. Let $M = (a_{ni})$ be nonnegative and regular. Then for each $n = 1, 2, \cdots$ let $s(n)$ be such that

$$\sum_{i=s(n)+1}^{\infty} a_{ni} < \frac{1}{n} \quad \text{and} \quad s(n+1) > s(n) .$$

For each $j = 1, 2, \cdots$ select $k(j)$ so that $n \geqq k(j)$ implies

$$\sum_{i=1}^{\infty} a_{ni}(\chi_{A_1}(i) + \cdots + \chi_{A_j}(i)) < \frac{1}{j} \quad \text{and} \quad k(j+1) > k(j) .$$

The existence of $k(j)$ follows from Proposition 2.3 (ii). Further, for $n \geqq k(1)$ let $p(n)$ be such that $k(p(n)) \leqq n < k(p(n)+1)$. Finally, for $m = 1, 2, \cdots$, we define

$$B_m = A_m \backslash \{1, 2, \cdots, s(k(m+1))\} .$$

Note that $B_m \sim A_m$ for $m = 1, 2, \cdots$. Letting $B = \bigcup_{i=1}^\infty B_i$, we now show that $\bar{\delta}(B) = 0$. Reasons for some of the steps in the following string of inequalities will be given immediately thereafter.

$$\begin{aligned}
&(1) && \bar{\delta}(B) = \limsup_{n\to\infty} (M \cdot \chi_B)_n \\
& && = \limsup_{n\to\infty} \sum_{i=1}^{\infty} a_{ni}\chi_B(i) \\
&(2) && \leqq \limsup_{n\to\infty} \left(\frac{1}{n} + \sum_{i=1}^{s(n)} a_{ni}\chi_B(i) \right) \\
& && = \limsup_{n\to\infty} \sum_{i=1}^{s(n)} a_{ni}\chi_B(i) \\
&(3) && = \limsup_{n\to\infty} \sum_{i=1}^{s(n)} a_{ni} \sum_{m=1}^{\infty} \chi_{B_m}(i) \\
&(4) && = \limsup_{n\to\infty} \sum_{i=1}^{s(n)} a_{ni} \sum_{m=1}^{p(n)} \chi_{B_m}(i) \\
& && \leqq \limsup_{n\to\infty} \sum_{i=1}^{s(n)} a_{ni} \sum_{m=1}^{p(n)} \chi_{A_m}(i) \\
& && \leqq \limsup_{n\to\infty} \sum_{i=1}^{\infty} a_{ni} \sum_{m=1}^{p(n)} \chi_{A_m}(i)
\end{aligned}$$

$$(5) \qquad \leqq \limsup_{n\to\infty} \frac{1}{p(n)}$$

$$(6) \qquad = 0 .$$

Reasons. (1) Proposition 3.1; (2) follows from the definition of $s(n)$; (3) Since the B_m's are disjoint, we can write $\chi_B(i) = \sum_{m=1}^{\infty} \chi_{B_m}(i)$; (4) We show that if $m > p(n)$ and $i \leqq s(n)$, then $\chi_{B_m}(i) = 0$: $m > p(n) \Rightarrow k(m+1) > k(p(n)+1) > n \Rightarrow s(k(m+1)) > s(n) \Rightarrow \{1, 2, \cdots, s(n)\} \cap B_m = \varnothing \Rightarrow \chi_{B_m}(i) = 0$ for $i \leqq s(n)$; (5) follows since $n \geqq k(p(n))$; (6) follows since $p(n) \to \infty$ as $n \to \infty$. This completes the proof.

We now define another density, one which is closely associated with the summability method introduced by G. G. Lorentz [5] called "almost convergence," and for which the (APO) fails. To this end let

$$u(A) = \lim_{n\to\infty}\left[\min_{m\geqq 0} \frac{1}{n} \sum_{i=m+1}^{m+n} \chi_A(i)\right].$$

We shall call u the (*lower*) *uniform density* of A.

Here we state without proof some facts concerning the density u: The limit in the definition of u exists for any set A. The function u is a lower asymptotic density (i.e., u satisfies (D.1)-(D.4) of § 2). The associated upper density $\bar{u}$ is

$$\bar{u}(A) = \lim_{n\to\infty}\left[\max_{m\geqq 0} \frac{1}{n} \sum_{i=m+1}^{m+n} \chi_A(i)\right].$$

It follows that $A \in \eta_u$ iff the sequence χ_A is almost convergent. Note that if a set A contains arbitrarily long consecutive strings of integers (i.e., if for each $N > 0$ there exists k such that $\{k+1, k+2, \cdots, k+N\} \subseteqq A$), then $\bar{u}(A) = 1$.

We use this last fact to show that the (APO) fails for u. Let $A = \{1, 2, 4, \cdots, 2^n, \cdots\}$ and let $A_i = \{i + a: a \in A\}$, $i = 1, 2, \cdots$. It is not difficult to show that each $A_i \in \eta_u^0$, since $\max_{m\geqq 0} n^{-1} \sum_{j=m+1}^{m+n} \chi_{A_i}(j)$ is approximately $(\log_2 n)/n$. In order to show that the (APO) fails, it suffices to show that, for any choice of sets $B_i \sim A_i$, $\bar{u}(\bigcup_{i=1}^{\infty} B_i) > 0$. Thus, if (B_i) is any sequence of sets with $B_i \sim A_i$, $i = 1, 2, \cdots$, and if $N > 0$, then for each i, $1 \leqq i \leqq N$, there exists k_i so that $2^n + i \in B_i$ whenever $n > k_i$. Taking $k = \max\{k_1, k_2, \cdots, k_N\}$ and $n > k$, we observe that $2^n + 1 \in B_1$, $2^n + 2 \in B_2, \cdots, 2^n + N \in B_N$ and therefore, $\{2^n + 1, 2^n + 2, \cdots, 2^n + N\} \subseteqq \bigcup_{i=1}^{\infty} B_i$. It follows that $\bigcup_{i=1}^{\infty} B_i$ contains arbitrarily long consecutive strings of integers and, consequently $\bar{u}(\bigcup_{i=1}^{\infty} B_i) > 0$.

4. Sequence spaces related to densities. If x is a sequence, l

is a real number and A is a set of natural numbers with $I\backslash A$ infinite, then by

$$x \xrightarrow[(A)]{} l$$

we shall mean that the sequence x converges to l in the ordinary sense if we ignore the terms indexed by A, that is,

DEFINITION 4.1. $x \xrightarrow[(A)]{} l$ in case for each $\varepsilon > 0$ there exists $N > 0$ such that $|x_n - l| < \varepsilon$ whenever $n \geqq N$, $n \notin A$.

DEFINITION 4.2. For any density δ, let

$$\omega_\delta = \{x \in \omega \colon \exists l \quad \text{real and} \quad A \subseteqq I \quad \text{with} \quad \bar{\delta}(A) = 0 \quad \text{and} \quad x \xrightarrow[(A)]{} l\} .$$

We call ω_δ the set of (δ-) *nearly convergent sequences.*

PROPOSITION 4.3. *For any density δ, ω_δ is a linear space of sequences with $c \subseteqq \omega_\delta$.*

Proof. The fact that $c \subseteqq \omega_\delta$ is immediate from the definition and Proposition 2.2 (ii). Let x and y be in ω_δ and let l_1, l_2, A, B be such that $\bar{\delta}(A) = \bar{\delta}(B) = 0$, $x \xrightarrow[(A)]{} l_1$ and $y \xrightarrow[(B)]{} l_2$. By Proposition 2.3 (ii) $\bar{\delta}(A \cup B) = 0$, and it is clear that $x + y \xrightarrow[(A \cup B)]{} l_1 + l_2$, consequently $x + y \in \omega_\delta$. The remaining linear space postulates follow easily.

PROPOSITION 4.4. *Let δ be a density. Then the* (APO) *holds for δ iff ω_δ is closed with respect to the topology of uniform convergence on ω.*

Proof. Note that the topology referred to in the proposition is not the usual linear topology of coordinatewise convergence. Assume first that the (APO) holds and let $y \in \overline{\omega_\delta}$. Then there is a sequence $\{x^n\}$ in ω_δ such that $x^n \to y$ uniformly. For each n there exists a real number l_n and a set of natural numbers A_n such that $\bar{\delta}(A_n) = 0$ and $x^n \xrightarrow[(A_n)]{} l_n$. We show that the sequence (l_n) is Cauchy and therefore converges, in the usual sense, to some limit l. Let $\varepsilon > 0$ be given. There exists $N > 0$ so that $n \geqq N$ implies $|x_i^n - y_i| < \varepsilon/4$ for $i = 1, 2, \cdots$. If we take $m, n \geqq N$, then there is an integer i such that $|x_i^n - l_n| < \varepsilon/4$ and $|x_i^m - l_m| < \varepsilon/4$ (choose $i \notin A_n \cup A_m$ and sufficiently large) and, therefore,

$$|l_n - l_m| \leqq |l_n - x_i^n| + |x_i^n - y_i| + |y_i - x_i^m| + |x_i^m - l_m| < \varepsilon .$$

We now use the (APO). There exist sets $B_n \sim A_n$ such that if $E = \bigcup_{i=1}^{\infty} B_i$, then $\bar{\delta}(E) = 0$. We claim that $y \xrightarrow[(E)]{} l$. If $\varepsilon > 0$ choose n such that for all i, $|x_i^n - y_i| < \varepsilon/3$ and such that $|l_n - l| < \varepsilon/3$. Since A_n is contained in E (except for at most finitely many points), there is an $N(=N(n))$ such that, if $i \geqq N$ and $i \notin E$, then $i \notin A_n$ and $|x_i^n - l_n| < \varepsilon/3$. Hence, for $i \geqq N$ and $i \notin E$ we have

$$|y_i - l| \leqq |y_i - x_i^n| + |x_i^n - l_n| + |l_n - l| < \varepsilon ,$$

and so $y \xrightarrow[(E)]{} l$. It follows that $y \in \omega_\delta$.

Now suppose the (APO) fails to hold for δ. Then there exists a sequence of disjoint sets $A_1, A_2, \cdots$ of natural numbers such that $\bar{\delta}(A_i) = 0$, $i = 1, 2, \cdots$, and also such that for any choice of (B_i) with $B_i \sim A_i$, $i = 1, 2, \cdots$, we have $\bar{\delta}(\bigcup_{i=1}^{\infty} B_i) > 0$. Evidently, if we remove one set, say B_j, from this union we still get the same conclusion—namely, for any $j = 1, 2, \cdots$, $\bar{\delta}(\bigcup_{i \neq j} B_i) > 0$. We define a sequence $\{x^n\}$ that converges uniformly to a sequence y, where each $x^n \in \omega_\delta$ and $y \notin \omega_\delta$. Let

$$x_i^n = \begin{cases} \dfrac{1}{j} \text{ if } i \in A_j, \text{ where } 1 \leqq j \leqq n \\ 0 \text{ otherwise} \end{cases}$$

and let

$$y_i = \begin{cases} \dfrac{1}{j} \text{ if there exists } j \text{ such that } i \in A_j \\ 0 \quad \text{otherwise.} \end{cases}$$

It is easily seen that $x^n \in \omega_\delta$ and that $|x_i^n - y_i| < 1/(n+1)$, $i = 1, 2, \cdots$, so that $\{x^n\} \to y$ (uniformly). However, if l is a real number and E is a set for which $y \xrightarrow[(E)]{} l$, then for all but at most one n (in the case that $l = 1/j$ for some j) we have, evidently,

$$(A_n \cap E) \sim A_n .$$

Letting $B_n = A_n \subset E$, $n = 1, 2, 3, \cdots$, we have $\bar{\delta}(E) \geqq \bar{\delta}(\bigcup_{n=1}^{\infty} B_n) > 0$. It follows that $y \notin \omega_\delta$. This completes the proof.

Since the sequences x^n and y defined above are bounded we immediately obtain the

COROLLARY 4.5. $\omega_\delta \cap m = \overline{\omega_\delta \cap m}$ iff δ *satisfies the* (APO).

We note that, in our prime example (ordinary asymptotic density and Cesàro summability), the space $\omega_\delta \cap m$ coincides with the space of bounded strongly Cesàro summable sequences. In this case the density and the summability method are related, both being produced from the Cesàro matrix. In relating densities and summability methods in general, we are led to investigate methods that we call R-type summability methods. If S is a summability method with domain, or convergence field, c_S, let the strong convergence field associated with S be

$$|c_S| = \{x \in \omega \colon \exists l \quad \text{such that} \quad S(|x - l|) = 0\} .$$

The set $|c_S|$ need not be a subspace of c_S. (For example, if c_S is the set of all convergent series with $S(x) = \sum_{i=1}^{\infty} x_i$, then $|c_S|$ is the set of all constant sequences.) It is a subspace, however, for those methods given in

DEFINITION 4.6. An *R-type summability method* (RSM) is a linear functional S with domain c_S, where c_S is a subspace of ω, and such that S satisfies the following two properties:

(P1) S is regular (i.e., $c \subseteqq c_S$ and $S(x) = \lim_k x_k$ for $x \in c$).

(P2) if $|x| \in c_S$ with $S(|x|) = 0$ and $y \in m$, then $y \cdot x \in c_S$ and $S(y \cdot x) = 0$.

We let

$$\begin{aligned} c_S^0 &= \{x \in \omega \colon S(x) = 0\} ; \\ |c_S| &= \{x \in \omega \colon \exists l \quad \text{such that} \quad |x - l| \in c_S^0\} ; \\ |c_S|^0 &= \{x \in \omega \colon |x| \in c_S^0\} . \end{aligned}$$

We remark that c_S^0 is a subspace of c_S and that (P2) can be briefly written: $m \cdot |c_S|^0 \subseteqq c_S^0$.

Our next three propositions establish that the summability methods under consideration are reasonable ones.

PROPOSITION 4.7. *For any summability method S, the condition* (P2) *is equivalent to the condition that $|c_S|^0$ be solid* (i.e., $m \cdot |c_S|^0 = |c_S|^0$).

Proof. Assume condition (P2) holds, let $y \in m$ and $x \in |c_S|^0$. Letting η be the sequence of ± 1's such that $|y \cdot x| = (\eta_i y_i x_i)$, and noting that $\eta \cdot y \in m$, we can write

$$|y \cdot x| = (\eta_i y_i x_i) = (\eta \cdot y) \cdot x \in m \cdot |c_S|^0 .$$

By (P2), $|y \cdot x| \in c_S^0$ and, therefore, $y \cdot x \in |c_S|^0$. Conversely, suppose that $|c_S|^0$ is solid and let $x \in |c_S|^0$. Writing $x = x^+ - x^-$, where $x_i^+ = \max\{x_i, 0\}$ and $x_i^- = -\min\{x_i, 0\}$, we observe that $x^+ = t \cdot x$, where $t \in m$. Thus $x^+ \in |c_S|^0$. Since $|x^+| = x^+$, it follows that $x^+ \in c_S^0$. Similarly, $x^- \in c_S^0$ and, since c_S^0 is a subspace, $x \in c_S^0$. Condition (P2) follows.

PROPOSITION 4.8. *If S is an* RSM, *then* $|c_S| \subset c_S$.

Proof. If $x \in |c_S|$ then, for some l, $|x - l| \in c_S^0$ which implies, by definition, that $x - l \in |c_S|^0$. It follows by (P2) that $x - l \in c_S^0$. It follows from (P1) and the linearity of S that $x \in c_S$ (and that $S(x) = l$).

PROPOSITION 4.9. *If S is an* RSM, *then $|c_S|$ and $|c_S|^0$ are subspaces of c_S and c_S^0, respectively. Furthermore, $c \subseteqq |c_S|$ and $c_0 \subseteqq |c_S|^0$.*

Proof. Condition (P2) implies that $|c_S|^0 \subseteqq c_S^0$ and Proposition 4.8 gives the inclusion $|c_S| \subseteqq c_S$. To show that $x + y \in |c_S|^0$ whenever $x \in |c_S|^0$ and $y \in |c_S|^0$, let η and τ be sequences of ± 1's such that

$$|x + y| = \eta \cdot x + \tau \cdot y \; .$$

Again (P2) implies that $\eta \cdot x$ and $\tau \cdot y$ are in c_S^0, and therefore since c_S^0 is a subspace, $|x + y| \in c_S^0$. By definition it follows that $x + y \in |c_S|^0$. We omit the other details, which are routine.

Note that if S is an RSM, then it follows from Propositions 4.8 and 4.9 that $c \subseteqq |c_S| \subseteqq c_S$. Furthermore, it is easy to see that if x is a convergent sequence and if l is any real number for which $|x - l| \in c_S^0$, then $l = S(x) = \lim_k x_k$. An RSM is therefore "strongly" regular. In case S is a matrix method of summability, this terminology agrees with that used in [6] (p. 191).

We now compare the sequence spaces ω_δ and $|c_S|$ as they relate to a density δ.

DEFINITION 4.9. A density δ and an RSM S are *related* in case, for each subset A of the natural numbers,

$$\bar{\delta}(A) = 0 \Longleftrightarrow \chi_A \in |c_S|^0 \; .$$

PROPOSITION 4.10. *If δ and S are a related density and* RSM, *then*

$$\omega_\delta \cap m \subseteqq |c_S| \subseteqq \overline{\omega_\delta}$$

(where $\bar{\omega}_\delta$ denotes the closure of ω_δ with respect to the topology of uniform convergence on ω).

Proof. Let $x \in \omega_\delta \cap m$. Then there exists a real number l and a set A of natural numbers such that $\bar{\delta}(A) = 0$ and $x \xrightarrow[(A)]{} l$. Since $\bar{\delta}(A) = 0$, we have $\chi_A \in |c_S|^0$. Writing

$$x - l = (x - l)\cdot\chi_A + (x - l)\cdot\chi_{I\setminus A}$$

and noting that $x - l \in m$, we have, by Proposition 4.7, $(x - l)\cdot\chi_A \in |c_S|^0$. Further, $(x - l)\cdot\chi_{I\setminus A} \in c_0 \subseteqq |c_S|^0$ by Proposition 4.9. It follows that $x - l \in |c_S|^0$ and, consequently, that $x \in |c_S|$.

Next consider any $x \in |c_S|$. If l is such that $|x - l| \in |c_S|^0$, define

$$A_n = \left\{ i \in I\colon |x_i - l| \geqq \frac{1}{n} \right\}, \quad n = 1, 2, \cdots .$$

We claim that $\bar{\delta}(A_n) = 0$. Define a sequence b by

$$b_i = \begin{cases} \dfrac{1}{x_i - l} & \text{if } i \in A_n \\ 0 & \text{otherwise .} \end{cases}$$

The sequence b is bounded and so $b\cdot(x - l) \in |c_S|^0$ by Proposition 4.7. But $b\cdot(x - l) = \chi_{A_n}$, thus $\chi_{A_n} \in |c_S|^0$. Since δ and S are related, this means that $\bar{\delta}(A_n) = 0$. Now define $\{y^n\}$ by

$$y_i^n = \begin{cases} x_i & \text{if } i \leqq n , \\ x_i & \text{if } i > n \text{ and } i \in \bigcup_{j=1}^n A_j , \\ l & \text{if } i > n \text{ and } i \notin \bigcup_{j=1}^n A_j . \end{cases}$$

If $E_n = \bigcup_{j=1}^n A_j$, then $\bar{\delta}(E_n) = 0$ and clearly $y^n \xrightarrow[(E_n)]{} l$, so that $y^n \in \omega_\delta$ for each $n = 1, 2, \cdots$. Also $|y_i^n - x_i| < 1/n$ and thus $\{y^n\} \to x$ uniformly. Therefore $x \in \overline{\omega_\delta}$.

COROLLARY 4.11. *If δ and S are a related density and* RSM, *and if the* (APO) *holds for δ, then*

$$|c_S| \cap m = \omega_\delta \cap m .$$

In case $|c_S|$ is a closed subspace of ω, then (independently of the validity of the (APO))

$$|c_S| \cap m = \overline{\omega_\delta \cap m} .$$

5. Concluding remarks. The results are illustrated most clearly in the case of matrices. If M is any nonnegative regular matrix, it is readily seen that the summability method defined by M is an RSM and that the method is related (in the sense of Definition

4.9) to the density δ_M (Proposition 3.1). By Proposition 3.2 the (APO) holds, and we therefore have $|c_M| \cap m = \omega_{\delta_M} \cap m = \overline{\omega_{\delta_M} \cap m}$. In the special case where $M = J$ (the identity matrix), we have $|c_J| \cap m = \omega_{\delta_J} \cap m = c \cap m = c$. It is interesting to note here that ω_{δ_J} is itself equal to c, since $\bar{\delta}_J(A) = 0$ if and only if A is finite. (Thus the condition that x converge to l except on a set $A \in \eta^0_{\delta_J}$ is equivalent to the ordinary convergence of x to l.)

In case $M = C_1$, the Cesàro matrix, we obtain $|\sigma_1| \subseteqq \omega_\delta$ and the relationship between strong Cesàro summability and ordinary asymptotic density mentioned several times previously ($|\sigma_1| \cap m = \omega_\delta \cap m$).

As a nonmatrix illustration of the results, if we let S be the summability method defined by $S(x) = l$ if and only if $(\sum_{i=m+1}^{m+n} x_i)/n \to l$ uniformly with respect to $m = 0, 1, 2, \cdots$, then the convergence field consists of the space of almost convergent sequence introduced by Lorentz [5]. The associated strong convergence field is the space $|AC|$ of strongly almost convergent sequences studied in [3] (x is strongly almost convergent in case there exists l such that $(\sum_{i=m+1}^{m+n} |x_i - l|)/n \to 0$ uniformly with respect to $m = 0, 1, 2, \cdots$). It can readily be checked that almost convergence is related to the uniform density function discussed in § 3. Since $|AC|$ is closed in the topology of uniform convergence, we have, by Corollary 4.1.1,

$$|AC| = |AC| \cap m = \overline{\omega_u \cap m} .$$

However, since the (APO) fails to hold for this density (see § 3) we have the strict inclusion

$$\omega_u \cap m \subsetneqq |AC| .$$

References

1. R. C. Buck, *Generalized asymptotic density*, Amer. J. Math., **75** (1953), 335–346.
2. A. R. Freedman, *On the additivity theorem for n-dimensional asymptotic density*, Pacific J. Math., **49**, No. 2 (1973), 357–363.
3. Allen R. Freedman, John J. Sember, and Marc Raphael, *Some Cesàro-type summability spaces*, Proc. London Math. Soc. (3) **37** (1978), 508–520.
4. Paul R. Halmos, *Lectures on Ergodic Theory*, Chelsea, New York, 1956.
5. G. G. Lorentz, *A contribution to the theory of divergent sequences*, Acta Mathematics, **80** (1948), 167–190.
6. I. J. Maddox, *Elements of Functional Analysis*, Cambridge, 1970.
7. P. Walters, *Ergdic Theory—Introductory Lectures*, Lecture Notes in Mathematics No. **458**, Springer-verlag, Berlin, Heidelberg, New York, 1975.

Received July 31, 1979.

SIMON FRASER UNIVERSITY
BURNABY, B. C., CANADA