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ε-COVERING DIMENSION

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A compact metric space T has Lebesgue covering dimension at most n if for each positive ε the space T has an ε-cover of order at most n. We show that if T is a compact subset of Euclidean n-space and T has an ε-cover of order at most $n-2$, then any two points whose distance from T is greater than ε can be joined by a path bounded away from T. This refines, and provides a constructive proof for, the theorem that the complement of an $(n-2)$-dimensional compact subset of Euclidean n-space is connected.

0. **Introduction.** In this paper we deal with arbitrary totally bounded metric spaces, rather than just compact ones, as completeness plays no role. Let T be a totally bounded metric space and F a finite family of subsets of T. If there is $s > 0$ so that each point in T is bounded away by s from all but at most $n+1$ sets of F, then we say that F has *order at most n with separation s* and write $o(F) \leqq n$. (This was written $o(F) \leqq n+1$ in [7] and [1].) If the union of F is dense in T, we say that F is a *cover* of T. A cover F is an *ε-cover* provided there is $\varepsilon' < \varepsilon$ such that if x and y are points in a set in F, then $d(x, y) < \varepsilon'$. Classically this means that $\operatorname{diam} U = \sup\{d(u, v): u, v \in U\} < \varepsilon$ for all U in F, but $\operatorname{diam} U$ may fail to be computable. Note for any $\varepsilon'' > \varepsilon'$ that F is an ε''-cover. We can now make precise the notion of approximate n-dimensionality.

DEFINITION 0.1. Let T be a totally bounded metric space and $\varepsilon > 0$. We say that T has *ε-covering dimension* at most n with separation s, and write $\varepsilon\text{-cov}\, T \leqq n$, if there is an ε-cover of T of order at most n with separation s.

A totally bounded metric space T has dimension at most n in the sense of Lebesgue if $\varepsilon\text{-cov}\, T \leqq n$ for all $\varepsilon > 0$. Thus if $\varepsilon\text{-cov}\, T \leqq n$, then T is approximately n-dimensional. For example the red yellow and black stripes of a coral snake form an ε-cover of its skin, showing the skin to have ε-dimension at most 1. However, when a coral snake swallows a mouse of cross-sectional diameter 2ε its ε-dimension increases. More precisely, the Jordan Brouwer theorem says that a homeomorph T of the 2-sphere divides 3-space into two connected components, but if $\varepsilon\text{-cov}\, T \leqq 1$, then there is no ε-ball inside. Thus the Little Prince was correct when he observed that a boa constrictor loses its one dimensionality when it swallows an elephant [8]. More generally we will prove

THEOREM A. *Let T be a totally bounded subset of R^n such that ε-cov $T \leqq n-2$ with separation s. If $d(\{p, q\}, T)$ is more than $\varepsilon/\sqrt{2}$ and if $0 < \theta < \phi = \inf\{s/2, (\sqrt{2}-1)(d(\{p, q\}, T) - \varepsilon/\sqrt{2})\}$, then there is a path joining p and q, bounded away from T by θ.*

These investigations were motivated by attempts to give a constructive proof that the complement of an $(n-2)$-dimensional subset of R^n is connected. Such a proof is given via Alexander duality and Cech cohomology in [5]. However, Theorem A is stronger than this result even from the ćlassical standpoint. Our treatment uses simplicial homology and, like [5], is constructive in the sense of Bishop [2], [3]. Menger's proof that the complement of an $(n-2)$-dimensional subset of R^n is connected uses inductive dimension and is not constructive [6].

1. Dimension theory. The basic references for constructive dimension theory are [7] and [1]. In these works the elements of an ε-cover were required to be totally bounded (located). This is occasionally inconvenient and, as we will show in this section, unnecessary.

Let K be an arbitrary subset of a metric space T. For $\theta > 0$ the *θ-neighborhood* of K is the open set

$$N_\theta(K) = \{y \in T\colon \text{there is } x \text{ in } K, \text{ with } d(x, y) < \theta\} .$$

A family F of subsets of a metric space T has a *Lebesgue number* $s > 0$ if for each x in T there is U in F with $N_s(x) \subset U$.

LEMMA 1.1. *Let F be an ε-cover of a totally bounded metric space T, such that $o(F) \leqq n$ with separation s. If θ is small enough, then $F' = \{N_\theta(U)\colon U \text{ in } F\}$ is an open ε-cover having Lebesgue number $\theta/2$, and order at most n with separation $s-\theta$.*

Proof. Choose $\varepsilon' < \varepsilon$ so that F is an ε'-cover and let $2\theta = \inf\{\varepsilon - \varepsilon', s\}$. To establish the order of F' we let $x \in T$, and let $U \in F$. Suppose $d(x, u) \geqq s$ for all u in U. Let $v \in N_\theta(U)$ and choose u in U with $d(v, u) \leqq \theta$. Then $d(x, v) \geqq d(x, u) - d(v, u) \geqq s - \theta$. Thus $o(F') \leqq n$ with separation $s - \theta$.

To obtain the Lebesgue number we let $x \in T$. Then there is U in F and u in U so that $d(x, u) < \theta/2$. If $d(x, y) < \theta/2$, then $d(y, u) < \theta$. Hence $N_{\theta/2}(x) \subset N_\theta(U)$ and F' has Lebesgue number $\theta/2$. □

Next we show that a finite family with Lebesgue number admits a partition of unity.

LEMMA 1.2. *Let F be a finite family of subsets of a totally bounded space T. Then F has a Lebesgue number if and only if there is a partition of unity subordinate to F. Moreover, the functions in the partition can be chosen with totally bounded support.*

Proof. Let $\{\phi_U\}$ be a partition of unity subordinate to F. Choose $s > 0$, so that if $d(x, y) < s$ then $d(\phi_U(x), \phi_U(y)) < 1/(1 + \operatorname{card} F)$ for all U in F. We shall show for fixed x that there is a U in F with $N_s(x)$ contained in U. As the sum of $\phi_U(x)$ over all U in F is 1, it follows that there is U in F with $\phi_U(x) > 1/(1 + \operatorname{card} F)$. Hence $\phi_U(y) > 0$ and so $y \in U$. Therefore $N_s(x)$ is contained in U.

Conversely let s be a Lebesgue number of F. Choose X a finite $(s/2)$-approximation to T. Let $x \in X$ and define

$$f_x(t) = \sup\{0, 1 - (1/s)d(t, x)\} .$$

Partition X into finite subsets X_U so that $x \in X_U$ implies $N_s(x) \subset U$.

Choose a positive number $\varepsilon < 1/(4 \operatorname{card} F)$, so that

$$\{t \in T: \sum_{x \in X_U} f_x(t) > \varepsilon\}$$

is totally bounded for each U in F [7, Theorem 0]. Define $\lambda_U(t) = \sup\{0, \sum_{x \in X_U} f_x(t) - \varepsilon\}$ for U in F. Then the support of λ_U is totally bounded and $\sum_{U \in F} \lambda_U \geqq \sum_{x \in X} f_x - \varepsilon \operatorname{card} F > 1/4$, as $\sum_{x \in X} f_x > 1/2$. Finally let $\phi_U(t) = \lambda_U(t)/\sum_{V \in F} \lambda_V(t)$. □

THEOREM 1.3. *Let T be a totally bounded metric space and F an ε-cover. Let $o(F) \leqq n$ with separation $s > 0$. Then there is an open ε-cover F' satisfying*:

(i) *$o(F') \leqq n$ with separation $s/2$.*

(ii) *Each U' in F' is totally bounded.*

(iii) *F' has a Lebesgue number.*

(iv) *Each set in F' is nonempty.*

Proof. By Lemma 1.1, we may assume that F is an open ε-cover such that $o(F) \leqq n$ with separation $s/2$ and has a Lebesgue number. By Lemma 1.2, there is a partition of unity $\{\phi_U\}$ so that the support U' of ϕ_U is totally bounded and is contained in U and so $F' = \{U': U \in F\}$ is an open ε-cover satisfying (i) and (ii). Note that $\{\phi_U\}$ is subordinate to F' so (iii) holds by Lemma 1.2. As each set in F' is totally bounded we may omit the empty ones. □

2. Simplicial homology. We employ the standard simplicial homology of triangulable spaces (the treatment in [4] is essentially constructive).

A point sufficiently far from a set is bounded away from its convex hull; more precisely we have:

LEMMA 2.1. *Let p be a point and X a subset of a real inner product space. Let $t \in X$ and $\varepsilon > 0$. If for some $\alpha > 0$ and each x in X we have $|t - x| \leqq \varepsilon$ and $|p - x| \geqq \alpha + \varepsilon/\sqrt{2}$ then $|p - q| \geqq \alpha$ for each q in the convex hull of X.*

Proof. Let the inner product be denoted by $\langle \ , \ \rangle$. We may assume $p = 0$. We will first show that if $x \in X$ then $\langle t, x\rangle \geqq |t|\alpha$. As radial projection onto the sphere of radius $\alpha + \varepsilon/\sqrt{2}$ around 0 decreases $|t - x|$ and $\langle t, x\rangle/|t|$, we may assume that $|t| = |x| = \alpha + \varepsilon/\sqrt{2}$. Then $\varepsilon^2 \geqq |t - x|^2 = \varepsilon^2 + 2\sqrt{2}\alpha\varepsilon + 2\alpha^2 - 2\langle t, x\rangle$. Thus $\langle t, x\rangle \geqq \alpha(\alpha + \sqrt{2}\varepsilon)$. Then $\langle t, x\rangle/|t| \geqq \alpha(\alpha + \sqrt{2}\varepsilon)/(\alpha + \varepsilon/\sqrt{2}) > \alpha$. So if q is a finite convex combination of points in X, then $|q| \geqq \langle t, q\rangle/|t| > \alpha$. □

We now relate ε-dimension to homology.

LEMMA 2.2. *Let T be a polyhedron in R^n such that ε-cov $T \leqq n - 2$. Let $p \in R^n$ and $d(p, T) \geqq \varepsilon/\sqrt{2}$. Then radial projection onto any sphere S with center p and radius at most $\alpha = d(p, T) - \varepsilon/\sqrt{2}$ induces the zero map from $H_{n-1}(T)$ to $H_{n-1}(S)$.*

Proof. By Theorem 1.3 there is an ε-cover F of T such that F has a Lebesgue number, $o(F) \leqq n - 2$, and each set in F is nonempty. Let $\{\phi_U\}$ be a partition of unity subordinate to F (Lemma 1.2). For each U in F choose x_U in U. Define a map $f: T \to R^n$ by $f(t) = \sum_{U \in F} \phi_U(t)x_U$. Define a homotopy $h: T \times I \to R^n$ by $h(t, \lambda) = \lambda t + (1 - \lambda)f(t)$. This is a homotopy between f and the injection, i, of T into R^n. If $t \in T$ and $\lambda \in I$, then $h(t, \lambda)$ is a convex combination of t and the x_U. Let $\varepsilon' < \varepsilon$ be such that F is an ε'-cover of T. Either $d(t, x_U) > \varepsilon'$ in which case $\phi_U(t) = 0$ so x_U does not enter into $h(t, \lambda)$, or $d(t, x_U) < \varepsilon$. Hence Lemma 2.1 applies, so $d(h(t, \lambda), p) \geqq \alpha$.

Let S be a sphere with center p and radius at most α and let r be the radial projection of the exterior of S onto S. As the domain of r contains the range of h the map $r \circ h$ is a homotopy between $r \circ i$ and $r \circ f$. But, since $o(F) \leqq n - 2$, the map f factors through a simplicial complex of dimension at most $n - 2$. Thus $r \circ f$, and therefore $r \circ i$, induces the trivial map from $H_{n-1}(T)$ to $H_{n-1}(S)$. □

3. Proof of the main theorem Choose θ', θ'', and m satisfying $\theta < \theta' < \theta' + m < \theta'' < \phi$. Let F be an ε-cover of T such that $o(F) \leqq n - 2$ with separation s. We first replace the totally bounded set T by a finite complex. Let Δ be an n-simplex containing a neighborhood of T and p. We will show that there is a path joining p to the boundary of Δ, bounded away from T by θ. Form $\Delta^{(k)}$, the kth derived complex of Δ, with k so large that the diameter of each simplex λ in $\Delta^{(k)}$ is less than m. By translating Δ slightly we may assume that p is in the interior of an n-simplex λ_0. Let T' be a set of n-simplices in $\Delta^{(k)}$ so that for each n-simplex λ in $\Delta^{(k)}$

$$\text{if } \lambda \in T', \text{ then } d(\lambda, T) < \theta' ,$$

and

$$\text{if } \lambda \notin T', \text{ then } d(\lambda, T) > \theta .$$

Let $F' = \{U' : U' = T' \cap N_{\theta''}(U) \text{ with } U \text{ in } F\}$.

We first show that F' has a Lebesgue number. If $x \in T'$, then there is t in T with $d(x, t) < \theta' + m$. There is U in F and u in U with $d(t, u) < (1/2)(\theta'' - \theta' - m) = \psi$. Then $N_{\theta''}(u)$ contains $N_\psi(x)$ so U' contains $T' \cap N_\psi(x)$. Hence ψ is a Lebesgue number of F'.

Next we show that F' is an $(\varepsilon + 2\theta'')$-cover. For $x, y \in U'$, there are u and v in U so that $d(x, u) < \theta''$ and $d(v, y) < \theta''$. Thus $d(x, y) \leqq d(x, u) + d(u, v) + d(v, y) < 2\theta'' + \varepsilon$.

Finally we show that the order of F' is at most $n - 2$. For x in T', there is t in T, so that $d(x, t) < \theta' + m$. If $d(t, u) \geqq s$ for all u in U, then for u' in U', we have $d(x, u') \geqq d(t, u') - d(x, t) \geqq (s - \theta'') - (\theta' + m)$. So $F' = \{U' : U \in F\}$ has order at most $n - 2$ with separation $(s - \theta'' - \theta' - m) > s - 2\theta'' > 0$.

As $d(p, T') \geqq d(p, T) - \theta'' \geqq (\varepsilon + 2\theta'')/\sqrt{2}$ we have $\alpha = d(p, T') - (\varepsilon + 2\theta'')/\sqrt{2} > 0$. By Lemma 2.2, with ε replaced by $\varepsilon + 2\theta''$, radial projection onto a small sphere $S \subset \lambda_0$ centered at p induces the zero map from $H_{n-1}(T')$ to $H_{n-1}(S)$.

Let G be the connected component of $\Delta^{(k)} \backslash T'$ containing p. Now H_{n-1} of the $(n-1)$-skeleton of G is the direct sum of the groups $H_{n-1}(\dot{\lambda})$ where λ ranges over the n-simplices of G. Radial projection onto $\dot{\lambda}_0$ induces the trivial map from $H_{n-1}(\dot{\lambda})$ to $H_{n-1}(\dot{\lambda}_0)$ for $\lambda \neq \lambda_0$, and the identity on $H_{n-1}(\dot{\lambda}_0)$. But the combinatorial boundary of the sum of the n-simplices of G is a cycle in $H_{n-1}(\dot{G})$ and has a nonzero coordinate in each $H_{n-1}(\dot{\lambda})$. Thus radial projection induces a nonzero map from $H_{n-1}(\dot{G})$ to $H_{n-1}(\dot{\lambda}_0)$.

If the boundary of G were contained in T', then radial projection would induce a nonzero map from $H_{n-1}(T')$ to $H_{n-1}(S)$, which is precluded. Thus there must be a point of G on the boundary of $\Delta^{(k)}$ and we are done, as $\lambda \notin T'$ implies $d(\lambda, T) > \theta$. □

4. Applications and questions. Theorem A gives a new proof of the Pflastersatz:

THEOREM B. *If F is a 0.5-cover of S^n, then $o(F) \geqq n$.*

Proof. Let $s > 0$ and assume that $o(F) \leqq n - 1$ with separation s, then the origin can be joined to infinity by a path which is bounded away from S^n, an impossibility. Thus it follows easily that $o(F) \geqq n$ [7, Theorem 1]. □

Theorem B indicates the scale at which the n-dimensionality of S^n manifests itself. This suggests that, for any totally bounded metric space, we define

$$\varepsilon_n(T) = \inf \{\varepsilon: \varepsilon\text{-cov } T \leqq n\}$$

if the infinimum exists. Note that $\varepsilon_0(T)$ is the diameter of T for a connected set T, that $\varepsilon_n(T) = 0$ if and only if $\text{cov } T \leqq n$, and that $\varepsilon_n(T) > 0$ implies $\text{cov } T > n$.

It seems likely that $\varepsilon_{n-1}(S^n) = 2$, and $\varepsilon_{n-1}([0, 1]^n) = 1$. This holds for $n=1$ and 2. If B^n is the n-ball, then $\varepsilon_1(B^2) = \sqrt{3}$. What is $\varepsilon_{n-1}(B^n)$?

Can the requirement that $d(p, T) > \varepsilon/\sqrt{2}$ in Theorem A be replaced by $d(p, T) > \varepsilon/2$? It can if $n = 2$.

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