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THE ROGERS-RAMANUJAN RECIPROCAL AND MINC'S PARTITION FUNCTION

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The reciprocals of the Rogers-Ramanujan identities are considered, and it it shown that the results yield identities for restricted compositions. The same technique is applied to obtain a generating function for partitions previously treated by H. Minc.

1. **Introduction.** The celebrated Rogers-Ramanujan identities were first presented in their analytic form as follows:

$$1+\frac{q}{1-q}+\frac{q^4}{(1-q)(1-q^2)}+\frac{q^9}{(1-q)(1-q^2)(1-q^3)}+\cdots = \prod_{n=0}^{\infty}\frac{1}{(1-q^{5n+1})(1-q^{5n+4})}\ ; \tag{1.1}$$

$$1+\frac{q^2}{1-q}+\frac{q^6}{(1-q)(1-q^2)}+\frac{q^{12}}{(1-q)(1-q^2)(1-q^3)}+\cdots = \prod_{n=0}^{\infty}\frac{1}{(1-q^{5n+2})(1-q^{5n+3})}\ . \tag{1.2}$$

The fascinating story of their discovery by L. J. Rogers [8] and their subsequent rediscovery by S. Ramanujan (see [5; p. 91]) and I.J. Schur [9] has been told many times [1; Ch. 7], [2; Ch. 3], [5; Ch. 6]. P. A. MacMahon [6] and I.J. Schur [9] observed that (1.1) and (1.2) are equivalent to the following assertions in additive number theory:

THEOREM R_1. *The number of partitions of n into parts that differ by at least 2 equals the number of partitions of n into parts of the forms $5m+1$ and $5m+4$.*

THEOREM R_2. *The number of partitions of n into parts that differ by at least 2 and contain no ones equals the number of partitions of n into parts of the forms $5m+2$ and $5m+3$.*

Apart from Schur's two ingenious proofs in [9], all other proofs effectively rely on establishing the following two variable result:

$$F_1(z) \equiv 1+\sum_{n=1}^{\infty}\frac{z^n q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \left\{\prod_{n=1}^{\infty}\frac{1}{(1-zq^n)}\right\}\left\{1+\sum_{n=1}^{\infty}\frac{(zq)_{n-1}(1-zq^{2n})(-z^2)^n q^{n(5n-1)/2}}{(q)_n}\right\} \tag{1.3}$$

where $(A)_n = (1-A)(1-Aq)\cdots(1-Aq^{n-1})$, $(A)_0 = 1$.

The reciprocal of $F_1(-zq^{-1})$ was utilized by Carlitz and Riordan [4; p. 386, eq. (10.7)] in their work on q-analogs of two element lattice permutation numbers; however they give no indication that in fact $1/F_1(-z)$ is the generating function for certain simply restricted compositions. In another paper Carlitz [3] treats classes of restricted compositions which he calls "up-down" and "down-up" partitions. These he shows are generated by reciprocals of q-analogs of the Olivier functions. In fact arguments similar to those given by Carlitz may be utilized to prove the following assertion.

THEOREM 1. *Let $C_d(m, n)$ denote the number of representations of n in the form*

$$n = c_1 + c_2 + \cdots + c_m\,, \quad \textit{where} \quad 1 \leqq c_{i+1} \leqq c_i + d\,.$$

Then for $d \geqq 0$,

$$\sum_{m,n\geqq 0} C_d(m, n) z^m q^n = \frac{1}{F_d(-z)}\,, \tag{1.4}$$

where

$$F_d(z) = \sum_{n=0}^{\infty} \frac{q^{d\binom{n}{2}+\binom{n+1}{2}}}{(q)_n}\,.$$

We note that $C_0(m, n)$ is just the number of partitions of n into m parts and (1.4) reduces to a well-known generating function identity [1; p. 16] since

$$F_0(z) = \prod_{n=1}^{\infty}(1 + zq^n)\,, \quad [1;\ \text{p. } 19]\,. \tag{1.6}$$

Let us call a representation of n of the form $c_1 + c_2 + \cdots + c_m$ where $1 \leqq c_{i+1} \leqq c_i + 1$ a *restricted composition*, and let $K_e(j; n)$ (resp. $K_0(j; n)$) denote the number of restricted compositions with each $c_i \geqq j$ and with an even (resp. odd) number of parts. Also let $L_e(j; n)$ (resp. $L_0(j; n)$) denote the number of partitions of n into an even (resp. odd) number of parts each $\equiv \pm j \pmod 5$. Then equations (1.1) and (1.2) together with Theorem 1 imply:

THEOREM 2. *For all $n \geqq 0$,*

$$K_e(1; n) - K_0(1; n) = L_e(1; n) - L_0(1; n)\,; \tag{1.7}$$

$$K_e(2; n) - K_0(2; n) = L_e(2; n) - L_0(2; n)\,. \tag{1.8}$$

Both Theorems 1 and 2 will be proved in § 2. In § 3 we apply these methods to H. Minc's partition function $\nu(1, n)$, the number of representations of n in the form $n = 1 + c_1 + c_2 + \cdots c_m$ where $1 = c_0$ and $c_{i+1} \leqq 2c_i$ for $0 \leqq i \leqq m-1$. Minc [7] reduced an enumeration problem in groupoids to the determination of $\nu(1, n)$, and he provided a recurrence whereby $\nu(1, n)$ could be computed. We shall present the generating function for $\nu(1, n)$:

THEOREM 3.

$$\sum_{n=1}^{\infty} \nu(1, n)q^n = \frac{q}{\sum_{j=0}^{\infty} \frac{(-)^j q^{2^{j+1}-j-2}}{(1-q)(1-q^3)(1-q^7)\cdots(1-q^{2^j-1})}}.$$

2. The Rogers-Ramanujan reciprocal. We begin by proving Theorem 1. From the definition of $C_d(m, n)$ we see that

$$\begin{aligned}
\sum_{n\geqq 0} C_d(m, n)q^n \equiv \gamma_m &= \sum_{c_1=1}^{\infty} \sum_{c_2=1}^{c_1+d} \sum_{c_3=1}^{c_2+d} \cdots \sum_{c_m=1}^{c_{m-1}+d} q^{c_1+c_2+\cdots+c_m} \\
&= \sum_{c_1=1}^{\infty} \sum_{c_2=1}^{c_1+d} \cdots \sum_{c_{m-1}=1}^{c_{m-2}+d} q^{c_1+c_2+\cdots+c_{m-1}} \frac{(q - q^{c_{m-1}+d+1})}{(1-q)} \\
&= \frac{q}{1-q}\gamma_{m-1} - \frac{q^{d+1}}{1-q} \sum_{c_1=1}^{\infty} \sum_{c_2=1}^{c_1+d} \cdots \sum_{c_{m-2}=1}^{c_{m-3}+d} q^{c_1+c_2+\cdots+c_{m-2}} \frac{(q^2 - q^{2c_{m-2}+2d+2})}{(1-q^2)} \\
&= \frac{q}{1-q}\gamma_{m-1} - \frac{q^{d+3}}{(1-q)(1-q^2)}\gamma_{m-2}
\end{aligned}$$

$$(2.1) \qquad + \frac{q^{3d+3}}{(1-q)(1-q^2)} \sum_{c_1=1}^{\infty} \sum_{c_2=1}^{c_1+d} \cdots \sum_{c_{m-3}=1}^{c_{m-4}+1} q^{c_1+c_2+\cdots+c_{m-3}} \frac{(q^3 - q^{3c_{m-3}+3d+3})}{(1-q^3)}$$

$$\begin{aligned}
&= \frac{q}{1-q}\gamma_{m-1} - \frac{q^{d+3}}{(1-q)(1-q^2)}\gamma_{m-2} + \frac{q^{3d+6}}{(1-q)(1-q^2)(1-q^3)}\gamma_{m-3} \\
&\quad - \frac{q^{6d+6}}{(1-q)(1-q^2)(1-q^3)} \sum_{c_1=1}^{\infty} \sum_{c_2=1}^{c_1+d} \cdots \sum_{c_{m-4}=1}^{c_{m-5}+d} q^{c_1+\cdots+c_{m-4}} \frac{(q^4 - q^{4c_{m-4}+4d+4})}{(1-q^4)} \\
&= \vdots \,.
\end{aligned}$$

Thus applying mathematical induction we may rigorously establish that the above iterative process yields

$$(2.2) \qquad \sum_{j=0}^{m} \gamma_{m-j} \frac{(-1)^j q^{d\binom{j}{2}+\binom{j+1}{2}}}{(q)_j} = \begin{cases} 0 & \text{if} \quad m > 0 \\ 1 & \text{if} \quad m = 0 \,. \end{cases}$$

Hence (2.2) is equivalent to

$$(2.3) \qquad \sum_{m=0}^{\infty} \gamma_m z^m \sum_{n=0}^{\infty} \frac{(-1)^n q^{d\binom{n}{2}+\binom{n+1}{2}}}{(q)_n} = 1 \,.$$

Consequently by (2.3),

$$\sum_{n,m\geqq 0} C_d(m, n)z^m q^n = \sum_{m\geqq 0} \gamma_m z^m \tag{2.4}$$
$$= \frac{1}{F_d(-z)} .$$

Therefore Theorem 1 is established.

As we remarked in the introduction, Theorem 2 follows immediately from Theorem 1 and the Rogers-Ramanujan identities. Namely

$$\begin{aligned} &\sum_{n=0}^{\infty} (K_e(1; n) - K_0(1. n))q^n \\ &= \sum_{n,m\geqq 0} C_1(m, n)(-1)^m q^n \qquad \text{(by definition)} \\ &= \frac{1}{F_1(1)} \qquad \text{(by Theorem 1)} \\ &= \prod_{n=0}^{\infty} (1 - q^{5n+1})(1 - q^{5n+4}) \quad \text{(by (1.1))} \\ &= \sum_{n=0}^{\infty} (L_e(1; n) - L_0(1; n))q^n . \end{aligned} \tag{2.5}$$

Equation (1.7) follows immediately from (2.5) when we compare coefficients of q^n in the extreme terms. Similarly for (1.8) we see that

$$\begin{aligned} &\sum_{n=0}^{\infty} (K_e(2; n) - K_0(2; n))q^n \\ &= \sum_{n,m\geqq 0} C_1(m, n)(-q)^m q^n \\ &= \frac{1}{F_1(q)} \\ &= \prod_{n=0}^{\infty} (1 - q^{5n+2})(1 - q^{5n+3}) \\ &= \sum_{n=0}^{\infty} (L_e(2; n) - L_0(2; n))q^n . \end{aligned}$$

3. Minc's partition function. If μ_m denotes the generating function for Minc's partitions with m parts then as in § 2:

$$\begin{aligned} \mu_m &= \sum_{c_1=1}^{2} \sum_{c_2=1}^{2c_1} \cdots \sum_{c_m=1}^{2c_{m-1}} q^{1+c_1+c_2+\cdots+c_m} \\ &= \sum_{c_1=1}^{2} \sum_{c_2=1}^{2c_1} \cdots \sum_{c_{m-1}=1}^{2c_{m-2}} q^{1+c_1+c_2+\cdots+c_{m-2}} \frac{(q - q^{2c_{m-1}+1})}{(1 - q)} \\ &= \frac{q}{1-q}\mu_{m-1} - \frac{q}{1-q} \sum_{c_1=1}^{2} \cdots \sum_{c_{m-2}=1}^{2c_{m-3}} q^{1+c_1+\cdots+c_{m-2}} \frac{(q^3 - q^{6c_{m-2}+3})}{(1 - q^3)} \\ &= \frac{q}{1-q}\mu_{m-1} - \frac{q^4}{(1-q)(1-q^3)}\mu_{m-2} \end{aligned} \tag{3.1}$$

$$+\frac{q^4}{(1-q)(1-q^3)}\sum_{c_1=1}^{2}\cdots\sum_{c_{m-3}=1}^{2c_{m-4}} q^{1+c_1+\cdots+c_{m-3}}\frac{(q^7-q^{14c_{m-3}+7})}{(1-q^7)}$$

$$= \vdots \,.$$

As before applying mathematical induction we may rigorously establish that the above iterative process yields

$$(3.2)\qquad \sum_{i=0}^{m}\mu_{m-i}\frac{(-1)^j q^{1+3+7+\cdots+(2_j-1)}}{(1-q)(1-q^3)(1-q^7)\cdots(1-q^{2-1})}=\begin{cases}0 & \text{for} \quad m>0\\ q & \text{for} \quad m=0\,.\end{cases}$$

Therefore as in Theorem 1

$$\sum_{n=1}^{\infty}\nu(1,n)q^n=\sum_{m=0}^{\infty}\mu_m=\frac{q}{\displaystyle\sum_{j=0}^{\infty}\frac{(-1)^j q^{1+3+7+\cdots+(2^j-1)}}{(1-q)(1-q^3)(1-q^7)\cdots(1-q^{2^j-1})}}$$

and this is clearly seen to be equivalent to Theorem 3 once we recall that $\sum_{j=0}^{s}(2^j-1)=2^{s+1}-s-2$.

4. **Conclusion.** The method here could obviously be applied more generally; for example, the role of 2 in Minc's partitions could clearly be played by any positive integer k. Of course similar methods are used by Carlitz [3] to treat up-down and down-up partitions. After first discovering Theorem 1, I had hoped that it might be possible to find similar results in general for

$$\frac{1}{f_{\mathscr{C}}(-z,q)}$$

where $f_{\mathscr{C}}(z,q)$ is the two variable generating function for the linked partition ideal $\mathscr{C}$ (see [1; Ch. 8] for an explanation of linked partition ideals). Unfortunately the coefficients are not even positive in general.

There is a natural way of providing a common generalization of Theorems 1 and 3. Namely the difference conditions bounding c_{i+1} can be extended to $1\leqq c_{i+1}\leqq d+a_0c_i+a_1c_{i-1}+\cdots+a_jc_{i-j}$. For example the generating function for representations of n of the form

$$n=1+1+c_1+c_2+\cdots+c_m$$

subject to $c_{-1}=c_0=1$ and $c_{i+1}\leqq c_i+c_{i-1}$ is

$$\frac{q^2}{\displaystyle\sum_{n=0}^{\infty}\frac{q^{u_1+\cdots+u_n-n}(-1)^n}{(1-q^{u_1-1})(1-q^{u_2-1})\cdots(1-q^{u_n-1})}}$$

where u_i are shifted Fibonacci numbers $u_1 = 2$, $u_2 = 3$, $u_n = u_{n-1} + u_{n-2}$ for $n > 2$. In general the Fibonacci exponent $u_i - 1$ in the generating function will be replaced by the sum of the 1st i terms of the recurrent sequence arising from the recurrence $c_{n+1} = d + a_0 c_n + a_1 c_{n-1} + \cdots + a_j c_{n-j}$.

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