PACIFIC JOURNAL OF MATHEMATICS
Vol. 97, No. 1, 1981

ON COMPACTLY PACKED RINGS

J. V. PAKALA AND T. S. SHORES

A commutative ring with identity is *compactly packed by primes* (briefly: a C. P.-ring) if whenever an ideal I of R is contained in the union of a family of prime ideals of R, then I is actually contained in one of the primes of the family. The aim of this note is to characterize Noetherian P. C.-rings ideal theoretically and, with suitable restrictions, in terms of Picard groups.

The notion of a C. P.-ring was introduced by C. Reis and T. Viswanathan in [6], where Noetherian C. P.-rings were characterized by the property that primes are radicals of principal ideals, and it was asked if it suffices to have maximal ideals be radicals of principal ideals. N. Popescu studied the torsion theoretic aspects of C. P.-rings in [4], where he extended the preceding result to semi-noetherian rings and asked if every semi-noetherian C. P.-ring must have Krull dimension at most one. Independently, W. Smith [8] characterized all C. P.-rings as those rings for which every prime ideal is the radical of a principal ideal. In this note we answer the preceding questions (the former in the affirmative and the latter in the negative) and examine a connection between torsion Picard groups and the C. P.-property.

We would like to express our thanks to William Krauter and Roger Wiegand for several enlightening observations regarding the connection between Picard groups and C. P.-rings.

As usual $\mathrm{Spec}(R)$ denotes the set of prime ideals of R, $V(I) = \{P \in \mathrm{Spec}(R) \mid I \subseteq P\}$ for subsets I of R and $D(I)$ denotes the complement of $V(I)$. Denote the radical of the ideal I by $\mathrm{rad}(I)$ and the residual $\{r \in R \mid rI \subseteq J\}$ by $(J : I)$. As we have noted, the first equivalence of the following theorem is due to Smith [8]; we supply a proof for convenience of the reader.

THEOREM 1. *The ring R is a* C. P.*-ring if and only if every prime ideal is the radical of a principal ideal, in which case every radical ideal is the radical of a principal ideal. Moreover the Noetherian ring R is a* C. P.*-ring if and only if every maximal ideal is the radical of a principal ideal.*

Proof. Suppose the R is a C. P.-ring and I is a radical ideal of R, i.e., $I = \mathrm{rad}(I)$. The C. P. property implies that $I \nsubseteq \cup D(I)$, whence there exists an element $x \in I$ such that $V(x) \subseteq V(I)$. But the converse inclusion is obvious, so $\operatorname{rad}(xR) = \operatorname{rad}(I) = I$. Conversely, suppose that every prime ideal of R is the radical of a principal ideal and that $I \subseteq \cup X$ for some ideal I of R and subset X of Spec(R). Let S be the multiplicatively closed set $R \backslash \cup X$ and expand I to an ideal P maximal with respect to avoiding S. Then P is necessarily prime, so the radical of a principal ideal, say xR. Now $x \in Q$ for some $Q \in X$, whence $I \subseteq Q$ as desired.

Finally suppose that R is a Noetherian ring whose maximal ideals are radicals of principal ideals. Without loss of generality, R is reduced. Moreover the Krull dimension of R is at most one by the Principal Ideal Theorem. So to show that R is C. P. it suffices to prove that every minimal prime of R is the radical of a principal ideal. Such primes are finite in number, say $Q_1, \cdots, Q_n$, so we may induct on n to prove the result. It suffices to prove the Q_1 is the radical of a principal ideal. But $(0: Q_1) = \cap \{Q_j \mid j \neq 1\}$, so $R/(0: Q_1)$ is C. P. by induction. Now use the first statement of the theorem to obtain an element $x \in Q_1$ such that $\operatorname{rad}(xR + (0: Q_1)) = \operatorname{rad}(Q_1 + (0: Q_1))$. Certainly $Q_1 \subseteq \operatorname{rad}(xR)$ and for the converse, let $y \in Q_1$ and choose n such that $y^n \in xR + (0: Q_1)$. Then $y^{n+1} \in xR$, as desired.

REMARK 2. In [6] it was shown that for a Dedekind domain the C. P.-property is equivalent to the condition that R has torsion ideal class group. R. Wiegand pointed out to us that this result may be generalized in the following way: Let R be a one dimensional Noetherian ring with finitely many singular maximal ideals (i.e., maximals at which the localization of R is not a valuation ring or field). Denote the Picard group of R by Pic(R). Lemma 3 of [9, p. 30] shows that if Pic(R) is torsion, then every maximal ideal is the radical of a principal ideal. W. Krauter has proved the converse of this result under the additional hypothesis that R is a domain, and with his kind permission we sketch his proof: note that if M is any nonsingular maximal ideal of R, then the only primary ideals associated to M are powers of M. So if M is the radical of the principal ideal xR, the xR is a power of M. Consequently, if I is any ideal of R not contained in any singular maximal, then the primary decomposition of I is a product of maximal ideals and some power or I is itself principal. Now use Lemma 4.3 of [2, p. 135] to obtain that if I is invertible, then there is an ideal J isomorphic to I such that J is not contained in any singular maximal ideal. Since Pic(R) is in this case the class group of R (invertible ideals modulo the subgroup of principal fractional ideals), Pic(R) is a torsion group as desired. The domain hypothesis on R may be removed by the following observations: first, for Noetherian R, Pic(R) is still the class group, where invertibility is with respect to the total quotient ring of R. Secondly, Lemma 4.3 of [2] is valid for nondomains, as we now demonstrate.

LEMMA 3. *Let I be an invertible ideal of the Noetherian ring R and $P_1, \cdots, P_n$ prime ideals of R. Then there is an invertible integral ideal $J \cong I$ such that $J \not\subseteq P_i$, $i = 1, \cdots, n$.*

Proof. Without loss of generality the P_i's are distinct maximal ideals such that every associated prime of R is contained in some P_i. Select for each index i elements a_i, b_i such that $a_i \in I^{-1}$, $Ia_i \not\subseteq P_i$ and b_i belongs to all P_j's except P_i. Set $c = a_1b_1 + \cdots + a_nb_n$ and obtain that $c \in I^{-1}$ but $Ic \not\subseteq P_i$ for any index i. In particular, $J = Ic$ is contained in no associated prime of R, so J contains a nonzero-divisor. Therefore c is a unit in the total quotient ring of R and $J \cong I$.

THEOREM 4. *Let R be a one dimensional Noetherian ring with only finitely many singular maximal ideals. Then R is a* C. P.*-ring if and only if* Pic(R) *is a torsion group. In this case the integral closure of R (in its total quotient ring) is also a* C. P.*-ring.*

Proof. In view of Remark 2 and Lemma 3, the property of torsion Picard group is equivalent to the condition that every maximal ideal be the radical of a principal ideal. So Theorem 1 yields the first statement. For the second statement, let asteriks denote integral closures and N the nilradical of R. Since nonzero-divisors of R map to nonzero-divisors of R/N, we have that R^*/NR^* embeds naturally in $(R/N)^*$. Therefore we may as well assume that R is a reduced C. P.-ring and show that any ring S between R and R^* is a C. P.-ring. Let $P_1, \cdots, P_n$ be the minimal primes of R, so that $R^* = \pi(R/P_i)^*$. Also let Q_i be a minimal prime of S lying over P_i so that $S/Q_i \subseteq (R/P_i)^*$. By the Krull-Akizuki Theorem, S/Q_i is a one dimensional Noetherian domain (or a field), and since $\cap Q_i = 0$ we deduce that S is Noetherian ring of dimension at most one. Moreover if M is a nonsingular maximal ideal of R, then $R_M = R_M^* = (R^*)_M$; thus $R_M = S_M$. So if the prime ideal P of S lies over M, one obtains by checking locally that $MS = P$. Since the only primary ideals for M are powers of M, the C. P.-property ensures that some power of M is a principal ideal. Consequently, the same is true of P. To complete the proof, let I be an invertible ideal of S and use Lemma 3 to obtain an isomorphic invertible integral ideal J which is contained only in maximal ideals of S lying over nonsingular maximals of R. As in Remark 2, J is a product of the maximal ideals containing it. Hence a suitable power of J is principal and the proof is complete.

EXAMPLE (1). It would be interesting to know if Theorem 4 holds for all Noetherian rings (even domains). In this connection there do exist examples of Noetherian C. P.-domains with infinitely many singular maximals, namely those obtained from M. Nagata's construction in §5 of [3, p. 11].

(2) There exist non-Noetherian C. P.-domains which are even Prüfer domains of Krull dimension one, yet do not have torsion Picard group. Such an example is constructed by W. Heinzer in [1, p. 139]. In fact the QR-domains studied in this paper are exactly the Prüfer C. P.-rings.

(3) Let R be a one dimensional affine algebra over a field k contained in the algebraic closure of a finite field. Then $\mathrm{Pic}(R)$ is torsion by Lemma 2 of [9, p. 30], so R is a C. P.-ring.

(4) Theorem 1 implies that any valuation domain of finite Krull dimension is a C. P.-ring. In particular let V be a discrete valuation ring of rank two (see [7] for examples); i.e., the prime ideals of V are $0 \subset P \subset M$ and the rings V_P and V/P are both discrete rank one valuation rings. We assert that every nonzero homomorphic image of V has nonempty set of associated primes. It then follows from Theorem 5.16 of [5, p. 352] that V is semi-noetherian. Therefore a semi-noetherian C. P.-ring can have Krull dimension greater than one, which answers a question of Popescu [4] in the negative.

To prove the assertion let V/I be a nonzero homomorphic image of V. If, for some $x \in V$, $(I: x)$ is a proper ideal of V strictly larger than P, then $(I: x)$ is a power of M since V/P is discrete rank one. But then $x + I$ generates a cyclic submodule of V/I isomorphic to V/M^n and M is an associated prime of V/I. If no such x exists then the kernel of the canonical map $V/I \to (V/I)_P$ is zero, so V/I embeds as an essential V-submodule of $(V/I)_P$. But $I_P = P_P^n$ for some $n \geqq 1$ since V_P is discrete rank one. Choose $x \in P^{n-1} \backslash I_P$ and obtain that $(I_P: x) = P$. Thus P is an associated prime of the V-module V_P/P_P^n, whence P is an associated prime of the essential submodule V/I. This proves the assertion in all cases.

REFERENCES

1. W. Heinzer, *Quotient overrings of integral domains*, Mathematika, **17** (1970), 139–148.
2. M. Murthy and R. Swan, *Vector bundles over affine surfaces*, Invent. Math., **36** (1976), 125–165.
3. M. Nagata, *On the closedness of the singular loci*, Publs. Math. Inst. Haut. Êtud. Sci., **2** (1959), 29–36.
4. N. Popescu, *Sur les C. P.-anneaux*, C. R. Acad. Sc. Paris, **272** (1971) Sér. A, 1493–1496.
5. ———, *Abelian Categories with Applications to Rings and Modules*, Academic

Press, 1973.
6. C. Reis and T. Viswanathan, *A compactness property of prime ideals in Noetherian rings*, Proc. Amer. Soc., **25** (1970), 353-356.
7. O. Schilling, *The theory of valuations*, Mathematical Surveys No. 4, A.M.S., (1950).
8. W. Smith, *A covering condition for prime ideals*, Proc. Amer. Math. Soc., **30** (1971), 451-452.
9. R. Wiegand, *Homeomorphisms of affine rurfaces over a finite field*, J. London Math. Soc., (2) **18** (1978), 28-32.

Received May 30, 1980 and in revised form January 15, 1981.

ANDHRA UNIVERSITY
WALTAIR, A. P. 530 003
INDIA
AND
UNIVERSITY OF NEBRASKA
LINCOLN, NE 68588