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ABSOLUTE CONVERGENCE FIELDS OF SOME TRIANGULAR MATRIX METHODS

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Recently Das [2] has obtained results on the comparison of the absolute convergence fields between the Nörlund matrix and its product with the Cesàro matrix. In the present paper a similar investigation for the Riesz matrix (($\bar{N}$, p_n) matrix) is made.

1. Let $A = (a_{n,k})$ be an infinite lower triangular matrix, that is $a_{n,k} = 0$, if $k > n$, transforming sequence $s \equiv \{s_n\}$ into the sequence $A(s)$ defined by

$$A(s) = \{A_n(s)\} = \left\{\sum_{k=0}^{n} a_{n,k} s_k\right\} .$$

The sequence s is said to be absolutely summable A or summable $|A|$, if the transformed sequence $A(s)$ is of bounded variation, that is if $\sum_{n=1}^{\infty} |A_n(s) - A_{n-1}(s)| < \infty$. The absolute convergence field of A, denoted by $|A|$, is the set of all sequences which are summable $|A|$. The matrix A is said to be absolute conservative if $|I| \subseteqq |A|$, where I is the identity matrix.

Let $\{p_n\}$ be a sequence of constants, real or complex, such that $P_n = \sum_{k=0}^{n} p_k \neq 0$. When $a_{n,k} = (p_{n-k})/P_n$, A is called the (N, p_n) matrix and for $a_{n,k} = p_k/P_n$, A is called the $(\bar{N}, p_n)$ matrix. The $(\bar{N}, p_n)$ matrix for $p_n > 0$ and $P_n \to \infty$ is also denoted by the $(R, P_n, 1)$ matrix. When the sequence $\{p_n\}$ is such that $p_n = 1$ for all n, both the (N, p_n) and the $(\bar{N}, p_n)$ matrices reduce to the $(C, 1)$ matrix.

For two matrix methods A and B, AB transform of s is defined by $A(B(s))$. In particular,

$$\bar{t}_m(p, q) = \frac{1}{P_m} \sum_{k=0}^{m} \frac{p_k}{Q_k} \sum_{n=0}^{k} q_n s_n , \tag{1.1}$$

where $\bar{t}_n(p, q)$ denotes $(\bar{N}, p_n)(\bar{N}, q_n)$ transform of s.

Throughout the present paper we write $P_n^{(1)} = \sum_{k=0}^{n} p_k$, and for any sequence $\{\theta_n\}$, $\Delta n\theta_n = \Delta\theta_n = \theta_n - \theta_{n+1}$ and $\theta_n = 0$, if $n < 0$; K denotes a positive constant, not necessarily the same at each occurrence.

2. Concerning the relative inclusion of the absolute convergence fields of (N, p_n) and $(N, p_n)(C, 1)$, the following is known (see Das [2], Theorem 2 and Theorem 5).

THEOREM A. *Let the sequence* $\{p_n\}$ *be such that* $p_n > 0$ *and*

$p_{n+1}/p_n \leqq p_{n+2}/p_{n+1} \leqq 1$. *Then* $|N, P_n| \subseteqq |(C, 1)(N, p_n)|$.

Silverman [5] has shown that the (N, p_n) matrix is not permutable with the $(C, 1)$ matrix unless the (N, p_n) matrix is a Cesàro matrix. However, it has been proved that Theorem A is true even if the $(C, 1)(N, p_n)$ is permuted ([2], Theorem 4).

It has been proved (see Prasad and Pati (4)) that the absolute Riesz summability $|R, \lambda_n, r|$ implies the summability $|R, \phi(\lambda_n), r|$, provided, roughly speaking, the $\phi(x)$ is reasonable regular and does not increase more rapidly than a power of x. But from Lemma 4 we see that

$$(1.2) \qquad |\bar{N}, P_n| \subseteqq |\bar{N}, p_n|$$

if and only if $p_n P_n^{(1)} = O((P_n)^2)$.

The following theorems which we prove in the present paper show that if we consider the product of $(C, 1)$ and $(\bar{N}, p_n)$ in place of $(\bar{N}, p_n)$ in (1.2) the relation (1.2) holds good for a fairly wider class of sequences $\{p_n\}$.

THEOREM 1. *Let* $\{p_n\}$ *be a nonnegative sequence. Then* $|\bar{N}, P_n| \subseteqq |(C, 1)(\bar{N}, p_n)|$, *if*

$$(1.3) \qquad \frac{k p_k P_k^{(1)}}{P_k} \sum_{n=k+1}^{\infty} \frac{1}{n^2 P_n} \leqq K\,, \quad k = 1, 2, \cdots.$$

THEOREM 2. *Let* $\{p_n\}$ *be a nonnegative sequence. Then*

$$|\bar{N}, P_n| \subseteqq |(\bar{N}, p_n)(C, 1)|\,.$$

The condition (1.3) seems to be quite less restrictive but it is not true even for all nonnegative sequences; for if we consider the sequence $\{p_n\}$ such that $P_0 = \log 2$ and for $n > 0$, p_n is chosen to be either 0 or 1 in such a way that $\log(n + 2) \sim P_n$. It is easy to see that for this case (1.3) is not satisfied.

Concerning the inclusion relation between the absolute convergence fields of the $(\bar{N}, q_n)$ and the $(C, 1)(N, p_n)$ methods we prove the following.

THEOREM 3. *Suppose that* $\{p_n\}$ *is nonnegative nonincreasing sequence and that* $\{q_n\}$ *is positive and nondecreasing sequence. Then*

$$|\bar{N}, q_n| \subseteqq |(C, 1)(N, p_n)|\,.$$

It is interesting to observe that the relation $|\bar{N}, q_n| \subseteqq |(N, p_n)(C, 1)|$ also holds good follows from Lemma 4. Since for nondecreasing sequence $\{q_n\}$, $Q_n \leqq (n+1)q_n$, we see that with $\{q_n\}$ in place of $\{p_n\}$ and $q_n = 1$ in Lemma 4, the hypotheses of Lemma 4 are satisfied. Hence

$$|\bar{N}, q_n| \subseteqq |C, 1| . \tag{1.4}$$

But for nonnegative nonincreasing sequence $\{p_n\}$ it follows from Lemma 3 that (N, p_n) is absolutely regular. Hence $|\bar{N}, q_n| \subseteqq |(N, p_n)(C, 1)|$.

2. For the proof of the theorems we need the following results. In what follows we write $\alpha_n = p_{n+1}P_n^{(1)}/P_{n+1}$ and $C_m = m + 1 - P_m^{(1)}/P_{m+1}$.

LEMMA 1. *In order that any* $\{x_n\} \in |I|$ implies $\{x_n\} \in |A|$, *where* $A = (a_{m,n})$, *it is necessary and sufficient that* $\sum_{k=0}^{\infty} a_{n,k}$ *converges for all* n *and*

$$\sum_{n=0}^{\infty} \left| \sum_{k=0}^{m} (a_{n+1,k} - a_{n,k}) \right| \leqq K , \quad m = 0, 1, 2, \cdots .$$

LEMMA 1 is contained in ([6], Theorem 3).

LEMMA 2. *For* $m, n = 0, 1, 2, \cdots$

$$\text{(i)} \qquad \sum_{k=0}^{m} \left(\Delta \frac{p_k}{P_k} \right) P_k^{(1)} = P_m - \alpha_m ;$$

$$\text{(ii)} \qquad \sum_{k=0}^{m} \left(\Delta_k \frac{p_{n-k}}{q_k} \right) Q_k = P_n - P_{n-m-1} - \frac{p_{n-m-1}Q_m}{q_{m+1}} .$$

The proof of Lemma 2 is direct. The following lemma is contained in [3].

LEMMA 3. *If* $\{p_n\}$ *is nonegative, nonincreasing, then for all* $k \geqq 0$ *and* $1 \leqq a \leqq b \leqq \infty$,

$$\sum_{n=a}^{b} P(n, k) = \sum_{n=a}^{b} \left(\frac{P_{n-k}}{P_n} - \frac{P_{n-k-1}}{P_{n-1}} \right) \leqq 1 ,$$

and, for any $n > 0$, $P(n, k) \geqq 0$.

LEMMA 4. *Let* $q_n > 0$ *and* $p_n \neq 0$. *Then in order that* $|\bar{N}, p_n| \subseteqq |\bar{N}, q_n|$, *it is necessary and sufficient that* $P_n/p_n = O(Q_n/q_n)$.

The sufficiency part of the lemma in a less general form is due to Sunouchi [6]. The present form is due to Bosanquet ([1], p. 654).

3. *Proof of Theorem* 1. Let $\bar{t}_n(P)$ denote the $(\bar{N}, P_n)$ transform of $\{s_n\}$. We have

$$P_n s_n = P_n^{(1)}\bar{t}_n(P) - P_{n-1}^{(1)}\bar{t}_{n-1}(P) , \quad n = 0, 1, 2, \cdots ,$$

so that

$$\begin{aligned} \bar{t}_n(1, p) &= \frac{1}{n+1}\sum_{s=0}^{n}\frac{1}{P_s}\sum_{r=0}^{s}\frac{p_r}{P_r}(P_r^{(1)}\bar{t}_r(P) - P_{r-1}^{(1)}\bar{t}_{r-1}(P)) \\ &= \frac{1}{n+1}\sum_{s=0}^{n}\frac{1}{P_s}\left\{\sum_{r=0}^{s}\left(\Delta\frac{p_r}{P_r}\right)P_r^{(1)}\bar{t}_r(P) + \alpha_s\bar{t}_s(P)\right\} \\ &= \frac{1}{n+1}\sum_{r=0}^{n}\left\{\sum_{s=r}^{n}\left(\Delta\frac{p_r}{P_r}\right)\frac{P_r^{(1)}}{P_s} + \frac{\alpha_r}{P_r}\right\}\bar{t}_r(P) \\ &= \sum_{r=0}^{n}a_{n,r}\bar{t}_r(P) , \end{aligned} \tag{3.1}$$

say. Writing $\beta_{n,m} = \sum_{r=0}^{m} a_{n,r}$ and observing that $a_{n,r} = 0$ for $r > n$ we see that $\beta_{n,m} = 1$ for $n \leqq m$ and for $n \geqq m$

$$\beta_{n,m} = \frac{1}{n+1}\sum_{r=0}^{m}\left\{\sum_{s=r}^{n}\left(\Delta\frac{p_r}{P_r}\right)\frac{P_r^{(1)}}{P_s} + \frac{\alpha_r}{P_r}\right\} .$$

We first simplify $\beta_{n,m}$ for $n \geqq m$. By virtue of the result (i) of Lemma 2, we have[1]

$$\begin{aligned} &\sum_{r=0}^{m}\sum_{s=r}^{n}\left(\Delta\frac{p_r}{P_r}\right)\frac{P_r^{(1)}}{P_s} \\ &\quad = \sum_{r=0}^{m}\left(\sum_{s=r}^{m} + \sum_{s=m+1}^{n}\right)\left(\Delta\frac{p_r}{P_r}\right)\frac{P_r^{(1)}}{P_s} \\ &\quad = \sum_{s=0}^{m}\frac{1}{P_s}\sum_{r=0}^{s}P_r^{(1)}\left(\Delta\frac{p_r}{P_r}\right) + \sum_{s=m+1}^{n}\frac{1}{P_s}\sum_{r=0}^{m}P_r^{(1)}\left(\Delta\frac{p_r}{P_r}\right) \\ &\quad = m + 1 + (P_m - \alpha_m)\sum_{s=m+1}^{n}1/P_s - \sum_{s=0}^{m}\alpha_s/P_s \end{aligned}$$

so that

$$\beta_{n,m} = \frac{m+1}{n+1} + \frac{1}{n+1}(P_m - \alpha_m)\sum_{s=m+1}^{n}\frac{1}{P_s} . \tag{3.2}$$

In order to prove the theorem, it is sufficient to show that the matrix $(a_{n,r})$ in (3.1) is absolutely conservative. From Lemma 1, we see that the matrix $(a_{n,r})$ will be absolutely conservative if

$$\sum \equiv \sum_{n=m}^{\infty}|\beta_{n+1,m} - \beta_{n,m}| \leqq K , \quad m = 0, 1, 2, \cdots , \tag{3.3}$$

since $\beta_{n,m} = 1$ for $n \leqq m$. From (3.2) we get

[1] We assume here onwards $\sum_a^b = 0$ if $b < a$.

$$\sum = \sum_{n=m}^{\infty} \left| \frac{m+1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)}(P_m - \alpha_m) \sum_{s=m+1}^{n} \frac{1}{P_s} - \frac{1}{(n+2)P_{n+1}}(P_m - \alpha_m) \right| .$$

Evidently,

$$\sum \leqq \sum_{n=m}^{\infty} (m+1)/(n+1)(n+2) + R(m) + L(m) , \tag{3.4}$$

where

$$R(m) = P_m \sum_{n=m}^{\infty} \left| \frac{1}{(n+1)(n+2)} \sum_{s=m+1}^{n} \frac{1}{P_s} - \frac{1}{(n+2)P_{n+1}} \right|$$

and $L(m) = \alpha_m R(m)/P_m$.

We have

$$\begin{aligned} R(m) &\leqq P_m \sum_{n=m}^{\infty} \frac{1}{(n+1)(n+2)} \left| \sum_{s=m+1}^{n} \frac{1}{P_s} - \frac{n-m}{P_{n+1}} \right| \\ &\quad + P_m \sum_{n=m}^{\infty} \frac{m+1}{(n+1)(n+2)P_{n+1}} \\ &= P_m X(m) + P_m Y(m) , \quad \text{say} . \end{aligned}$$

In view of the fact that for nonnegative sequence $\{p_n\}$, $\{P_n\}$ is nondecreasing, it follows that

$$P_m Y(m) \leqq K . \tag{3.5}$$

Observing that $\sum_{s=m}^{n} 1/P_s > (n-m)/P_n$, we obtain

$$X(m) = \sum_{n=m}^{\infty} \left\{ \frac{1}{(n+1)(n+2)} \sum_{s=m+1}^{n} \frac{1}{P_s} - \frac{1}{(n+2)P_{n+1}} \right\} + Y(m) .$$

We now prove that $P_m X(m) \leqq K$. For, we first estimate

$$X^*(N, m) = \sum_{n=m}^{N} \left\{ \frac{1}{(n+1)(n+2)} \sum_{s=m+1}^{n} \frac{1}{P_s} - \frac{1}{(n+2)P_{n+1}} \right\} .$$

First changing the order of summation and then using that $\sum_{n=s}^{N} 1(n+1)n = 1/s - 1/(N+1)$, we get

$$X^*(N, m) = \sum_{s=m+1}^{N} \frac{1}{(s+1)P_s} - \frac{1}{N+2} \sum_{s=m+1}^{N} \frac{1}{P_s} - \sum_{n=m+1}^{N+1} \frac{1}{(n+1)P_n} . \tag{3.6}$$

If $P_n \to \infty$ as $n \to \infty$, it follows from (3.6) that $X^*(N, m) \to 0$ as $N \to \infty$. If $P_n \not\to \infty$, then, since $\{P_n\}$ is nondecreasing, $P_n \to a$ finite limit P, say, as $n \to \infty$; and in this case $X^*(N, m) \to -1/P$. In view of this and (3.5) it follows that

$$P_m X(m) \leqq K . \tag{3.7}$$

From (3.5) and (3.7) we get

$$R(m) \leqq K . \tag{3.8}$$

Now we estimate $L(m)$. We have

$$L(m) \leqq \alpha_m(X(m) + Y(m)) .$$

From the hypothesis (1.3) we see that α_n is bounded whenever $P_n \leqq K$. Using this fact and the observations made just after (3.5), we see that $\alpha_m X(m) \leqq K$. That $\alpha_m y(m) \leqq K$ follows from the hypothesis (1.3). Thus $L(m) \leqq K$. This together with (3.8) and (3.4) yields $\Sigma = O(1)$, since the first term in (3.4) is bounded. This proves Theorem 1.

Proof of Theorem 2. Following closely the proof for (3.1) we see that

$$\begin{aligned}\bar{t}_n(p, 1) &= \frac{1}{P_n}\sum_{r=0}^{n}\left\{\sum_{s=r}^{n}\frac{p_s P_r^{(1)}}{s+1}\left(\Delta\frac{1}{P_r}\right) + \frac{p_r P_r^{(1)}}{(r+1)P_{r+1}}\right\}\bar{t}_r(P) \\ &= \sum_{r=0}^{n} a_{n,r}\bar{t}_r(P) .\end{aligned}$$

Thus, in this case

$$\beta_{n,m} = \frac{1}{P_n}\sum_{r=0}^{m}\left\{\sum_{s=r}^{n}\frac{p_s P_r^{(1)}}{s+1}\left(\Delta\frac{1}{P_r}\right) + \frac{p_r P_r^{(1)}}{(r+1)P_{r+1}}\right\}$$

for $n \geqq m$ and $\beta_{n,m} = 1$ for $m \geqq n$.

Using the technique, with the result (ii) in place of (1) of Lemma 2, for obtaining (3.2) we see that

$$\beta_{n,m} = P_m/P_n + (C_m/P_n)\sum_{s=m+1}^{n} p_s/(s+1) .$$

Now we proceed to prove that for this case also (3.3) holds good. We have

$$\begin{aligned}\Sigma &\equiv \sum_{n=m}^{\infty}\left|\frac{p_{n+1}P_m}{P_nP_{n+1}} + C_m\frac{p_{n+1}}{P_{n+1}}\left(\frac{1}{P_n}\sum_{s=m+1}^{n}\frac{p_s}{s+1} - \frac{1}{n+2}\right)\right| \\ &\leqq C_m\sum_{n=m}^{\infty}\frac{p_{n+1}}{P_{n+1}}\left|\frac{1}{P_n}\sum_{s=m+1}^{n}\frac{p_s}{s+1} - \frac{P_n - P_m}{(n+2)P_n}\right| \\ &\quad + C_mP_m\sum_{n=m}^{\infty}\frac{p_{n+1}}{(n+2)P_nP_{n+1}} + P_m\sum_{n=m+1}^{\infty}\frac{p_{n+1}}{P_nP_{n+1}} \\ &= \Sigma_1 + \Sigma_2 + \Sigma_3 .\end{aligned}$$

To prove that Σ_1 is bounded we first consider the following sum

$$\Sigma(N) = \sum_{n=m+1}^{N}\frac{p_{n+1}}{P_{n+1}}\left|\frac{1}{P_n}\sum_{s=m+1}^{n}\frac{p_s}{s+1} - \frac{P_n - P_m}{(n+2)P_n}\right| .$$

Observing that $(n+2)\sum_{s=m+1}^{n} p_s/(s+1) > P_n - P_m$ we see that the expression under the modulus sign in $\Sigma(N)$ is nonnegative. Hence by a change of order of summation we get

$$\Sigma(N) = \sum_{s=m+1}^{N} \frac{p_s}{s+1} \sum_{n=s}^{N} \frac{p_{n+1}}{P_n P_{n+1}} - \sum_{n=m+1}^{N} \frac{p_{n+1}}{(n+2)P_{n+1}} + P_m \sum_{s=m+1}^{N} \frac{p_{n+1}}{(n+2)P_n P_{n+1}} .$$

Since

$$\sum_{n=s}^{N} p_{n+1}/P_n P_{n+1} = 1/P_s - 1/P_{N+1} \tag{3.9}$$

and $p_n/P_n \leqq 1$, we have

$$\begin{aligned} \Sigma(N) &\leqq \frac{K}{m+1} + \frac{1}{P_{N+1}} \sum_{s=m}^{N} \frac{p_s}{s+1} + \frac{P_m}{m+3} \sum_{s=m+1}^{N} \frac{p_{n+1}}{P_n P_{n+1}} \\ &\leqq \frac{K}{m+1} + \frac{1}{(m+1)P_{N+1}} \sum_{s=0}^{N} p_s \leqq \frac{K}{m+1} . \end{aligned} \tag{3.10}$$

It is clear that the term of Σ_1 for $n = m$ is bounded. In view of this and (3.10) we get that Σ_1 is bounded, since $C_m \leqq Km$ by the fact that $P_m^{(1)} \leqq (m+1)P_m$. That Σ_2 and Σ_3 are bounded follows from (3.9). Thus we get that $\Sigma \leqq K$ for all m. This completes the proof of the theorem.

Proof of Theorem 3. It is interesting to observe from the result (1.4) that to prove Theorem 3, it is sufficient to show that $|C, 1| \subseteqq |(C, 1)(N, p_n)|$, which is just special case of Theorem 3 when $(\bar{N}, q_n)$ is $(C, 1)$. But to prove this special case we require the same argument (except minor simplification of the method of the proof) as for the general case. In order to give a direct proof we consider the general case.

Let $t_n(1, p)$ denote $(C, 1)(N, p_n)$ transform of the sequence $\{s_n\}$. We have

$$t_n(1, p) = \frac{1}{n+1} \sum_{r=0}^{n} \left\{ \sum_{k=r}^{n} \left(\Delta_r \frac{p_{k-r}}{q_r} \right) \frac{Q_r}{P_k} \right\} \bar{t}_r(q) = \sum_{r=0}^{n} a_{n,r} \bar{t}_r(q) .$$

So far the case

$$\beta_{n,m} = \frac{1}{n+1} \sum_{r=0}^{m} \sum_{k=r}^{n} \left(\Delta_r \frac{p_{k-r}}{q_r} \right) \frac{Q_r}{P_k} .$$

It is clear that $\beta_{n,m} = 1$ if $m \geqq n$. Simplifying by using Lemma 2(ii) we see that for $m \leqq n$

$$\beta_{n,m} = 1 - \frac{1}{n+1}\sum_{k=m+1}^{n}\left(\frac{P_{k-m-1}}{P_k} + \frac{Q_m p_{k-m-1}}{q_{m+1}P_k}\right).$$

Now we prove that (3.3) is true for this case also. We have

$$\begin{aligned}
\Sigma &\equiv \sum_{n=m}^{\infty}\left|\frac{1}{n+2}\left(\frac{1}{n+1}\sum_{k=m+1}^{n}\frac{P_{k-m-1}}{P_k} - \frac{P_{n-m}}{P_{n+1}}\right)\right. \\
&\qquad \left. - \frac{Q_m}{q_{m+1}}\left(\frac{p_{n-m}}{(n+2)P_{n+1}} - \frac{1}{(n+1)(n+2)}\sum_{k=m+1}^{n}\frac{p_{k-m-1}}{P_k}\right)\right| \\
&\leqq \lim_{M\to\infty}\sum_{n=m}^{M}\left|\frac{P_{n-m}}{(n+2)P_{n+1}} - \frac{1}{(n+2)(n+1)}\sum_{k=m+1}^{n}\frac{P_{k-m-1}}{P_k}\right| \\
&\qquad + \frac{Q_m}{q_{m+1}}\sum_{n=m}^{\infty}\frac{p_{n-m}}{(n+2)P_{n+1}} + \frac{Q_m}{q_{m+1}}\sum_{n=m}^{\infty}\frac{1}{(n+1)(n+2)}\sum_{k=m+1}^{n}\frac{p_{k-m-1}}{P_k} \\
&= \lim_{M\to\infty}\Sigma'(M) + \Sigma'' + \Sigma''' ,
\end{aligned} \tag{3.11}$$

say. We first consider $\Sigma'(M)$. Since for nondecreasing sequence $\{p_n\}$, $\{P_{k-m-1}/P_k\}$ is nondecreasing in k for $k > m$, we get that the expression under the modulus sign in $\Sigma'(M)$ is nonnegative. By a change of order of summation we obtain

$$\begin{aligned}
\Sigma'(M) &= \sum_{n=m+1}^{M}\frac{P_{n-m}}{(n+2)P_{n+1}} - \sum_{k=m+1}^{M}\frac{P_{k-m-1}}{(k+1)P_k} + \frac{1}{M+2}\sum_{k=m+1}^{M}\frac{P_{k-m-1}}{P_k} \\
&\leqq \frac{P_{M-m}}{(M+2)P_{M+1}} - \frac{p_0}{(m+2)P_{m+1}} + \frac{1}{M+2}\sum_{k=m+1}^{M}1 \leqq K .
\end{aligned}$$

Hence

$$\sum{}'(M) = O(1) . \tag{3.12}$$

To prove the boundedness of Σ'' and Σ''' we first estimate the following sum. Observing that $(n-m+1)p_{n-m} \leqq P_{n-m} \leqq P_{n+1}$ we see that for $a = 1$ or 2

$$\begin{aligned}
\sum_{n=m}^{\infty}\frac{p_{n-m}}{(n+a)P_{n+1}} &= \sum_{n=m}^{2m}\frac{p_{n-m}}{(n+a)P_{n+1}} + \sum_{n=2m+1}^{\infty}\frac{p_{n-m}}{(n+a)P_{n+1}} \\
&\leqq \frac{P_m}{(m+a)P_{m+1}} + \sum_{n=2m+1}^{\infty}\frac{1}{(n-m)(n-m+1)} \leqq \frac{K}{m+1} .
\end{aligned} \tag{3.13}$$

Since for nondecreasing sequence $\{q_n\}$, $Q_n \leqq (n+1)q_n$, we obtain from (3.13) that

$$\Sigma'' < \infty . \tag{3.14}$$

Applying the above reasoning after a change of order of summation, we see that

$$\Sigma''' = \frac{Q_m}{q_{m+1}}\sum_{k=m+1}^{\infty}\frac{p_{k-m-1}}{(k+1)P_k} < \infty . \tag{3.15}$$

That Σ is bounded follows when we use (3.12), (3.14) and (3.15) in (3.11). This completes the proof of Theorem 3.

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