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ORDERS OF FINITE ALGEBRAIC GROUPS

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Let $\bar{G}$ be a simply connected simple algebraic group over a finite field F_q of q elements. The order of the group $\bar{G}(F_q)$ of F_q-rational points of $\bar{G}$ is well-known (cf: Steinberg, Carter). The proof makes use of the Bruhat decomposition and the study of polynomials invariant under the action the Weyl group. In this paper we deduce the order of $\bar{G}(F_q)$ from an explicit formula for the integral $M(s, \Lambda)$ which occurs in Langlands' theory of Eisenstein series.

First of all, according to a theorem of Lang $\bar{G}$ is quasi-split (cf: Lang [9], Satake [13] p. 105) and from Steinberg's theorem (cf: Steinberg [14], Kneser [6] p. 255) $\bar{G}$ is either a Chevalley group or a twisted group of one of the following types: ${}^2A_l(l \geqq 2)$, ${}^2D_l(l \geqq 4)$, ${}^2E_6$, ${}^3D_4$, ${}^2B_2$, ${}^2G_2$ and ${}^2F_4$. To simplify matters we shall assume that the characteristic of F_q is not 2 and 3 and exclude groups of the type ${}^2B_2$, ${}^2G_2$ and ${}^2F_4$. Furthermore we can assume that there exists a quasi-split simple algebraic group G defined over a p-adic number field F such that the residue field of F is isomorphic to F_q, G splits over an unramified Galois extension E of F and G reduces modulo p to $\bar{G}$ (cf: Weil [17]).

1. Fix a Haar measure dx on F such that the volume of the ring R of p-adic integers in F is one. Let ω be a left invariant highest F-differential form on G. Then ω and dx determines a Haar measure on $G(F)$ which will also be denoted by ω (cf: Weil [17]).

LEMMA 1. *Let m be the dimension of $\bar{G}$ and $|\bar{G}(F_q)|$ be the order of $\bar{G}(F_q)$. Then*

$$|\bar{G}(F_q)| = q^m \int_{G(R)} \omega \,. \tag{1}$$

This is proved in Weil [17] p. 22.

2. Let B be a Borel subgroup of G defined over F and A a maximal torus of G in B. Then by assumption the Galois group $\mathrm{Gal}\,(E/F)$ acts on the group $X(A)$ of rational characters of A. This gives rise to a representation

$$\pi\colon \mathrm{Gal}\,(E/F) \longrightarrow \mathrm{Eng}\,(X(A) \otimes_{Z} Q) \,.$$

For $s \in \boldsymbol{C}$, let

$$L(s, A) = (\det (I - q^{-s}\pi(\sigma)))^{-1}\ , \tag{2}$$

where σ is the Frobenius automorphism in $\mathrm{Gal}(E/F)$.

Let N (resp. $\bar{N}$) be the unipotent radical of B (resp. the Borel subgroup opposite to B). For $g \in G(F)$, if $g = nak$ with $n \in N(F)$, $a \in A(F)$ and $k \in G(R)$, is the Iwasawa decomposition of g, then we denote a by $a(g)$.

The data (G, B, A) determined a root system Σ, a subset Σ^+ of positive roots, and a basis Δ of Σ. Let ρ be the half sum of the positive roots in Σ. Then ρ defines a homomorphism on $A(F)$. (We denote this homomorphism also by ρ.) Let

$$M = \int_{\bar{N}} \rho^2(a(\bar{n}))d\bar{n}\ . \tag{3}$$

M is in fact a special value of the linear transformation $M(s, \Lambda)$ (cf: Langlands [10] p. 237) in the case Λ is ρ and s is the Weyl group element which takes all the positive roots to negative roots. Rapoport ([12] p. 4–10) showed that

$$M = L(1, A)\int_{G(R)} \omega\ . \tag{4}$$

Comparing with (1) we get

$$|\bar{G}(\boldsymbol{F}_q)| = q^m M L(1, A)^{-1}\ . \tag{5}$$

We shall use this formula to calculate $|\bar{G}(\boldsymbol{F}_q)|$.

3. As we have already pointed out Steinberg's theorem implies that G is obtained by twisting a F-split group $\tilde{G}$ by a one-cocycle

$$\sigma \longrightarrow \phi_\sigma \in Z^1(\mathrm{Gal}(E/F),\ \mathrm{Aut}\,\tilde{G})\ .$$

And ϕ_σ comes from the action of the Frobenius σ on the Dynkin diagram of G. In fact, if we denote the action of σ on $\alpha \in \Delta$ by $\sigma\alpha$, then this means that $\alpha \circ \phi_\sigma = \sigma\alpha$. Moreover, since Δ forms a basis of $X(A) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$, the representation π in §2 is determined by the effect of $\pi(\sigma)$ on Δ. And for $\alpha \in \Delta$ we have

$$\pi(\sigma)\alpha = \sigma\alpha\ . \tag{6}$$

LEMMA 2. *Let Ω_0 be the set of orbits of σ in Δ. For $\mathcal{O} \in \Omega_0$, $|\mathcal{O}|$ denotes the order of the orbit $\mathcal{O}$. Then for $s \in \boldsymbol{C}$,*

$$L(s, A) = \prod_{\mathcal{O} \in \Omega_0} (1 - q^{-s|\mathcal{O}|})^{-1} . \tag{7}$$

Proof. Suppose $\mathcal{O} \in \Omega_0$ and $|\mathcal{O}| = n$, then we can write

$$\mathcal{O} = \{\alpha, \sigma\alpha, \cdots, \sigma^{n-1}\alpha\}, \text{ for some } \alpha \in \Delta . \tag{8}$$

Let $\pi(\sigma, \mathcal{O})$ be the restriction of $\pi(\sigma)$ to the subspace of $X(A) \otimes \boldsymbol{Q}$ spanned by $\mathcal{O}$. Then with the basis in the order listed in (8), it is a trivial consequence of (6) that

$$\begin{aligned} \det(1 - q^{-s}\pi(\sigma, \mathcal{O})) \\ = 1 - q^{-sn} . \end{aligned} \tag{9}$$

Finally noting that the action of σ is broken into orbits, we see that the matrix of $\pi(\sigma)$ is broken into blocks on the diagonal. Since each block has determinant similar to that given in (9), the lemma is proved.

4. In $X(A) \otimes \boldsymbol{R}$ we choose a scalar product $(\cdot, \cdot)$ invariant under the Weyl group of the root system Σ. For every root $\alpha \in \Sigma$, we denote by $\hat{\alpha}: X(A) \otimes \boldsymbol{R} \to \boldsymbol{R}$ the linear form defined by

$$\hat{\alpha}(\xi) = 2(\xi, \alpha)/(\alpha, \alpha) .$$

Let $\hat{\Sigma} = \{\hat{\alpha} | \alpha \in \Sigma\}$ and $\hat{\Delta} = \{\hat{\alpha} | \alpha \in \Delta\}$. Then $\hat{\Sigma}$ is a root system with base $\hat{\Delta}$. It is well-known that there exist a complex semisimple Lie algebra $\hat{\mathfrak{g}}$ and a Cartan subalgebra $\hat{\mathfrak{a}}$ of $\hat{\mathfrak{g}}$ such that the root system of $(\hat{\mathfrak{g}}, \hat{\mathfrak{a}})$ is $\hat{\Sigma}$ and the Cartan matrix of $\hat{\mathfrak{g}}$ is the transpose of

$$(2(\alpha_i, \alpha_j)/(\alpha_j, \alpha_j))_{i \leqq i, j \leqq l}$$

if $\Delta = \{\alpha_1, \cdots, \alpha_l\}$.

Choose $H_1, \cdots H_l \in \hat{\mathfrak{a}}$ so that $\xi(H_i) = \xi(\alpha_i)$ for all $\xi \in \mathrm{Hom}_{\boldsymbol{Z}}(X(A), \boldsymbol{Z})$. In $\hat{\mathfrak{g}}$ choose root vectors $X_{\pm\hat{\alpha}_i}$ belonging to $\pm\hat{\alpha}_i$ such that

$$[X_{\hat{\alpha}_i}, X_{-\hat{\alpha}_i}] = H_i \quad 1 \leqq i \leqq l .$$

If $\sigma \in \mathrm{Gal}(E/F)$, let $\sigma\hat{\alpha} = \widehat{\sigma\alpha}$ for $\alpha \in \Delta$. Extend this action linearly to $\hat{\Sigma}$. This means that the orbits of σ in $\hat{\Sigma}$ and Σ corresponds bijectively. Moreover there is an automorphism (also denoted by) σ of the Lie algebra $\hat{\mathfrak{g}}$ so that

$$\sigma(H_i) = H_j \text{ and } \sigma X_{\hat{\alpha}_i} = X_{\hat{\alpha}_j} ,$$

if $\sigma\hat{\alpha}_i = \hat{\alpha}_j$ (cf: Jacobson [5] Chap. VII).

Let $\hat{\Omega}$ (resp. $\hat{\Omega}_0$) denote the set of orbits of the Frobenius automorphism $\sigma(\varepsilon\ \mathrm{Gal}(E/F))$ acting on $\hat{\Sigma}^+$ (resp. $\hat{\Delta}$). Let $\hat{\Omega}_1$ be the set of those elements of $\hat{\Omega}$ not in $\hat{\Omega}_0$. We shall pick a representative in each orbit and label each orbit by the corresponding representative. Suppose

$$\mathcal{O}(\hat{\alpha}) = \{\hat{\alpha}, \sigma\hat{\alpha}, \cdots, \sigma^{n-1}\hat{\alpha}\}$$

is such an orbit, we use $n(\hat{\alpha})$ to denote the order of the orbit $\mathcal{O}(\hat{\alpha})$ and define the number $\eta(\hat{\alpha})$ by

$$\sigma^n X_{\hat{\alpha}} = \eta(\hat{\alpha}) X_{\hat{\alpha}} . \tag{10}$$

5. Following Bhanu-Murthy [1], Gindikin-Karpelevich [4] and Langlands [11], Lai [8] (p. 56) has calculated the integral (3)—in the notations of previous section—

$$M = \prod_{\mathcal{O}(\hat{\alpha}) \in \hat{\Omega}} \frac{1 - \eta(\hat{\alpha}) q^{-n(\hat{\alpha})(1+\hat{\alpha}(\rho))}}{1 - \eta(\hat{\alpha}) q^{-n}(\hat{\alpha})\hat{\alpha}(\rho)} . \tag{11}$$

THEOREM. *Let $\bar{G}$ be a simply connected simple algebraic group over a finite field $\boldsymbol{F}_q$ of q elements. $\bar{G}$ not of the types ${}^2B_2$, ${}^2G_2$ and ${}^2F_4$. Then the order of the $\boldsymbol{F}_q$-rational points is given by*

$$|\bar{G}(\boldsymbol{F}_q)| = q^m \frac{\prod_{\mathcal{O}(\hat{\alpha}) \in \hat{\Omega}} (1 - \eta(\hat{\alpha}) q^{-n(\hat{\alpha})(1+\hat{\alpha}(\rho))})}{\prod_{\mathcal{O}(\hat{\alpha}) \in \hat{\Omega}_1} (1 - \eta(\hat{\alpha}) q^{-n(\hat{\alpha})\hat{\alpha}(\rho)})} . \tag{12}$$

More specifically, if $\bar{G}$ is a Chevalley group and d_i, $1 \leqq i \leqq l$ its exponents then

$$|\bar{G}(\boldsymbol{F}_q)| = q^m \prod_{i=1}^{l} (1 - q^{-d_i}) . \tag{13}$$

For the twisted groups we have

$$|{}^2A_l(\boldsymbol{F}_q)| = q^{l(l+1)/2} \prod_{i=2}^{l+1} (q^i - (-1)^i) \tag{14}$$

$$|{}^2D_l(\boldsymbol{F}_q)| = q^{l(l-1)}(q^l + 1) \prod_{i=1}^{l-1} (q^{2i} - 1) \tag{15}$$

$$|{}^3D_4(\boldsymbol{F}_q)| = q^{12}(q^2 - 1)(q^6 - 1)(q^8 + q^4 + 1) \tag{16}$$

$$|{}^2E_6(\boldsymbol{F}_q)| = q^{36}(q^2 - 1)(q^5 + 1)(q^6 - 1)(q^8 - 1)(q^9 + 1)(q^{12} - 1) . \tag{17}$$

Proof. As we have already pointed out the orbits of σ in Σ and $\hat{\Sigma}$ corresponds, so we can substitute (7) and (11) into (5) to get (12) if we note that $\hat{\alpha}(\rho) = 1$ and $\eta(\hat{\alpha}) = 1$ for $\hat{\alpha} \in \hat{\Delta}$. We shall use (12) to calculate $|\bar{G}(\boldsymbol{F}_q)|$ case by case.

6. In case G is a Chevalley group the calculation is well-known.

We shall only sketch the arguments. The exponents d_i are integers so that $\prod_{i=1}^{l}(1+t^{2d_i-1})$ is the Poincaré polynomial of the compact real form of G. According to an observation of Shapiro and of Steinberg [15] (for proof see Kostant [7]), the set of numbers

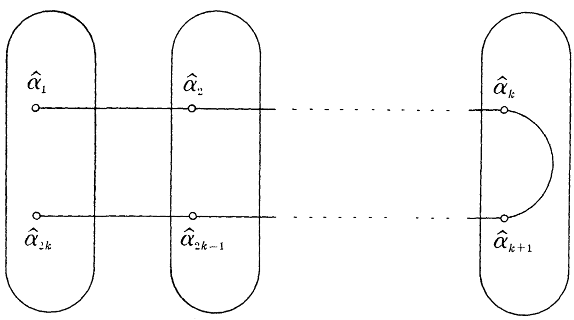
$$\{1+\hat{\alpha}(\rho)\}_{\hat{\alpha}\in\hat{\Sigma}^+}$$

is the same as the set of numbers

$$\{\hat{\alpha}(\rho)\}_{\hat{\alpha}\in\hat{\Sigma}^+-\hat{\Delta}} \cup \{d_i\}_{1\leq i\leq l}\ .$$

Together with the fact that σ acts trivially so that $\hat{\Omega}=\hat{\Sigma}^+$, $\hat{\Omega}_0=\hat{\Delta}$ and $n(\hat{\alpha})=\eta(\hat{\alpha})=1$, we get from (12) immediately the formula (13). This formula is due to Chevalley [3].

7. The case of the twisted group ${}^2A_{2k}$ i.e., the group obtained from a group of type A_{2k} by a twisting by the Frobenius of the quadratic extension E/F. The Dynkin diagram is



We have circled together the elements of the same orbit.

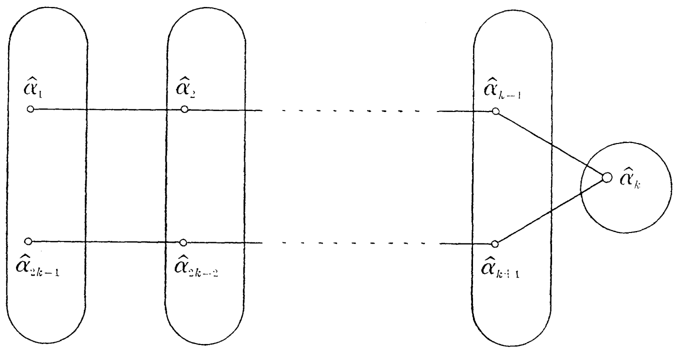
Hereafter we shall use an abbreviation for a root by indicating only the coefficients. For example in the case of ${}^2A_4$, the symbol (1 1 0 0) denotes the root $\hat{\alpha}_1+\hat{\alpha}_2$; in the case of ${}^2A_{2k}$, the symbol $(0\cdots 0\,1\,1\,\vdots\,1\,1\,0\cdots 0)$ denotes the root $\hat{\alpha}_{k-1}+\hat{\alpha}_k+\hat{\alpha}_{k+1}+\hat{\alpha}_{k+2}$.

By looking at root table it is easy to see that the orbits in $\hat{\Sigma}^+$ have either one or two roots. In fact the one element orbits: $\mathcal{O}(\hat{\alpha})$ are $(0\cdots 0\,1\,\vdots\,1\,0\cdots 0)$, $(0\cdots 0\,1\,1\,\vdots\,1\,1\,0\cdots 0)$, $\cdots(0\,1\cdots 1\,\vdots\,1\cdots 1\,0)$ and $(1\cdots 1\,1\cdots 1)$; the respective values of $\hat{\alpha}(\rho)$ are $2, 4, \cdots, 2k-1$ and $2k$. The value of $\eta(\hat{\alpha})$ in each of these cases is -1. The rest of the orbits have two roots in each of them and in all these cases $\eta(\hat{\alpha})=1$. It is easy to see that $\hat{\alpha}(\rho)$ equal to the number of 1's in the symbol of $\hat{\alpha}$. One now writes down all the factors occurring in the right hand side of (12). After removing those terms that occur both in the numerator and denominator we get

$$\frac{q^m(1-q^{-4})(1+q^{-3})\cdots(1-q^{-4k})(1+q^{-(2k+1)})}{(1+q^{-2})\cdots(1+q^{-2k})}=q^m\prod_{i=2}^{2k+1}(1-(-1)^i q^{-i})\ .$$

Since $m = 4(k^2 + k)$, we get (14) for the case $l = 2k$.

8. Similar computations can be made in the case G is of the type ${}^2A_{2k-1}$. The orbits of the twisted action on the Dynkin diagram are



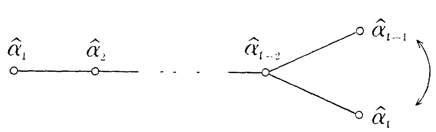
The right hand side of (12) after removing like terms from the numerator and denominator is

$$\frac{q^m(1-q^{-2})(1-q^{-6})\cdots(1-q^{-2(2k-1)})(1-q^{-2k})}{(1-q^{-3})\cdots(1-q^{-(2k-1)})} = q^m \prod_{i=2}^{2k}(1-(-1)^i q^{-i})$$

$$= q^{k(2k-1)} \prod_{i=2}^{2k}(q^i-(-1)^i) .$$

This is just (14) when $l = 2k - 1$.

9. As for the case of ${}^2D_l$, the Dynkin diagram is

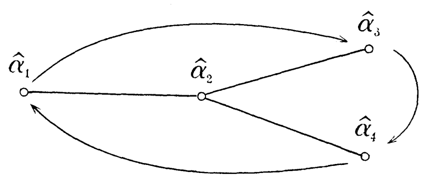


We have used an arrow to indicate the action of σ. That is σ interchanges $\hat{\alpha}_{l-1}$ and $\hat{\alpha}_l$ and fixes the rest of the simple roots. Using root tables to calculate the right hand side of (12), we found, after removing the like terms from the numerator and denominator, that the denominator has only one term $1 - q^{-l}$ corresponding to the root $\hat{\alpha}_1 + \cdots + \hat{\alpha}_l$; whereas, the numerator contains the terms $1 - q^{-2}$, $1 - q^{-4}$, $1 - q^{-6}$, $1 - q^{-8}$, $\cdots$, $1 - q^{-(2l-2)}$ corresponding to the roots $\hat{\alpha}_1$, $\hat{\alpha}_1 + \hat{\alpha}_2 + \hat{\alpha}_3$, $\hat{\alpha}_{l-3} + 2\hat{\alpha}_{l-2} + \hat{\alpha}_{l-1} + \hat{\alpha}_l$, $\hat{\alpha}_{l-4} + 2(\hat{\alpha}_{l-3} + \hat{\alpha}_{l-2}) + \hat{\alpha}_{l-1} + \hat{\alpha}_l$, $\cdots$, $\hat{\alpha}_1 + 2(\hat{\alpha}_2 + \cdots + \hat{\alpha}_{l-2}) + \hat{\alpha}_{l-1} + \hat{\alpha}_l$ respectively, and the term $1 - q^{-2l}$ corresponding to the orbit $\{(\hat{\alpha}_1 + \cdots + \hat{\alpha}_{l-1}, \hat{\alpha}_l + \cdots + \hat{\alpha}_{l-2} + \hat{\alpha}_l)$. As a result we get

$${}^2D_l(F_q) = q^m(1-q^{-2})\cdots(1-q^{-(2l-2)})(1-q^{-2l}(1-q^{-l})^{-1} .$$

Since $m = 2l^2 - l$ we get (15) immediately.

10. The next case is ${}^3D_4$. The Dynkin diagram with the action of Frobenius indicated by arrows is:



Direct calculation shows that for all orbits $\eta(\hat{\alpha}) = 1$. The values of $n(\hat{\alpha})$ and $\hat{\alpha}(\rho)$ are given in the following table.

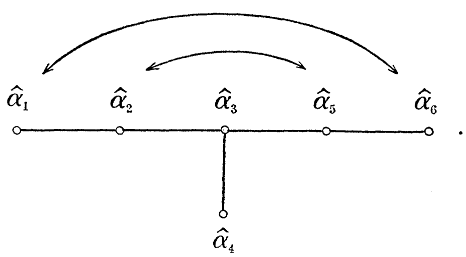
	$\mathcal{O}(\hat{\alpha})_1$	$\mathcal{O}(\hat{\alpha}_2)$	$\mathcal{O}(\hat{\alpha}_1+\hat{\alpha}_2)$	$\mathcal{O}(\hat{\alpha}_1+\hat{\alpha}_2+\hat{\alpha}_3)$	$\mathcal{O}(\hat{\alpha}_1+\hat{\alpha}_2+\hat{\alpha}_3+\hat{\alpha}_4)$	$\mathcal{O}(\hat{\alpha}_1+2\hat{\alpha}_2+\hat{\alpha}_3+\hat{\alpha}_4)$
$n(\hat{\alpha})$	3	1	3	3	1	1
$\hat{\alpha}(\rho)$	1	1	2	3	4	5

In this case $m = 28$. The right hand side of (12) is

$$\frac{q^{28}(1-q^{-6})(1-q^{-2})(1-q^{-9})(1-q^{-12})(1-q^{-5})(1-q^{-6})}{(1-q^{-6})(1-q^{-9})(1-q^{-4})(1-q^{-5})}$$
$$= q^{12}(q^2-1)(q^6-1)(q^8+q^4+1)$$

and we get (16).

11. The last case is ${}^2E_6$. The Dynkin diagram with Galois action indicated by arrows is



$\hat{\Sigma}^+$ has 36 elements breaking up into 24 orbits with atmost 2 elements in each orbit; $\eta(\hat{\alpha})$ is always 1. Let us write $(k_1 \cdots k_6)$ for the root

$$\hat{\alpha} = k_1\hat{\alpha}_1 + \cdots + k_6\hat{\alpha}_6 .$$

Then $\hat{\alpha}(\rho) = k_1 + \cdots + k_6$. The 2 elements orbit are:

$$\begin{array}{ll} \{(100000),\ (000001)\}\,, & \{(010000),\ (000010)\}\,, \\ \{(110000),\ (000011)\}\,, & \{(111000),\ (001011)\}\,, \\ \{(111100),\ (001111)\}\,, & \{(111110),\ (011111)\}\,, \\ \{(011000),\ (001010)\}\,, & \{(011100),\ (001110)\}\,, \\ \{(011011),\ (111010)\}\,, & \{(122110),\ (012121)\}\,, \\ \{(112110),\ (012111)\}\,, & \{(112121),\ (122111)\}\,. \end{array}$$

The rest of the orbits are

$$\begin{array}{l} (001000),\ (000100),\ (111111),\ (111011)\,, \\ (011010),\ (001100),\ (012110),\ (011110)\,, \\ (123221),\ (123121),\ (122121),\ (112111)\,. \end{array}$$

Evaluating the right hand side of (12) leads to

$$q^{78}(1-q^{-2})(1+q^{-5})(1-q^{-6})(1-q^{-8})(1+q^{-9})(1-q^{-12})$$

which is just (17). This completes the proof of the theorem.

12. It is sometimes convenient to express formula (12) in terms of determinants. Let h be an element in the adjoint group of $\hat{\mathfrak{g}}$ such that

$$h\cdot X_{\hat{\alpha}}=q^{-1}X_{\hat{\alpha}} \tag{18}$$

for $\hat{\alpha}\in\hat{\Delta}$. Take an orbit $\mathcal{O}(\hat{\alpha})$ of the action of the Frobenius automorphism σ. As σ is an isometry on the real vector space spanned by $\hat{\Sigma}$ (cf: Carter [2] p. 201), we have

$$h\cdot X_{\hat{\beta}}=q^{-\hat{\alpha}(\rho)}X_{\hat{\beta}}$$

for $\hat{\beta}\in\mathcal{O}(\hat{\alpha})$.

Let $\hat{\mathfrak{n}}(\hat{\alpha})$, $\hat{\mathfrak{n}}$ be the subalgebras of $\hat{\mathfrak{g}}$ spanned by $\{X_{\hat{\beta}}\,|\,\hat{\beta}\in\mathcal{O}(\hat{\alpha})\}$, $\{X_{\hat{\beta}}\,|\,\hat{\beta}\in\hat{\Sigma}^{+}\}$ respectively. Then σ on $\hat{\mathfrak{n}}(\hat{\alpha})$ has matrix

$$\left(\begin{array}{c|c} 0\cdots 0 & \eta(\hat{\alpha}) \\ \hline & 0 \\ I & \vdots \\ & 0 \end{array}\right)$$

where I is the identity matrix. The matrix of σ on $\hat{\mathfrak{n}}$ is broken into such blocks on the diagonal and the blocks are parametrized by $\hat{\Omega}$. Thus we get

$$\prod_{\mathcal{O}(\hat{\alpha})\in\hat{\Omega}}(1-\eta(\hat{\alpha})q^{-n(\hat{\alpha})(1+\hat{\alpha}(\rho))})$$
$$=\det{}_{\hat{\mathfrak{n}}}(1-q^{-1}\sigma h)$$

where $\det_{\hat{\mathfrak{n}}}$ means the determinant of $1-q^{-1}\sigma h$ acting on $\hat{\mathfrak{n}}$.

Let $\hat{\mathfrak{n}}_1$ be the subalgebra of $\hat{\mathfrak{g}}$ spanned by $X_{\hat{\beta}}$ for $\hat{\beta}\in\mathcal{O}(\hat{\alpha})$ and $\mathcal{O}(\hat{\alpha})\in\hat{\Omega}_1$. Then similar argument leads to

$$\prod_{\mathcal{O}(\hat{\alpha})\in\hat{\Omega}_1}(1-\eta(\hat{\alpha})q^{-n(\hat{\alpha})\hat{\alpha}(\rho)})$$
$$=\det{}_{\hat{\mathfrak{n}}_1}(1-\sigma h)\ .$$

The following corollary is now immediate.

COROLLARY.

$$|\tilde{G}(\boldsymbol{F}_q)|=\frac{q^m\det_{\hat{\mathfrak{n}}}(1-q^{-1}\sigma h)}{\det_{\hat{\mathfrak{n}}_1}(1-\sigma h)}\ . \tag{19}$$

13. Finally we would like to point out that our formula agrees with the standard formula for the order of finite twisted groups (as given for example in Carter [2]).

For $1\leqq i\leqq \mathfrak{l}$, we can assume that $\{H_i, X_{\hat{\alpha}_i}, X_{-\hat{\alpha}_i}\}$ forms a basis of a 3-dimensional simple Lie algebra. Put

$$H=\sum_i H_i,\ X=\sum_i X_{\hat{\alpha}_i},\ Y=\sum_i X_{-\hat{\alpha}_i}\ .$$

and let $\hat{\mathfrak{s}}$ be the 3-dimensional subalgebra with basis $\{H, X, Y\}$. If $d_1,\cdots,d_{\mathfrak{l}}$ are the exponents of G, then according to Kostant [7], $\hat{\mathfrak{g}}$ as an $\hat{\mathfrak{s}}$-module under the adjoint representation decomposes into irreducible submodules $\hat{\mathfrak{b}}_i$ of dimensions $2d_i-1$ for $1\leqq i\leqq\mathfrak{l}$. On the other hand it is clear from paragraph 4 that σ leaves invariant H, X and Y and hence we can arrange the decomposition $\hat{\mathfrak{g}}=\bigoplus\hat{\mathfrak{b}}_i$ in such a way that σ leaves each factor $\hat{\mathfrak{b}}_i$ invariant and acts on $\hat{\mathfrak{b}}_i$ by the root of unity ε_i. Moreover the element h of paragraph 13 belongs to the connected subgroup of $Ad(\hat{\mathfrak{g}})$ with Lie algebra $\hat{\mathfrak{s}}$ and so h also leaves each submodule $\hat{\mathfrak{b}}_i$ invariant.

If $\hat{\alpha}\in\hat{\Sigma}^+$ let $o(\alpha)$ be the sum of the coefficients of $\hat{\alpha}$ relative to the basis $\hat{\Delta}$. Then we group the eigenvalues $\{q^{-o(\alpha)}\,|\,\hat{\alpha}\in\hat{\Sigma}^+\}$ of the restriction of h to $\hat{\mathfrak{n}}$ according to the decomposition $\hat{\mathfrak{g}}=\bigoplus\hat{\mathfrak{b}}_i$. In fact, let Δ_i be the set of $\hat{\alpha}\in\hat{\Sigma}^+$ such that the eigenvector corresponding to the eigenvalue $q^{-o(\hat{\alpha})}$ lies in $\hat{\mathfrak{b}}_i$. The corresponding set with $\hat{\Sigma}^+\backslash\Delta^+$ replacing $\hat{\Sigma}^+$ will be denoted by Δ_i'. Then we get immediately,

$$\det_{\hat{n}}(1 - q^{-1}\sigma h) = \prod_{i=1}^{l} \prod_{\hat{\alpha} \in \Delta_i} (1 - \varepsilon_i q^{-(o(\hat{\alpha})+1)})$$

and

$$\det_{n_1}(1 - \sigma h) = \prod_{i=1}^{l} \prod_{\hat{\alpha} \in \Delta'_i} (1 - \varepsilon_i q^{-o(\alpha)}) .$$

The procedure of Shapiro as proved in Kostant [7] says that for $1 \leqq i \leqq l$, the set of integers

$$\{1 + o(\hat{\alpha})\}_{\hat{\alpha} \in \Delta_i}$$

is the same as the set of integers

$$\{o(\hat{\alpha})\}_{\hat{\alpha} \in \Delta'_i} \cup \{d_i\} .$$

It is immediate from these formulas that (19) yields

$$|\bar{G}(\boldsymbol{F}_q)| = q^m \prod_{i=1}^{l} (1 - \varepsilon_i q^{-d_i}) .$$

This is the standard formula for the order of finite twisted groups.

14. We would like to thank the referee for his suggestions in particular with regard to paragraphs 12 and 13.

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