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SECOND NOTE ON ARTIN'S SOLUTION OF HILBERT'S 17TH PROBLEM. ORDER SPACES

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We consider real varieties in K^n/k, where K/k is maximally ordered and k is dense in K. Our principal results are:

THEOREM 1. **Assume V is irreducible. A. The closure of the set of all simple points equals the set of all central points — $z \in V$ is central if some order of $k(V)/k$ contains every function which is positive at z. B. If $f+r$ is totally positive in $k(V)/k$ for every positive r in k, then f itself is totally positive.**

THEOREM 2. **For a semi-algebraic set S in K^n defined by polynomial relations $b_j(x) = 0$, $g_i(x) > 0$ $(1 \leqq i, j \leqq m)$, define $B = (b_1, \cdots, b_m)$, $G = \{g_1, \cdots, g_m\}$. Then every irreducible component of $V_k(B)$ contains central points on S if and only if (*) for $0 < p_i \in k$, $g_{ij} \in G$, $a_i \in k[X]$, $\sum_i p_i \prod_j g_{ij}\ a_i^2 \in {}^R\sqrt{B}$ implies every $a_i \in {}^R\sqrt{B}$.**

For an ordered ground field k and a formally real extension field F/k, the set $\Omega(F/k)$ of all orders of F/k admits two natural topologies. The standard topology, which originates in Harrison's 1966 Memoir [15], has basic open set $\Omega(E)$, while the weak topology uses basic open sets $\Omega_N(E)$, where E ranges in each case over the finite subsets of $\dot{F} = F\backslash\{0\}$, and

$$\Omega(E) = \{P \in \Omega(F/k); P \supset E\}\ ,$$

$$\Omega_N(E) = \{P \in \Omega(F/k); U_P \supset E\}\ ,$$

$$U_P = P \cap B_P \backslash J_P\ .$$

An order P is the positive cone of an ordering while B_P and J_P, respectively, denote the valuation ring of all elements finite over k, and the maximal ideal of all infinitesimal elements. U_P represents the group of all positive units in B_P. The real place associated with P is denoted h_P. The order space $\Omega(F/k)$ is applied here in a setting which dates back to Abraham Robinson's 1955–1956 papers on ordered fields and definite functions [23], [24]. An existence theorem from Lang's 1953 paper [18] is one of our most important tools. We consider the real variety $V = \mathscr{V}_K(A)$ of all zeros in K^n, K being a real closed ordered extension of k, of the real prime ideal A in the polynomial ring $k[X] = k[X_1, \cdots, X_n]$ in n variables. Thus the ring $k[x] = k[X]/A$ is a formally real domain over k and its field of quotients $k(x)$ is a formally real field over k. By means of the reelprimnullstellensatz, i.e., any polynomial which vanishes all over V must belong to A, of which we give a new and easy proof[1], we know that $k(x)$ is equal to $k(V)$, the function field of V. The paper is devoted mostly to this field $F/k = k(V)/k = k(x)/k$. K is always assumed to be real closed. For any ordered field k, its real closure is written $\bar{k}$.

For any subset H of V we define $\Omega(H)$ as the subset of $\Omega(F/k)$ which consists of all those orders which have a center on H, and we say that z is a *center* of the order P provided for all $f(x)$ in F, if $f(z)$ is positive, then $f(x)$ belongs to P. The concept of center is basic. The most important cases of H are V itself, the set V_c of all centers on V of orders (*central points*), the set V^0 of all zero-dimensional points of V, the strong closure V_1 of the set of all simple points on V, and semi-algebraic sets of the form:

$$\mathscr{H}(E) = \{z \in V; e_i(z) > 0 \text{ for all } i \in T\} ,$$

where $E = \{e_i; i \in T\}$ is an indexed of $\dot{F}$, T being finite. Assuming that k is real closed, Robinson (loc. cit.) proved that the set $\mathscr{D}(\mathscr{H}(E))$ of all members of F which are positive definite over $\mathscr{H}(E)$ is included in $\sigma(E)$:

$$\sigma(E) = \{\sum_{i \in T} \hat{e}_i(x) f_i(x)^2; \hat{e}_i \in \hat{E}, f_i \in F\} ,$$
$$\hat{E} = \{p \prod_{j \in U} e_k(x); U \subset T, 0 < p \in k\} .$$

This generalizes Artin's solution to Hilbert's 17th problem. For a subset L of V, L_c is the set of all central points on L, $\mathscr{D}(L)$ is the set of all members of F which are positive definite over L, $\mathscr{P}(L)$ is the intersection of all orders which have center on L. Also, $\mathscr{P}(F/k)$ is the intersection of all orders in $\Omega(F/k)$. We prove, for finite E in F, that

$$\mathscr{D}(\mathscr{H}(E)_c) = \mathscr{P}(\mathscr{H}(E)) = \sigma(E) ,$$

which generalizes and strengthens Robinson's generalization. We do not assume k to be real closed. It is shown that $\Omega(\mathscr{H}(E))$ is an (AC/k) family of orders and that F/k is an *Archimedes-Clifford*, or $(A - C)$ field — for any formally real extension M/k a family Δ in $\Omega(M/k)$ is (AC/k) provided for all f in M, if $f + p$ belongs to the intersection $\cap \Delta$ of all orders in Δ for every strictly positive p in k, then f itself belongs to $\cap \Delta$; the field M/k is $(A - C)$ provided $\Omega(M/k)$ is an (AC/k) family.

In §3 we examine centering in more detail. It is shown that $\Omega(V)$ is (strongly) dense in $\Omega(F/k)$. The natural relation

[1] This is due to T. Y. Lam, The Theory of Ordered Fields, Preprint, 1979.

$$\mathrm{Cen}_V = \{(P, z) \in \Omega(V) \times V;\ z \text{ is a center of } P\}$$

is actually a map in case k is dense in K; it is strongly continuous while its contraction $\Omega(V) \to V_c$ is weakly open. Assuming again that k is dense in K we show equality of the sets V_1, V_c and V_s. The first two are defined above while V_s is the set of all points on V whose every (strong) neighborhood on V is Zariski-dense in V. Inclusion of V_1 in V_c (i.e., every limit point of simple points is the center of an order) is proved in [10], §2; this same section includes a proof of inclusion of V_s in V_c and of V_1 in V_s. Inclusion of $\mathscr{D}(V)$ in $\mathscr{P}(V)$ results from a standard, easy modification of the proof in the same section that $\mathscr{D}(V)$ is a subset of $\mathscr{P}(F/k)$ — cf. [13].

The case of an Archimedean ground field k is taken up in §4. Then for our $H = \mathscr{H}(E)$, the set $\mathscr{P}(H)$ is an (AC) cone in the sense of [6] and [9]—see also Becker [3]. The subset $\Omega(V)$ is open. The structure theory of the paper just quoted is applicable, but now we work with the space $\mathscr{X}(F/k)$, defined by

$$\mathscr{X}(F/k) = \Omega(F/k)/\sim$$
$$\text{“}P \sim Q\text{” means “}h_P = h_Q\text{”},$$

furnished with the quotient topology; $\mathscr{X}(F/k)$ is defined thus for any formally real F/k, any ordered k. In the Archimedean case we have

$$h_P f = \sup\{f \in \boldsymbol{Q};\ f - r \in P\},$$

with $+\infty$ assigned as value in each of the infinity-producing cases. Theorems of this section reveal numerous connections between the abstract ordered ground field results of the first three sections and the theorems of Kadison, Becker and the present author for the case of Archimedean ground field.

Section 5 treats fields having the $(A - C)$, or *Archimedes-Clifford* property. The class of all such fields is so large as to include every finite algebraic extension of a pure transcendental extension of k (see Theorem 9).

Semi-algebraic sets, i.e., solution sets of finite systems of relations of the form $f_j(X) = 0$ and $g_i(X) > 0$, which need not lie on any irreducible variety, are the subject of §6. We prove a criterion for existence of central points in a semi-algebraic set. The condition, which is called *compatibility* of the set $G = \{g_i(X)\}$ with the ideal generated by the $f_j(X)$, is a generalization of the old result known to Baer, Artin and Robinson, that $\sigma(G)$ is equal to $\mathscr{P}(\mathscr{H}(G))$.

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Historical note. An excellent account of the history of Hilbert's 17th problem is given by Pfister [19]. Ribenboim [21] gives a lively account of the reelnullstellensatz in its various forms and of the early (1969) development of the subject of real commutative algebra.

2. Definiteness and the reelprimnullstellensatz. The reelnullstellensatz for real prime ideals, and Robinson's generalizations, are given very brief proofs. Then we prove some generalizations and strengthened forms of Robinson's theorems. By an application of Lang's Theorem 8 we prove our chief tool theorem, the algebraic order theorem. As usual, $\bar{k}$ denotes the real closure of k.

THEOREM 1. *The reelprimnullstellensatz* ([12]; cf. [8], [21], [22]). *Assume A is a real prime ideal in $k[X]$. Then for $f(X)$ in $k[X]$, $f(x)$ vanishes at every point of $\mathscr{V}_{\bar{k}}(A)$ if and only if $f(X)$ belongs to A. Hence, $\mathscr{I}(\mathscr{V}_{\bar{k}}(A)) = A$ and $k(V) = k(x)$.*

Proof. Assume $f(X) \notin A$. From reality of A we have reality of $k(x)$, and accordingly choose an order P of $k(x)/k$. Since $f(x) \neq 0$ holds, we may as well assume $f(x)$ is strictly positive by P. As allowed by Lang's Theorem 8, which is stated below, we select an algebraic real place h on $k(x)/k$ which is finite at every x_i, and positive at $f(x)$. Then $z = h(x) \equiv (hx_1, \cdots, hx_n)$ lies on $\mathscr{V}_{\bar{k}}(A)$ and $f(z) = f(hx) = hf(x)$ is positive. Hence, $f(X)$ is not in $\mathscr{I}(\mathscr{V}_{\bar{k}}(A))$. This establishes the nontrivial inclusion of the assertion.

Lang's Theorem 8 (*slightly generalized*), [18]. *Along with the hypotheses of Theorem* 1, *assume that E is a finite subset of $\dot{F}$ and that P is an order of F which contains E. Then there exists an algebraic real place on $k(x)/k$ which is finite and positive at every member of E and which is finite at every x_i.*

Note. Although Lang assumes that k is real closed, the generalized form stated above is an easy extension of his Theorem 8. See Elman, Lam and Wadsworth [14], §4 bis.

The set of all zero-dimensional points on V is denoted by V^0; it is equal to the set $\mathscr{V}_{\bar{k}}(A)$. The Krull place associated with an order P is denoted h_P. The order P is *rational*, or *algebraic*, according as h_P is rational, or algebraic, and dim P is defined as dim h_P, rank P is equal to rank h_P.

For our purposes the following corollary to Lang's Theorem 8 has a particularly useful form.

Algebraic order theorem. Keeping the hypotheses of Lang's Theorem 8, *there exists an algebraic order P of $k(x)/k$ which is finite on every x_i and centered on $\mathscr{H}(E)$.*

Proof. Using Lang's Theorem 8 we select a real algebraic place h on F/k which is finite on the x_i and positive over E. By means of Krull's construction we obtain an order P which is compatible with h (i.e., $f \in U_P$ if and only if $hf > 0$). Then the point $z = h(x)$ is an algebraic center of P and it lies on $\mathscr{H}(E)$. Moreover, since the residual field of h is Archimedean over k, h and h_P are isomorphic with each other. Hence P is algebraic.

THEOREM 2. *Assume E is a finite subset of $\dot{F}(F = k(V))$, and that $k \subset K$, K being real closed. Then ($\bar{S}$ is the closure of S)*

A. $\Omega_N(E) \cap \Omega(V) \subset \Omega(\mathscr{H}(E)) \subset \overline{\Omega^0(\mathscr{H}(E))} = \Omega(E) = \overline{\Omega_N(E)}.$

A′. *If K/k is Archimedean then* $\Omega_N(E) \cap \Omega(V) = \Omega(\mathscr{H}(E))$.

B. $\sigma(E) = \mathscr{P}(\mathscr{H}(E)) = \mathscr{D}(\mathscr{H}(E) \cap V_c^0) = \cap\, \Omega(E)$. *This includes Robinson's theorem.*

C. $\overline{\Omega^0(V)} = \Omega(k(V)) = \overline{\Omega(V)}$.

D. $\mathscr{P}(V) = \mathscr{D}(V_c^0) = \mathscr{P}(k(V)) = \sigma(\{1\})$.

Proof. First we observe that C is a consequence of A by taking $E = \{1\}$. Similarly, D follows from B and C.

The last equality in A is left to the reader. For the first inclusion of A, assume P is in $\Omega_N(E)$ with center at z on V, which implies that $E \subset U_p$. Hence $z \in \mathscr{H}(E)$, and $P \in \Omega(\mathscr{H}(E))$. Next we prove the reverse inclusion, assuming K/k is Archimedean, which will complete the proof of A′. Now if P is centered at $z \in \mathscr{H}(E)$ then for every e in E, $e(z)$ is positive and neither infinite nor infinitesimal, whence $P \in \Omega_N(E) \cap \Omega(V)$. As to the second inclusion of A, we note that it follows immediately from the algebraic order theorem.

The algebraic order theorem also shows immediately that $\Omega(E)$ is a subset of $\Omega^0(\mathscr{H}(E))$. For the reverse inclusion, let P be an order outside of $\Omega(E)$, so that E is not included in P. We decompose E as follows:

$$E = E' \cup E'' , \qquad E' = E \cap P , \qquad E'' = E \cap (-P) ,$$

and set

$$E''' = E' \cup (-E'') .$$

By noting that E'' is not empty we see that the neighborhood $\Omega(E''')$ of P is disjoint from $\Omega(\mathscr{H}(E))$, and thus P is not in the closure of $\Omega(\mathscr{H}(E))$. This completes the proof of Part A.

For part B, observe first that the equality of $\sigma(E)$ with $\cap \Omega(E)$ was known to Baer [2] and Artin [1]. Applications of the algebraic order theorem yield easily the equality of $\mathscr{P}(\mathscr{H}(E))$ with $\cap \Omega E$ and also the middle equality of Part B. Thus the theorem is proved.

THEOREM 3. *Assume $\Omega(E)$ is not empty, E being a finite subset of $\dot{F}$. Then $\Omega(\mathscr{H}(E))$ is an (AC/k) family. The field F/k is an Archimedes-Clifford field. Compare Dubois* [6].

Proof. Let $f(x)$ be outside $\mathscr{P}(\mathscr{H}(E))$. There is then an order P which excludes $f(x)$ and which has a center in $\mathscr{H}(E)$. By the algebraic order theorem there exists an algebraic order P' which includes $-f(x)$ and which has a center z in $\mathscr{H}(E)$. Hence, $f(z) < 0$ holds. There exists a positive element r in k such that $f(z) + r < 0$, whence $f(x) + r$ is outside of P'. Thus $f(x) + r$ is outside of $\mathscr{P}(\mathscr{H}(E))$. This proves the contrapositive form of the condition (AC/k) for $\Omega(\mathscr{H}(E))$.

3. Centering and simplicity. To the order space properties already used and deduced in §2, we add some deeper results, including a proof that, in case k is dense in K, Cen_V is continuous, and that a point of V is central if and only if it is inner (i.e., a member of the strong closure of the set of all simple points on V).

THEOREM 4. *Every $\Omega_N(E)$, for finite E, is open in $\Omega(F/k)$. The set of all such $\Omega_N(E)$ is a base for a topology in $\Omega(F/k)$.*

Proof. Let r, s be positive elements of k, and set

$$E(r, s) = \{e - r; e \in E\} \cup \{s - e; e \in E\} .$$

Then

$$\Omega_N(E) = \bigcup_{r\ s} \Omega E(r, s) .$$

Hence $\Omega_N(E)$ is open. Routine computations show that the $\Omega_N(E)$ form a base for a topology.

The topology just alluded to is the *weak topology* in $\Omega(F/k)$.

We turn next to the centering relation,

$$\mathrm{Cen}_V = \{(P, z); P \in \Omega(F/k),\ z \text{ is a center of } P\} .$$

It is not necessarily a mapping. Except for an occasional explicit use of Zariski's topology on V, we use the standard strong topology inherited from K^n.

THEOREM 5. *Assume k is dense in K. Then* Cen_V *is a strongly continuous map. The contraction $\Omega(V) \to V_c$ is a weakly open map.*

Proof. Let G be an open set in V. The density hypothesis implies

$$G = \bigcup_{\mathscr{H}(E) \subset G} \mathscr{H}(E) .$$

Hence, by means of Theorem 2, we deduce

$$\Omega(G) = \bigcup_{\mathscr{H}(E) \subset G} (\Omega_N(E) \cap \Omega(V)) .$$

This is open by virtue of Theorem 4; it is, in fact, open in the weak topology in $\Omega(V)$. Thus the set $\mathrm{Cen}_V^{-1}(G) \equiv \Omega(G)$ is a weak open set, for arbitrary open G, whence Cen_V is strongly continuous.

Let $\Omega_N(E) \cap \Omega(V)$ be an arbitrary weak open set in $\Omega(V)$. Applying Theorem 2 again we get

$$\begin{aligned} \mathrm{Cen}_V (\Omega_N(E) \cap \Omega(V)) &= \mathrm{Cen}_V \, \Omega(\mathscr{H}(E)) \\ &= \mathscr{H}(E) \cap \Omega(V) . \end{aligned}$$

This proves the weak openness (observe that $\mathscr{H}(E)$ is open on V).

We denote by V_s the set of all "strong" points of V, i.e., points every strong neighborhood of which is Zariski-dense in V.

THEOREM 6. *Assume k is dense in K. Then $V_s = V_c = V_1$,* i.e., *for a point to be central it is necessary and sufficient that it be an inner point or that no (strong) neighborhood of it be included in any proper subvariety.*

Proof. That V_1 is a subset of V_s is proved in [10], from the case $k = \boldsymbol{R}$ (Dubois-Efroymson [12]), by means of an instant application of Tarski's principle. The inclusion $V_s \subset V_1$ is an easy exercise with Jacobians. In [10] (cf. [11]) we proved $V_s \subset V_c$. Now we prove $V_c \subset V_s$. Assume that z' is a point of $V \backslash V_s$. There exists a neighborhood Y of z' on V, which may be taken in the form

$$Y = \mathscr{H}(E) , \quad \text{for some finite } E \subset \dot{F} ,$$

by virtue of the density hypothesis, and a nonzero polynomial function $g(x)$ in $k[x]$ such that $g(Y) = \{0\}$. Then each of $g(x)$ and $-g(x)$ is positive definite over $\mathscr{H}(E)$, whence each of these belongs to

$\mathscr{P}(\mathscr{H}(E))$, according to Theorem 2. Since $g(x)$ is not zero, there are no orders centered on $\mathscr{H}(E)$ and, in particular, none centered at z'. This completes the proof of the theorem.

4. Archimedean ground fields. For any orders P and Q of any formally real extension F/k, we say that P and Q are equivalent to each other and write "$P \sim Q$", whenever $h_P = h_Q$. The correspondence $P \mapsto \operatorname{cls} P$ maps $\Omega(F/k)$ into the set of all real places on F/k, its image being the set $\mathscr{X}(F/k)$, of all real places of the form h_P. The set $\mathscr{X}(F/k)$ is topologized by the quotient topology for the map above. For any subset Δ of $\Omega(F/k)$, we use $\mathscr{X}(\Delta)$ to denote $\Delta/\sim$; in case $F = k(V)$, V being a real variety, we write $\mathscr{X}(V)$ in place of $\Omega(V)/\sim$. The easy proof of the following lemma is omitted.

LEMMA. *Assume that k is an ordered subfield of $K = \boldsymbol{R}$. As usual, $V = \mathscr{V}_K(\mathscr{A})$ for a real prime ideal A in $k[X]$. Then $\Omega(V)$ and $\mathscr{X}(V)$ are open sets.*

With the hypotheses of the lemma, recall that

$$h_P f = \sup\{r \in \boldsymbol{Q}; f - r \in P\} .$$

From here on in §4, we resume the notations of §1, taking $K = \boldsymbol{R}$, and $V = \mathscr{V}_{\boldsymbol{R}}(\mathscr{A})$. Assume also that E is a finite subset of $\dot{F}(F = k(V))$ with $H = \mathscr{H}(E)$. The notation $B(H)$ denotes the ring of all $f(x)$ which are bounded on H, partially ordered by the positive cone $\mathscr{P}(H)$.

Assume H contains a simple point. From Theorem 6 we know that there are orders centered on H and any such order belongs to $\Omega(E)$. It follows (Theorem 3) that $\mathscr{H}(E)$ is an (AC/k) family and so $\mathscr{P}(H)$ is an (AC/k) positive cone. Since k is Archimedean, $\mathscr{P}(H)$ satisfies also the condition (AC) defined in Dubois [6]. The norm "$\|\ \ \|$" is defined there as follows:

$$\|f(x)\| = \sup\{r \in \boldsymbol{Q};\ f + r \text{ and } f - r \text{ belong to } \mathscr{P}(H)\} .$$

The completion of $B(H)$ in this norm is denoted $B^*(H)$.

Results of the above paper can now be applied to prove the theorem following. Constructions of the maps and spaces mentioned in the theorem are sketched after the statement. The proofs are omitted. The key is that the ring $B(H)$ is, by virtue of Theorem 3, a Stone ring (see [7] and Becker [3]).

THEOREM 7. *Assume $k \subset \boldsymbol{R} = K$. Assume that V is an irreducible real variety in $\boldsymbol{R}^n$ over k and that $H = \mathscr{H}(E)$, for a finite subset E of $\dot{F}(F = k(V)/k)$, contains a simple point of V. Then there* *exists, for* $i = 1, 2$, *a compact Hausdorff space* T_i *and a strongly order-preserving isomorphism* ψ_i *of* $B^*(H)$ *onto the pointwise partially ordered ring* $C(T_i)$ *of all continuous real functions on* T_i. *For any nonzero* $b(x)$ *in* $B(H)$ *the support of* $\psi_i(b(x))$ *is all of* T_i. *The spaces* T_i *are canonically homeomorphic by a map*

$$\gamma\colon T_1 \longrightarrow T_2$$

and the induced map

$$\gamma'\colon C(T_1) \longrightarrow C(T_2)$$

in an isomorphism of partially ordered rings. This γ' *is characterized as the unique solution for* α *to the equation*

$$\alpha \circ \psi_1 = \psi_2 \ .$$

Constructions. For T_1 we take the maximal ideal space of $B^*(H)$, and for T_2, we take $\mathscr{H}(E) = \Omega(E)/\sim$. The maps are defined as follows:

$$\psi_1 b = \{(M, r); r \in \boldsymbol{R} \cap (b + M),\ M \in T_1\} \ .$$

This $\psi_1 b$ is actually a function on T_2, continuous and real. To define ψ_2, let $b \in B(H)$. Then for $h \in \mathscr{H}(E)$,

$$(\psi_2 b)h = hb \ ;$$

now ψ_2 is extended to all of $B^*(H)$ by continuity.

For any maximal ideal M in T_1 and any order P in $\Omega(H)$, we say "*P is centered at M*", or "*P is associated with M*", provided P contains every $b(x)$ whose coset $\operatorname{mod} M$ contains a positive real number—i.e., $\psi_1 b$ is positive at M. Then the correspondence:

$$M \longmapsto h_P \text{ providen } P \text{ is centered at } M \ ,$$

defines a map, which is the γ of the theorem. To say that ψ_i is strongly-order-preserving means simply that

$$B^*(H)^+ = \psi_i^{-1}(C(T_i)^+) \ .$$

THEOREM 8. *Let* $M \in T_1$, $h_P = \gamma M$ *(hence* P *is centered at* M*).*

Case 1. Assume P belongs to $\Omega(V)$, say z is the center of P on V. Then for all $b(x) \in B(H)$,

A. $b(z) = 0 \Leftrightarrow b(x) \in J_P \Leftrightarrow b(x) \in M$.

B. $b(z) > 0 \Leftrightarrow b(x) \in U_P \Leftrightarrow \psi_1(b)M > 0$.

C. For all $e(x)$ in E, $e(z) \geqq 0$.

Case 2. Assume P has no center on V. Then some x_i is infinitely large by P.

Because F/k is an Archimedes-Clifford field and $k \subset \boldsymbol{R} = K$, Theorems 7 and 8 have birational analogues for $\Omega(F/k)$, $\mathscr{P}(F/k)$, $\mathscr{X}(F/k) = \Omega(F/k)/\sim$, with $B(F/k)$ defined as follows:

$$B(F/k) = \{f(x);\ \text{for some } r \in \boldsymbol{Q},\ f \pm \mathscr{P}(F/k)\}\ .$$

The proofs and statements require only routine modifications (see [9], §6); the analogues follow.

THEOREM 7′. (*Birational form of Theorem* 7) *In the statement of Theorem* 7, *delete* E, H, $\mathscr{H}(E)$. *Replace* $B^*(H)$ *by* $B^*(F/k)$. *The conclusions remain valid, ij the constructions below are made.*

Constructions. Replace $B(H)$ by $B(F/k)$, $\mathscr{X}(E)$ by $\mathscr{X}(F/k)$, $\Omega(E)$ by $\Omega(F/k)$, $B^*(H)$ by $B^*(F/k)$.

THEOREM 8′. (*Birational form of Theorem* 8) *Let the statement of Theorem* 8 *stand. The conclusion is still valid for the constructions above.*

Finally we note that by introducing extended real functions with values in the one-point compactification $\boldsymbol{R}_1$ of $\boldsymbol{R}$, improved versions of Theorems 7, 8 and 7′, and 8′ can be obtained. The routine details may be supplied by the reader. See our [9].

5. Snow-fields and Archimedes-Clifford fields. Let F/k be an arbitrary formally real extension field, k being ordered, as usual. Let Δ be a subspace of $\Omega(F/k)$. We define J_Δ as follows:

$$J_\Delta = \cap \{J_P;\ P \in \Delta\}\ .$$

DEFINITION. The subspace Δ is a *snow-pack* on F/k provided for every (relatively) open subset Γ of Δ, $J_\Gamma = \{0\}$. Equivalently, Δ is a snow-pack provided for every f, if f is infinitesimal by every order in some nonempty open subset of Δ, then f is zero. The field F/k is a *snow-field* provided $\Omega(F/k)$ is a snow-pack on F/k.

LEMMA (cf. [6], §3). *If* Δ *is a snow-pack on* F/k, *then* Δ *is an* (AC/k) *family. In case* k *is Archimedean-ordered, the converse is valid.*

Proof. The converse is an immediate consequence of Theorem 7 in §4. Assume Δ is not an (AC/k) family. It is easily verified (see [6], Theorem 3.2), that there exists f in F, and Q in Δ such that

$$f \in \bigcap_{P \in \Delta} (P \cup J_P) \backslash Q \ .$$

The set $\Gamma = \{P; P \in \Delta \text{ and } f \notin P\}$ is an open set containing Q. Also f belongs to J_Γ, but since $f \notin Q$, f is not zero. Thus Δ is not a snow-pack.

COROLLARY. *Every snow-field is an Archimedes-Clifford field. If the ground field is Archimedean ordered, the converse is varid.*

THEOREM 9. *Assume that F/k is a formally real finite algebraic extension of a pure transcendental extension of the ordered field k. Then F/k is an Archimedes-Cliford field: and if k is Archimedean ordered, then F/k is a snow-field.*

Proof. It is verified with small difficulty that the hypotheses permit expressing F in the form $F = K(U)$, with $K = k(T)$, where T is a finite set and U is a transcendence base of F over K (not k). Since K/k is finitely generated, it, along with every real finitely generated extension of it, is an $(A - C)$ field, by the corollary. The theorem will be proved if we show the following: *if K/k has the property that each of its finitely generated pure transcendental formally real extensions is $(A - C)$, then every pure transcendental extension of K is an $(A - C)$ field.*

To prove the italicized statement, assume $F = K(U)$, and that U is an infinite transcendence base of F/k. Let $\mathscr{F}$ be the family of all subfields of F of the form $K(S)$, where S is a finite subset of U. By hypothesis this is a family of $(A - C)$ fields. The family is clearly totally directed, i.e., it satisfies the following conditions:

(i) For every E_1 and E_2 in $\mathscr{F}$ there exists E_3 in $\mathscr{F}$ which is an extension of each of E_1 and E_2.

(ii) For every E_1 and E_2 in $\mathscr{F}$, if E_2 is a field extension of E_1, then E_2 is a total extension of E_1, i.e., the restriction map ε_{E_2/E_1} which maps each order in $\Omega(E_2/k)$ to its intersection in E_1 is surjective (every order of E_1/k extends).

We next show that every totally directed family also satisfies:

(iii) The union E_∞ of all members of $\mathscr{F}$ is a total extension field of every E_i in the family $\mathscr{F}$.

The proof, sketched below, is a straightforward transfinite induction. First, using Zermelo's theorem, we index the members of $\mathscr{F}$ by means of a well-ordered set. Now, given any E in $\mathscr{F}$, a re-indexing produces, by means of condition (i), a well-ordered ascending chain E_α $(E_\alpha \supset E)$, of members of $\mathscr{F}$ whose union is E_∞: $E_0 = E$, and for $\alpha > 0$, E_α is the first member of the original well-ordered set which contains all preceding E_β. In a similar way, given an order P of E, an ascending chain of P_α is constructed by taking $P_0 = P$, and using (ii) for passing from β to $\beta + 1$, while for a limit ordinal α, P_α is the union of all P_β for $\beta < \alpha$. Then the union of all P_α is an order of E_∞ which extends P_0. In this way condition (iii) is verified.

Since $F = K(U)$ is the union of all the $(A - C)$ fields in $\mathscr{F}$, the theorem will be proved if we prove the following:

LEMMA. *The union $\bar{E}/k$ of any totally directed family $\mathscr{F}$ of $(A - C)$ fields is itself an $(A - C)$ field.*

To prove the lemma, assume f belongs to $\bar{E}$ and that for all positive r in k, $f + r$ belongs to $P(\bar{E}/k)$. There is a field E in $\mathscr{F}$ which contains f. Let P be an order of E/k, and let $\bar{P}$ be an extension of P to E (such $\bar{P}$ exists by virtue of condition (iii) above). Now $f + r$ belongs to $\bar{P}$ for every positive r in k and hence $f + r$ belongs to $P = \bar{P} \cap E$. This latter condition is valid for every order P of E; in other words, $f + r$ belongs to $\mathscr{P}(E/k)$ for every positive r in k. Since E is by hypothesis an $(A - C)$ field, we have f itself in $\mathscr{P}(\bar{E}/k)$. By Artin's easy theorem [1], f is a positive combination of squares in E. Hence f belongs to $\mathscr{P}(\bar{E}/k)$. This proves that $\bar{E}$ is $(A - C)$, and the theorem is all proved.

Note. The finiteness restriction on the algebraic extension is not superfluous, as shown in Dubois [9], §6.17.

THEOREM 10. Cf. *Lang*, loc.-cit. *Theorem* 9. *For the same hypotheses on F/k, for every f in F, f is positive definite over $\mathscr{X}(F/k)$ if and only if f belongs to $\mathscr{P}(F/k)$.*

Proof. (By Anonymous). Assume $f \notin P(F/k)$. Take the K of Theorem 9 above to contain f. Then $f \notin \mathscr{P}(K/k)$. Lang's Theorem 9 gives us an order Q such that $h_Q f < 0$ holds, whence for some (and every) extension of Q to an order P of F/k, we have $h_P f < 0$. This proves half of the assertion, and the converse is obvious.

6. Compatibility.

Let k be an ordered field, $k[X]$ the polynomial ring in n variables as in the early paragraphs. Let $\mathscr{T}$ be a finite set of indices. Consider the relations

$$\mathscr{S}\begin{cases} f_j(X) = 0\,, & 1 \leqq j \leqq v \\ g_i(X) > 0\,, & i \in \mathscr{T}\,, \end{cases}$$

where f_j and g_i belongs to $k[X]$. The solution set to $\mathscr{S}$ is a semi-algebraic set. Let B be the ideal generated by the $f_j(X)$ and let G be the (indexed) set of all $g_i(X)$. By a *central solutiou* to $\mathscr{S}$ we mean a solution z in K^n which is a central point on one or more of the irreducible real components of $\mathscr{V}_K(B)$. Since we are not assuming that B is either real or prime, we digress briefly to the situation of a commutative ring A with unity, without (ordered) ground field k. For an ideal B in A the *real radical* of B is the intersection of all real prime ideals over B. For this definition and the lemmas below, see our 1970 paper [12], §1. In case A has no ground field, all references below to p_i and k should be deleted.

LEMMA 1. *The real radical* $\sqrt[R]{B}$ *is the set of all* x *in* A *which saisfy a relation of the form*

$$x^{2m} + \sum p_i a_i^2 \in B ,$$

for some natural number m, *positive* p_i *in* k, *and* a_i *in* A.

For a set $G = \{g_i; i \in T\}$, where T is finite, the set $\hat{G}$ (compare §1) is

$$\hat{G} = \{p \cdot \prod_{i \in U} g_i; 0 < p \in k, U \subset T\} .$$

For an ideal B in A, G is compatible with B provided for all $\hat{g}_i$ in $\hat{G}$ and all a_i in A,

$$\sum \hat{g}_i a_i^2 \in \sqrt[R]{B} \quad \text{implies every} \quad a_i \in \sqrt[R]{B} .$$

THEOREM 11. *The condition that every irreducible real component of* $\mathscr{V}_K(B)$ *contain a central solution to* S *is equivalent to compatibility of* G *with* B.

Proof. By means of Lemma 2 below the proof is reduced to the case where B is a real prime ideal, and that case is settled by Lemma 3 below.

LEMMA 2. *Let* A *be a unitary commutative ring, let* B *be an ideal in* A *and let* G *be a finitely indexed subset of* A.

A. *Assume* G *is compatible with* B. *Then* G *is compatible with every minimal real prime over* B.

B. *Assume* A *is Noetherian. Then the converse of* A *is valid.*

C. *Assume* G *is compatible with every real prime over* B. *Then* G *is compatible with* B.

Proof. We make liberal use of the results of [12]; theorems, lemmas, etc., referred to below are from that paper.

A. Let P be a minimal real prime over B — see Theorem 1.3G. Then P is also a minimal real prime over $\sqrt[R]{B}$, hence it is associated with $\sqrt[R]{B}$. We choose an element c outside of $\sqrt[R]{B}$ so that $P = \sqrt{[\sqrt[R]{B}: c]}$. Now assume for $\hat{g}_i$ in $\hat{G}$, a_i in A, that $\sum \hat{g}_i a_i^2$ belongs to $P = \sqrt[R]{P}$. Then for some m,

$$[\sum \hat{g}_i(ca_i)^2]^{2m} = [c^2 \sum \hat{g}_i a_i^2]^{2m} = c^{4m}(\sum \hat{g}_i a_i)^{2m} \in \sqrt[R]{B} .$$

Reality of $\sqrt[R]{\mathscr{B}}$ now implies that $\sum \hat{g}_i(ca_i)^2$ belongs to $\sqrt[R]{B}$. Compatibility implies that each ca_i belongs to $\sqrt[R]{B}$, which is contained in P. Since P is prime and c is not in it, a_i belongs to P for every i. Hence G is compatible with P.

B. Assume that $y = \sum \hat{g}_i a_i^2$ belongs to $\sqrt[R]{B}$. Then y belongs to every real prime over B. Now assume that G is compatible with every minimal real prime of B. Then every a_i belongs to every minimal real prime, and hence, by the Noetherian assumption, every a_i belongs to $\sqrt[R]{B}$ — see Theorem 1.4b.

C. This is now obvious.

LEMMA 3. *Now let $A = k[X] = k[X_1, \cdots, X_n]$, with ordered field k. Assume B is a real prime ideal in A, and that $G = \{g_i; i \in T\}$ is a finite indexed subset of A. Assume G is compatible with B. Then the system $\mathscr{S}$ has a central solution. The converse is also valid.*

Proof. We take an order of $k(x)$ which contains $G/B = \{g_i(x); i \in T\}$ as allowed by the assumed compatibility. The algebraic order theorem guarantees that $\mathscr{S}$ has a central solution.

Conversely, assume that z is a central solution to $\mathscr{S}$. Assume further that $\sum \hat{g}_i(X)a_i(X)^2$ belongs to B. Let Q be an order of $k(x)$ which is centered at z. Then $\hat{g}_i(z) > 0$ holds for every i whence $\hat{g}_i(x)$ is a nonzero member of Q. Also we have $\sum \hat{g}_i a_i(x)^2 = 0$. Hence every $a_i(x)$ is zero so $a_i(X)$ belongs to B. This proves compatibility. The lemma, and with it the theorem, is proved.

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