PACIFIC JOURNAL OF MATHEMATICS
Vol. 97, No. 2, 1981

ON POLYNOMIAL INVARIANTS OF FIBERED 2-KNOTS

KOUHEI ASANO AND KATSUYUKI YOSHIKAWA

Given any polynomial $\lambda(t) = \sum_{j=0}^{m} c_j t^j$ satisfying the conditions that c_j is an integer, $\lambda(1) = \pm 1$, $c_0 = 1$ and $c_m = \pm 1$, we will construct a fibered 2-knot in the 4-sphere with the invariants $\{\lambda_i^q(t)\}$ such that $\lambda_1^q(t) = \lambda(t)$ and $\lambda_i^q(t) = 1$ for $i > 1$ and $q = 1, 2$.

1. Introduction. An *n-knot* K is a smooth submanifold of the $(n+2)$-sphere S^{n+2} which is homeomorphic to S^n. By the *exterior* of K, we mean the complement of an open tubular neighborhood of K in S^{n+2}. If the exterior of K fibers over a 1-sphere, K is called a *fibered n-knot.*

Let E be the exterior of an n-knot and $\tilde{E}$ the infinite cyclic covering of E with $\langle t \rangle$ as the covering transformation group. Let Λ denote the integral group ring of $\langle t \rangle$ and $\Gamma = \Lambda \otimes_Z Q$ the rational group ring of $\langle t \rangle$. Since Γ is a principal ideal domain, $H_q(\tilde{E}, Q) \cong \Gamma/\lambda_1^q(t) \oplus \cdots \oplus \Gamma/\lambda_{r_q}^q(t)$. In this decomposition, we can take $\lambda_i^q(t)$ so that

(i) $\lambda_i^q(t)$ is a primitive element and $\lambda_{i+1}^q(t) \mid \lambda_i^q(t)$ in Λ. Then $\{\lambda_i^q(t): 1 \leqq i \leqq r_q\}$ are called the polynomial invariants of K in dimension q, for $1 \leqq q \leqq n$ [3], [6], [7].

In [7], it is shown that polynomial invariants $\{\lambda_i^q(t): 1 \leqq i \leqq r_q$, $1 \leqq q \leqq n$ of a fibered n-knot have the following properties:

(ii) If $\lambda_i^q(t) = \sum_{j=0}^{m} c_j t^j$, then $c_0 = \pm 1$ and $c_m = \pm 1$.

(iii) $\lambda_i^q(1) = \pm 1$.

(iv) $\lambda_i^q(t) = \varepsilon t^\alpha \lambda_i^{n-q+1}(t^{-1})$, $\varepsilon = \pm 1$ and α is an integer.

(v) If $n = 2q - 1$, q is even, $\Delta(t) = \lambda_1^q(t) \cdots \lambda_{r_q}^q(t)$ is in normal form, i.e., $\Delta(t) = \Delta(t^{-1})$ and $\Delta(1) > 0$, then $\Delta(-1)$ is an odd square.

Furthermore, the family $\{\lambda_i^q(t)\}$ satisfying (i)-(v) can be realized as the invariants of a fibered n-knot, if $\lambda_1^1(t) = \lambda_1^n(t) = 1$. In this paper, we will prove that

THEOREM. *Given any polynomial $\lambda(t) = \sum_{j=0}^{m} c_j t^j$ satisfying $\lambda(t) \in \Lambda$, $\lambda(1) = \pm 1$, $c_0 = 1$ and $c_m = \pm 1$, there exists a fibered 2-knot in the 4-sphere with the invariants $\{\lambda_i^q(t)\}$ such that $\lambda_1^q(t) = \lambda(t)$ and $\lambda_i^q(t) = 1$, for $i > 1$ and $q = 1, 2$.*

Using our theorem and the argument in [7], we can show that, for given family $\{\lambda_i^q(t)\}$ which satisfy (i)-(v), there is a fibered n-knot with $\{\lambda_i^q(t)\}$ as its invariants.

The authors wish to express their gratitude to Prof. A. Kawauchi for helpful suggestions.

2. *Proof of Theorem*. Let $V_m = B^4 \cup \bigcup\{h_i^{(1)}: 1 \leqq i \leqq m\}$, where B^4 is a 4-ball and $h_i^{(1)}$ is a 1-handle. Then $\pi_1(V_m)$ is a free group freely generated by the elements $x_1, \cdots, x_m$ corresponding to $h_1^{(1)}, \cdots, h_m^{(1)}$, respectively. By φ, we denote an automorphism of $\pi_1(V_m)$ defined by

$$\varphi(x_i) = \begin{cases} x_{i+1}, & 1 \leqq i \leqq m-1, \\ (x_1^{c_0} x_2^{c_1} \cdots x_m^{c_{m-1}})^{-c_m}, & i = m. \end{cases}$$

Clearly, there exists an autohomeomorphism $\hat{\varphi}$ of V_m which induces the automorphism φ of $\pi_1(V_m)$. Without loss of generalities, we may assume that φ has a fixed point p in ∂V_m. Let X be the 5-manifold obtained from $V_m \times [0, 1]$ by identifying $V_m \times \{0\}$ and $V_m \times \{1\}$ via a homeomorphism $\hat{\varphi}$. More precisely, X is the quotient of $V_m \times [0, 1]$ by the equivalence relation $(x, 0) \sim (\hat{\varphi}(x), 1)$.

We can show that $\pi_1(X)$ has a presentation

$$\langle t, x_1, \cdots, x_m : t x_1 t^{-1} x_2^{-1}, \cdots, t x_{m-1} t^{-1} x_m^{-1}, t x_m t^{-1} (x_1^{c_0} \cdots x_m^{c_{m-1}})^{c_m} \rangle .$$

As in [5], $H_1(\tilde{X}, Q)$ is isomorphic to $\Gamma/\lambda(t)$, as a Γ-module, where $\tilde{X}$ denotes the infinite cyclic covering of X with $\langle t \rangle$ as the covering transformation group. Adding a 2-handle $H_0^{(2)}$ to X along a simple closed curve $\alpha = p \times [0, 1]/\sim$ representing t in $\pi_1(X)$, we obtain a simply connected 5-manifold Y. We will show that Y is homeomorphic to a 5-ball.

Let $H_i^{(2)} = h_i^{(1)} \times [0, 1/2]/\sim$, $H_i^{(1)} = h_i^{(1)} \times [1/2, 1]/\sim$, for $1 \leqq i \leqq m$, $H_0^{(1)} = B^4 \times [0, 1/2]/\sim$ and $H_0^{(0)} = B^4 \times [1/2, 1]/\sim$. Then

$$Y = H_0^{(0)} \cup \bigcup\{H_i^{(1)}: 0 \leqq i \leqq m\} \cup \bigcup\{H_i^{(2)}: 0 \leqq i \leqq m\},$$

is a handle decomposition of Y such that $H_i^{(j)}$ is a j-handle.

Let $W_{m+1} = H_0^{(0)} \cup \bigcup\{H_i^{(1)}: 0 \leqq i \leqq m\}$. If we denote the elements of $\pi_1(W_{m+1})$ corresponding to $H_0^{(1)}, H_1^{(1)}, \cdots, H_m^{(1)}$ by $t, x_1, \cdots, x_m$, respectively, $\pi_1(W_{m+1})$ is a free group generated by $t, x_1, \cdots, x_m$. For $1 \leqq i \leqq m$, the attaching sphere of $H_i^{(2)}$ represents $t x_i t^{-1} \varphi(x_i)^{-1}$.

The following transformations of a presentation $\langle y_1, \cdots, y_s : r_1, \cdots, r_t \rangle$ are called *Andrews-Curtis moves* [1], [2], [4]:

(i) Replace r_i by r_i^{-1}.

(ii) Replace r_i by $w r_i w^{-1}$, where w is a word in $y_1, \cdots, y_s$.

(iii) Replace r_i by $r_i r_j$, for $i \neq j$.

(iv) Add a generator y and a relator $y w^{-1}$, where w is a word in $y_1, \cdots, y_s$.

(v) Inverse transformation of (iv).

It is not difficult to show that a presentation

$$\langle t, x_1, \cdots, x_m: t, tx_1t^{-1}x_2^{-1}, \cdots, tx_{m-1}t^{-1}x_m^{-1}, tx_mt^{-1}(x_1^{c_0} \cdots x_m^{c_{m-1}})^{c_m}\rangle$$

can be transformed to the trivial presentation by Andrews-Curtis moves. Hence one can slide 2-handles $\{H_i^{(2)}\}$ to cancel 1-handles $\{H_i^{(1)}\}$ [1]. Thus Y is homeomorphic to a 5-ball.

Let B^3 be a co-core of $H_0^{(2)}$. Then $H_0^{(2)}$ can be considered as a tubular neighborhood of B^3 in Y. Hence the exterior of a 2-knot ∂B^3 in ∂Y is $(\overline{\partial X - (\partial X \cap H_0^{(2)})})$. Since $(\overline{\partial X - (\partial X \cap H_0^{(2)})})$ fibers over a 1-sphere and

$$\pi_1(X) \cong \pi_1(\partial X) \cong \pi_1(\overline{\partial X - (\partial X \cap H_0^{(2)})}) ,$$

the proof is completed.

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Received November 3, 1980.

KWANSEI GAKUIN UNIVERSITY,
NISHINOMIYA, HYOGO 662,
JAPAN