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A RECIPROCITY LAW FOR RAMANUJAN SUMS

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Certain arithmetic functions are representable by two types of series involving Ramanujan sums. A reciprocity law for Ramanujan's sum is derived which relates these series.

Ramanujan [7] and Hardy [4] found representations for several arithmetic functions of n, of the form

$$F(n) = \sum_{k=1}^{\infty} a(k)c_k(n) , \tag{1}$$

where $c_k(n)$ is Ramanujan's sum. Rearick [8] showed that such representations exist for a large class of arithmetic functions, and called them C-series representations. In this paper we show that functions of a certain class are also representable as series of the form

$$F(k) = \sum_{n=1}^{\infty} b(n, k)c_k(n) . \tag{2}$$

Let us call representations like (2) C'-series. The class of functions represented by C'-series is smaller than the class of functions representable by C-series; however we can use a reciprocity law for Ramanujan sums to show that (1) and (2) are equivalent under certain conditions.

Ramanujan proved

$$c_k(n) = \sum_{d\mid(k,n)} d\mu(k/d) . \tag{3}$$

Rearick and Donovan [3] showed that for fixed square-free k, the function $\mu(k)c_k(n)$ is multiplicative in n. After this observation it is easy to prove

LEMMA 1. *For fixed square-free k,*

$$\mu(k)c_k(n) = \sum_{d\mid(k,n)} d\mu(d) . \tag{4}$$

Proof. We need only show that the right hand side of (4) is multiplicative and then demonstrate the equality when n is a power of a prime.

DEFINITION 2. Let $\bar{k}$ denote the largest square-free divisor of

k (the core of k), and let $k^* = k/\bar{k}$.

Hardy [4] proved that $c_k(n) = 0$ unless $k^* \mid n$. Using this fact, one may work directly from (3) to prove

LEMMA 3. $c_k(nk^*) = k^* c_{\bar{k}}(n)$.

THEOREM 4. *Let F be an arithmetic function. Let $A(k)$ be the function which is zero whenever k is not square-free, and for square-free values of k is defined recursively by the equation*

$$F(k) = \sum_{d \mid k} d\mu(d) A(d) . \tag{5}$$

Suppose

$$\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} |A(kn)| < \infty . \tag{6}$$

Define

$$b(k) = \sum_{n=1}^{\infty} \mu(n) A(nk) . \tag{7}$$

Then

$$\mu(k) F(k) = \sum_{n=1}^{\infty} b(n) c_k(n) , \tag{8}$$

for all square-free k, and this series converges absolutely. On the other hand, if F has a series representation like (8) with $\sum_{k=1}^{\infty} |b(k)| < \infty$, and $b(k) = 0$ if k is not square-free, then the coefficient function is given by (7).

Proof. Using a well known inversion property of the Möbius function ([5, Theorem 270]),

$$A(k) = \sum_{n=1}^{\infty} b(nk) . \tag{9}$$

Using this in (5) we obtain

$$\begin{aligned} F(k) &= \sum_{d \mid k} d\mu(d) \sum_{n=1}^{\infty} b(nd) \\ &= \sum_{n=1}^{\infty} b(n) \sum_{d \mid (k,n)} d\mu(d) \\ &= \sum_{n=1}^{\infty} b(n) c_k(n) \mu(k) , \qquad \text{by Lemma 1} \end{aligned}$$

since k is square-free. So $\mu(k) F(k) = \sum_{n=1}^{\infty} b(n) c_k(n)$. The absolute convergence of (8) follows from (6).

On the other hand, given (8) with $b(k) = 0$ for all non square-free k, and $\sum_{k=1}^{\infty} |b(k)| < \infty$, we can reverse the steps to obtain (9), which after (6) is equivalent to (7), so the coefficient function is uniquely determined.

The restriction of the argument of the coefficient function to the square-free integers is necessary for uniqueness. For example, if we expand $\sum_{n=1}^{\infty} (\mu(k)c_k(n)/n^s)$ and $\sum_{n=1}^{\infty}(|\mu(n)|\mu(k)c_k(n)/\varphi_s(n))$ in Euler products ($s = \sigma + it$) for $\sigma > 1$ we find

$$\varphi_{s-1}(k) = \frac{\mu(k)k^{s-1}}{\zeta(s)} \sum_{n=1}^{\infty} \frac{c_k(n)}{n^s}$$

and also

$$\varphi_{s-1}(k) = \frac{\mu(k)k^{s-1}}{\zeta(s)} \sum_{n=1}^{\infty} \frac{|\mu(n)|c_k(n)}{\varphi_s(n)} ,$$

for any square-free value of k (see [6]). Here $\varphi_s(k) = k^s \prod_{p|k} (1-(1/p^s))$.

Since the sum in Lemma 1 is symmetric in k and n, we have

LEMMA 5. *For all square-free k and n,*

$$\mu(k)c_k(n) = \mu(n)c_n(k) .$$

We may now obtain the general reciprocity law for Ramanujan sums.

THEOREM 6. *For all k and n,*

$$\frac{\mu(\bar{k})}{k^*}c_k(nk^*) = \frac{\mu(\bar{n})}{n^*}c_n(kn^*) .$$

Proof. By Lemma 5, $\mu(\bar{k})c_{\bar{k}}(\bar{n}) = \mu(\bar{n})c_{\bar{n}}(\bar{k})$. From (3) it follows that $c_{\bar{k}}(\bar{n}) = c_{\bar{k}}(n)$, so $\mu(\bar{k})c_{\bar{k}}(n) = \mu(\bar{n})c_{\bar{n}}(k)$. Upon application of Lemma 3 to both sides of this equation the reciprocity law is proved.

Using the reciprocity law for Ramanujan sums, we can show that C'-series exist which represent certain functions for all integral values of their argument. However, in these representations the coefficient function becomes dependent on both variables. These C'-series representations are equivalent to a class of C-series representations.

THEOREM 7. *A function F has an absolutely convergent C-series representation $F(n) = \sum_{k=1}^{\infty} a(k)c_k(n)$, with $a(k) = 0$ if k is not square-free, if and only if $F(n)n^*\mu(\bar{n})$ has an absolutely convergent C'-series representation $F(n)n^*\mu(\bar{n}) = \sum_{k=1}^{\infty} b(k/n^*)c_n(k)$.*

Proof. Suppose $F(n) = \sum_{k=1}^{\infty} a(k)c_k(n)$ is absolutely convergent and $a(k) = 0$ if k is not square-free. Then

$$\begin{aligned} F(n) &= \sum_{k=1}^{\infty} a(k)c_k(n) \\ &= \sum_{k=1}^{\infty} \mu(k)a(k)\mu(k)c_k(n)\,, \qquad \text{since } k \text{ is square-free} \\ &= \sum_{k=1}^{\infty} \mu(k)a(k)\frac{\mu(\bar{n})}{n^*}c_n(kn^*)\,, \qquad \text{by reciprocity} \\ &= \frac{\mu(\bar{n})}{n^*}\sum_{k=1}^{\infty} b(k)c_n(kn^*)\,, \qquad \text{where } b(k) = \mu(k)a(k) \\ &= \frac{\mu(\bar{n})}{n^*}\sum_{n=1}^{\infty} b(k/n^*)c_n(k)\,. \end{aligned}$$

Conversely,

$$\begin{aligned} F(n) &= \frac{\mu(\bar{n})}{n^*}\sum_{k=1}^{\infty} b(k/n^*)c_n(k) \\ &= \frac{\mu(\bar{n})}{n^*}\sum_{k=1}^{\infty} b(k)c_n(kn^*) \\ &= \sum_{k=1}^{\infty} \frac{b(k)\mu(\bar{k})}{k^*}c_k(nk^*)\,, \qquad \text{by reciprocity} \\ &= \sum_{k=1}^{\infty} \alpha(k)c_k(nk^*)\,, \qquad \text{where } \alpha(k) = \frac{b(k)\mu(\bar{k})}{k^*} \\ &= \sum_{k=1}^{\infty} \alpha(k)k^*c_{\bar{k}}(n)\,, \qquad \text{by Lemma 3}\,. \end{aligned}$$

If we now let $H(k) = \{m \in \boldsymbol{Z}^+ \colon \bar{m} = k\}$ and put

$$a(k) = \sum_{m \in H(k)} m^*\alpha(m)$$

(this converges since the original sum converged absolutely), then

$$F(n) = \sum_{k=1}^{\infty} a(k)c_k(n)\,,$$

with $a(k) = 0$ if k is not square-free.

For example, in [7] Ramanujan proved

$$\varphi_{s-1}(n) = \frac{n^{s-1}}{\zeta(s)}\sum_{k=1}^{\infty} \frac{\mu(k)c_k(n)}{\varphi_s(k)}\,. \tag{10}$$

If we now apply the reciprocity law to (10) we obtain

$$\varphi_{s-1}(n) = \frac{n^{s-1}\mu(\bar{n})}{\zeta(s)n^*}\sum_{k=1}^{\infty} \frac{|\mu(k/n^*)|c_n(k)}{\varphi_s(k/n^*)}\,,$$

which is a C'-series representation of φ_{s-1}. This representation is now valid for all n, generalizing our earlier result for φ_{s-1}.

Following Anderson and Apostol [1], we turn our attention to a generalized Ramanujan sum, defined as follows.

DEFINITION 8. For any two arithmetic functions f and g let $s_k(n) = \sum_{d\mid(k,n)} f(d)g(k/d)$.

THEOREM 9. *Suppose f and g are multiplicative and $g(n) = \pm 1$ for all n. Then for fixed n the function $s_k(n)$ is multiplicative in the variable k, while for fixed k the function $g(k)s_k(n)$ is multiplicative in the variable n.*

Proof. The first assertion is easy and requires only that f and g be multiplicative (see [2, Lemma 2.1]). To prove the second assertion, fix k and choose $(n, m) = 1$. We have

$$g(k)s_k(m)g(k)s_k(n) = s_k(m)s_k(n) , \tag{11}$$

since $g^2(k) = 1$. Both sides of (11) are multiplicative in k so we may assume k is a prime power. Then

$$s_k(m)s_k(n) = \sum_{d_1\mid(k,m)} f(d_1)g(k/d_1) \sum_{d_2\mid(k,n)} f(d_2)g(k/d_2) .$$

Since $(m, n) = 1$ and k is a prime power, either d_1 or d_2 must be 1, so

$$g(k/d_1)g(k/d_2) = g(k)g(k/d_1d_2)$$

and thus

$$\begin{aligned} s_k(m)s_k(n) &= \sum_{d_1d_2\mid(k,m)} f(d_1d_2)g(k)g(k/d_1d_2) \\ &= g(k)\sum_{d\mid(k,mn)} f(d)g(k/d) \\ &= g(k)s_k(mn) , \end{aligned}$$

thus establishing the theorem.

In order to obtain a satisfactory reciprocity law for $s_k(n)$, we must now either restrict k and n to the square-free integers, or restrict g to the class of completely multiplicative functions. We choose the latter course.

LEMMA 10. *Suppose f is multiplicative, g completely multiplicative, and $g(n) = \pm 1$ for all n. Then*

$$g(k)s_k(n) = \sum_{d\mid(k,n)} f(d)g(d) .$$

Proof.

$$g(k)s_k(n) = g(k) \sum_{d\mid(k,n)} f(d)g(d)$$

$$= g(k) \sum_{d\mid(k,n)} f(d)\frac{g(k)}{g(d)}$$

since g is completely multiplicative,

$$= \sum_{d\mid(k,n)} f(d)g(d) ,$$

since $g(n) = \pm 1$ for all n.

Under our assumptions on f and g, the reciprocity law for $s_k(n)$ is somewhat simpler than that for $c_k(n)$, namely:

THEOREM 11. *With the hypotheses of Lemma* 10,

$$g(n)s_n(k) = g(k)s_k(n) .$$

Proof. The sum in Lemma 10 is unchanged if we interchange k and n.

Fnally, using the same proof as for Theorem 4, we can derive general series expansions of certain arithmetic functions in terms of either variable of $s_k(n)$, and the expansions are equivalent due to reciprocity. We have

THEOREM 12. *Let F, f, and g be arithmetic functions, with f and g satisfying the hypotheses of Lemma* 10. *Let $A(n)$ be the function which is zero whenever $f(n) = 0$, and for other values of n is defined recursively by the equation $F(n) = \sum_{d\mid n} f(d)g(d)A(d)$. Suppose*

$$\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} |A(kn)| < \infty . \tag{12}$$

Also define

$$b(n) = \sum_{k=1}^{\infty} A(kn)\mu(k) \tag{13}$$

and

$$a(n) = g(n)b(n) . \tag{14}$$

Then for all n such that $f(n) \neq 0$,

$$F(n) = \sum_{k=1}^{\infty} a(k)s_k(n) \tag{15}$$

and

$$g(n)F(n) = \sum_{k=1}^{\infty} b(k)s_n(k) \tag{16}$$

where both sums converge absolutely. On the other hand, if $a(n)$ *or* $b(n)$ *is the coefficient function in a series of the form* (15) *or* (16) *representing* F *whose associated function* A *satisfies* (12), *with* $\sum_{n=1}^{\infty}|a(n)| < \infty$, *and* $A(n) = 0$ *if* $f(n) = 0$, *then the coefficient functions are given by* (13) *and* (14).

For example, if we let $s_k(n) = \sum_{d|(k,n)} \mu(d)\lambda(k/d)$, we can expand $\sum_{k=1}^{\infty}(\mu(k)s_k(n)/\varphi_s(k))$ in an Euler product for $\sigma > 1$, and obtain (for all square-free n)

$$\sigma_s(n) = \frac{n^s}{\zeta(s)}\sum_{k=1}^{\infty}\frac{\mu(k)s_k(n)}{\varphi_s(k)}\,.$$

($\sigma_s(n)$ is the sum of the sth powers of the divisors of n.)

In light of reciprocity and the fact that $|\mu(k)| = \mu(k)\lambda(k)$, we may also write

$$\lambda(n)\sigma_s(n) = \frac{n^s}{\zeta(s)}\sum_{k=1}^{\infty}\frac{|\mu(k)|\,s_n(k)}{\varphi_s(k)}\,.$$

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