

## A GEOMETRIC CHARACTERIZATION OF $n$ TH ORDER CONVEX FUNCTIONS

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A new geometric characterization is presented for a function convex of order  $n$  on an open interval, distinct from the whole of  $\mathbf{R}$ . We shall prove that if  $f: (a, b) \rightarrow \mathbf{R}$  with  $b < +\infty$ , if  $\alpha$  is an arbitrarily fixed number,  $\alpha \geq b$ , and if  $F(x)$  denotes the ordinate of the point of intersection in the  $x, y$ -plane between the vertical line  $x = \alpha$  and the osculating parabola of order  $n$  to the graph of  $f$  at the point  $(x, f(x))$ , then  $f$  is convex of order  $n$  on  $(a, b)$  iff  $F$  is increasing thereon.

1. Introduction. Let  $f$  be a real-valued convex function on the interval  $(a, b)$  where  $a \geq 0$ : it is stated in [1; p. I. 51, Exercise 7], and is indeed elementary to prove, that the function  $F(x) \equiv f(x) - xf'_x(x)$  is decreasing on  $(a, b)$ ,  $f'_x$  denoting the right derivative of  $f$ .  $F(x)$  is none other than the "ordinate at the origin" of the right tangent line to the graph of  $f$  at the point  $(x, f(x))$ . Besides proving the converse of this proposition in this paper we shall extend the result to  $n$ th order convex functions, the role of the tangent line being played by the osculating parabola of order  $n$ . The result we present provides a meaningful geometric characterization of such a class of functions (Theorem 2.1 below).

The notion of higher-order convexity is classical: its systematic study essentially began with a paper by Popoviciu [4] and was continued in many other works by the same author. The entire theory is surveyed in his monograph [5]. Other properties can be found in books [2; Chp. 4 §3 and Chp. 3 §2] and [3; Chp. XI] in the context of generalized convex functions.

Many characterizations of  $n$ th order convex functions can be obtained from these references; in order to establish our main result we shall only need a few of such characterizations and shall state them here for the sake of convenience.

DEFINITION. A function  $f: (a, b) \rightarrow \mathbf{R}$ ,  $-\infty \leq a < b \leq +\infty$ , is said to be convex of order  $n$  ( $n \in \mathbf{N}$ ) on the open interval  $(a, b)$  if for all choices of  $\{x_i\}_{i=0}^{n+1}$  satisfying  $a < x_0 < x_1 < \dots < x_{n+1} < b$  the inequality below holds if in (1.1) the strict sign always prevails then  $f$  is said to be strictly convex. For  $n = 1$  we have the usual convex functions.

$$(1.1) \quad \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_0 & x_1 & \cdots & x_{n+1} \\ x_0^2 & x_1^2 & \cdots & x_{n+1}^2 \\ \vdots & \vdots & \vdots & \vdots \\ x_0^n & x_1^n & \cdots & x_{n+1}^n \\ f(x_0) & f(x_1) & \cdots & f(x_{n+1}) \end{vmatrix} \cong 0$$

**THEOREM 1.1.** For  $f: (a, b) \rightarrow \mathbf{R}$  the following are equivalent properties:

- (1)  $f$  is [strictly] convex of order  $n$  on  $(a, b)$ ;
- (2)  $f \in C^{n-1}(a, b)$  and there exists a denumerable set  $N \subset (a, b)$  such that  $f^{(n)}(x)$  exists and is [strictly] increasing on  $(a, b) \setminus N$ ;
- (3)  $f \in C^{n-1}(a, b)$  and  $f^{(n-1)}(x)$  has a right derivative  $f_R^{(n)}$  which is right-continuous and [strictly] increasing on  $(a, b)$ .

*Proof.* This theorem easily follows from some known characterizations of  $n$ th order convex functions but, since the present version is not explicitly stated in the literature, we give a sketch of its proof. The whole proof is based on the fact that  $f$  is [strictly] convex of order  $n$  on  $(a, b)$  iff  $f \in C^{n-1}(a, b)$  and  $f^{(n-1)}$  is [strictly] convex of order 1 on  $(a, b)$ : see [4; p. 41] or [3; Th. 2.1, p. 386].

(1)  $\Leftrightarrow$  (2). This is explicitly stated for  $n = 1$  in [1; Prop. 8, p. I. 38] and follows for  $n \geq 2$  from the case  $n = 1$  and the above remark.

(1)  $\Leftrightarrow$  (3). This is a particular case of Th. 2.1 in [3; p. 386] for nonstrict convexity when the condition that  $f^{(n-1)}(x)$  has a left-continuous left derivative on  $(a, b)$  is added.

Now let  $n = 1$  and let  $f$  be a usual [strict] convex function on  $(a, b)$ : the implication (1)  $\Rightarrow$  (3) is classical: see [1; Prop. 6 and Cor. 2, pp. I. 36–I. 37]. For the converse notice that if  $f \in C^0(a, b)$  and  $f'_R$  is merely supposed to exist and to be increasing on  $(a, b)$  then there exists a denumerable set  $N \subset (a, b)$  such that  $f'_R$  is continuous on  $(a, b) \setminus N$ . Hence  $f'$  exists on  $(a, b) \setminus N$  and is obviously [strictly] increasing thereon: i.e., (3)  $\Rightarrow$  (2). For  $n \geq 2$  one combines the case  $n = 1$  and the initial remark.

**2. The main result.** Let  $f$  be  $(n - 1)$  times differentiable on  $(a, b)$  and let there exist the right derivative of  $f^{(n-1)}$  at some point  $\xi \in (a, b)$ :  $f_R^{(n)}(\xi)$ .

The right osculating parabola of order  $n$  to the graph of  $f$  at the point  $(\xi, f(\xi))$  is defined as the curve whose equation in the plane

referred to rectangular cartesian coordinates  $x, y$  is

$$(2.1) \quad y = f(\xi) + \frac{f'(\xi)}{1!}(x - \xi) + \cdots + \frac{f^{(n-1)}(\xi)}{(n-1)!}(x - \xi)^{n-1} + \frac{f_R^{(n)}(\xi)}{n!}(x - \xi)^n .$$

If  $\alpha$  is a fixed number, the quantity

$$(2.2) \quad F(\xi) \equiv f(\xi) + \frac{(\alpha - \xi)}{1!}f'(\xi) + \cdots + \frac{(\alpha - \xi)^{n-1}}{(n-1)!}f^{(n-1)}(\xi) + \frac{(\alpha - \xi)^n}{n!}f_R^{(n)}(\xi)$$

represents the ordinate of the point of intersection between the vertical line  $x = \alpha$  and the curve (2.1). We shall prove the following result which stresses the connection between the convexity character of a function  $f$  and the monotonicity character of the associated function  $F$ .

**THEOREM 2.1.** *Let  $(a, b)$  be an open interval distinct from the whole of  $\mathbf{R}$ ,  $n$  an integer ( $n \geq 1$ ) and  $\alpha$  an arbitrarily fixed number outside  $(a, b)$ . Let  $f \in C^{n-1}(a, b)$  and suppose that  $f^{(n-1)}$  is absolutely continuous on every compact subinterval of  $(a, b)$  and that it has a right derivative  $f_R^{(n)}$  which is right-continuous on  $(a, b)$ . Then the following are equivalent properties:*

- (1)  $f$  is [strictly] convex of order  $n$  on  $(a, b)$ ;
- (2)  $(-1)^n F$  is [strictly] increasing on  $(a, b)$  (in the case  $\alpha \leq a$ );
- (3)  $F$  is [strictly] increasing on  $(a, b)$  (in the case  $\alpha \geq b$ ).

*Proof.* If  $f$  is assumed to be  $(n + 1)$  times differentiable on  $(a, b)$  the assertion follows trivially from Th. 1.1 and the identity

$$(2.3) \quad F'(x) = \frac{(\alpha - x)^n}{n!}f^{(n+1)}(x) = \frac{(-1)^n}{n!}(x - \alpha)^n f^{(n+1)}(x), \quad x \in (a, b) .$$

For the general case we shall present two entirely different proofs: the first uses arguments from classical analysis (some integration formulas and the second mean value theorem of integral calculus) while the second, briefer than the first, is based on distribution theory and patterned after the above-mentioned trivial proof.

*First proof.* In order to use Th. 1.1 we need a connection between  $f_R^{(n)}$  and  $F$  and we shall in fact prove a kind of mean value formula, namely that for each couple of points  $x_1 < x_2$  in  $(a, b)$  there exists a  $\xi \in (x_1, x_2)$  such that

$$(2.4) \quad f_R^{(n)}(x_2) - f_R^{(n)}(x_1) = \frac{n!}{(\alpha - \xi)^n} [F(x_2) - F(x_1)].$$

Hence it follows

$$(2.5) \quad \begin{aligned} \text{sign}[f_R^{(n)}(x_2) - f_R^{(n)}(x_1)] &= (-1)^n \text{sign}[F(x_2) - F(x_1)] & \text{if } \alpha \leq a, \\ &= \text{sign}[F(x_2) - F(x_1)] & \text{if } \alpha \geq b. \end{aligned}$$

We shall show that (2.4) holds when any one of the two functions  $f_R^{(n)}$  and  $F$  is supposed to be monotonic: the assertion of Th. 2.1 will then follow immediately from (2.5) and Th. 1.1.

Suppose firstly that  $f_R^{(n)}$  is monotonic on  $(a, b)$  and let  $x_1, x_2 \in (a, b)$ ,  $x_1 < x_2$ . From the assumptions made for  $f$  it follows that  $f^{(n)}$  exists and is continuous on  $(a, b) \setminus N$ , where  $N$  is a suitable denumerable set, hence a classical result of the Riemann-Stieltjes integration theory ensures the equality between the two measures  $df^{(n-1)}(x)$  and  $f_R^{(n)}(x)dx$ . Now, using repeated integration by parts for Riemann-Stieltjes integrals, we may at once verify the validity of the formula

$$(2.6) \quad \frac{1}{(n-1)!} \int_{x_1}^{x_2} (\alpha - t)^{n-1} f_R^{(n)}(t) dt = \sum_{k=0}^{n-1} \left[ \frac{(\alpha - t)^k f^{(k)}(t)}{k!} \right]_{t=x_1}^{t=x_2}.$$

From (2.2) and (2.6) we derive

$$(2.7) \quad \begin{aligned} F(x_2) - F(x_1) &= \frac{1}{(n-1)!} \int_{x_1}^{x_2} (\alpha - t)^{n-1} f_R^{(n)}(t) dt \\ &+ \frac{(\alpha - x_2)^n f_R^{(n)}(x_2) - (\alpha - x_1)^n f_R^{(n)}(x_1)}{n!}. \end{aligned}$$

Applying the second mean value theorem of the integral calculus to the integral at the righthand side of (2.7) we obtain for a suitable  $\xi \in (x_1, x_2)$ :

$$\begin{aligned} F(x_2) - F(x_1) &= \frac{1}{(n-1)!} \left\{ f_R^{(n)}(x_1) \int_{x_1}^{\xi} (\alpha - t)^{n-1} dt + f_R^{(n)}(x_2) \int_{\xi}^{x_2} (\alpha - t)^{n-1} dt \right. \\ &\quad \left. + \frac{(\alpha - x_2)^n f_R^{(n)}(x_2) - (\alpha - x_1)^n f_R^{(n)}(x_1)}{n} \right\} \\ &= \frac{(\alpha - \xi)^n}{n!} [f_R^{(n)}(x_2) - f_R^{(n)}(x_1)], \end{aligned}$$

that is (2.4). Notice that when  $n = 1$  an elementary proof, which does not invoke the second mean value theorem, is derived by substituting the classical inequality  $f(x_1) \geq f(x_2) + f'_R(x_2)(x_1 - x_2)$  in the explicit expression of the difference  $F(x_1) - F(x_2)$ . But the same kind of argument does not work for  $n \geq 2$ .

Suppose now that  $F$  is monotonic; we will express  $f$  in terms of  $F$ . The starting point is the following identity

$$(2.8) \quad F(x) = \frac{(\alpha - x)^{n+1}}{n!} \frac{d^n}{dx^n} \left( \frac{f(x)}{\alpha - x} \right)$$

valid at each point  $x$  where  $f$  is  $n$  times differentiable. This identity is checked by applying the Leibniz rule to the product  $(\alpha - x)^{-1}f(x)$ . By the assumptions the function  $d^{n-1}/dx^{n-1}(f(x)/(\alpha - x))$  is absolutely continuous hence relation (2.8) is valid almost everywhere and we can invert it by applying a familiar integration formula thus inferring that, given any  $T \in (a, b)$ , there exist  $n$  constants  $c_1, \dots, c_n$  such that

$$(2.9) \quad f(x) = c_1(\alpha - x) + \dots + c_n(\alpha - x)^n + n(\alpha - x) \int_T^x (x - t)^{n-1} \frac{F(t)}{(\alpha - t)^{n+1}} dt, \\ x \in (a, b).$$

As  $F$  is right-continuous we derive from (2.9)

$$(2.10) \quad f_R^{(n)}(x) = (-1)^n c_n n! + n! \frac{F(x)}{(\alpha - x)^n} - n \cdot n! \int_T^x \frac{F(t)}{(\alpha - t)^{n+1}} dt, \\ x \in (a, b).$$

If now  $x_1, x_2$  are arbitrary points in  $(a, b)$ ,  $x_1 < x_2$ , we have

$$(2.11) \quad \frac{1}{n!} [f_R^{(n)}(x_2) - f_R^{(n)}(x_1)] = -n \int_{x_1}^{x_2} \frac{F(t)}{(\alpha - t)^{n+1}} dt \\ + \frac{F(x_2)}{(\alpha - x_2)^n} - \frac{F(x_1)}{(\alpha - x_1)^n}.$$

Since  $F$  is monotonic we may appeal to the second mean value theorem in estimating the integral in (2.11) and we find that the righthand side of (2.11) equals

$$-nF(x_1) \int_{x_1}^{\xi} \frac{dt}{(\alpha - t)^{n+1}} - nF(x_2) \int_{\xi}^{x_2} \frac{dt}{(\alpha - t)^{n+1}} \\ + \frac{F(x_2)}{(\alpha - x_2)^n} - \frac{F(x_1)}{(\alpha - x_1)^n} = \frac{1}{(\alpha - \xi)^n} [F(x_2) - F(x_1)],$$

that is (2.4). The proof is then complete.

*Second proof.* The reader is now supposed familiar with the definitions and results in [6; Chps. I and II] or in [7; Chps. 21 and 24]; from now on equalities and inequalities are to be understood in the sense of distributions. We shall prove two preliminary lemmas. It is known—[6; Chp. I, p. 54]—that a distribution on  $(a, b)$  is an increasing function iff its derivative is a positive distribution on  $(a, b)$ ; the next lemma is analogous, in the framework of distribution theory, to the elementary characterization of strict monotonicity of differentiable functions.

LEMMA A. *If  $T \in \mathcal{D}'(a, b)$  then  $T$  is a strictly increasing function on  $(a, b)$  iff its derivative  $DT$  is a positive distribution on  $(a, b)$  and  $(a, b) \setminus \text{supp } DT$  has no interior points.*

*Proof of Lemma A.* If  $T$  is a strictly increasing function on  $(a, b)$  then  $DT$  is a positive distribution on  $(a, b)$ ; if then there existed a nonvoid open interval  $(\alpha, \beta) \subset (a, b) \setminus \text{supp } DT$  we would have, by the very definition of  $\text{supp } DT$ , that  $DT$  vanishes on  $(\alpha, \beta)$ , i.e., the restriction of  $DT$  to  $(\alpha, \beta)$  is the zero distribution; hence  $T$ , as a primitive of  $DT$  on  $(\alpha, \beta)$ , is a constant function thereon which contradicts the assumption. Vice versa if  $DT$  is a positive distribution on  $(a, b)$  then  $T$  is an increasing function, say  $f$ , on  $(a, b)$ . If we now suppose that  $f$  is not strictly increasing then it is constant on some nonvoid open interval  $(\alpha, \beta) \subset (a, b)$ . Let now  $\phi$  be a test function,  $\phi \in C_0^\infty(a, b)$ , such that  $\text{supp } \phi \subset (\alpha, \beta)$ . We have

$$(2.12) \quad \begin{aligned} \langle Df, \phi \rangle &= -\langle f, \phi' \rangle = -\int_a^b f(x)\phi'(x)dx \\ &= -\int_a^\beta f(x)\phi'(x)dx = \int_a^\beta \phi(x)df(x) = 0, \end{aligned}$$

where integration by parts for Riemann-Stieltjes integrals has been used. Relation (2.12) simply means that  $Df$  vanishes on  $(\alpha, \beta)$  and hence  $(\alpha, \beta) \subset (a, b) \setminus \text{supp } DT$ : a contradiction.

The second lemma is a trivial consequence of the definition of the product of a distribution and an infinitely differentiable function, see [6; Chp. V and Th. I, p. 118].

LEMMA B. *Let  $T \in \mathcal{D}'(a, b)$  and let  $\phi$  be some function strictly positive and infinitely differentiable on  $(a, b)$ . Then we have  $T \geq 0$  on  $(a, b)$  iff  $\phi T \geq 0$  on  $(a, b)$ ; further  $\text{supp } \phi T = \text{supp } T$ .*

Turning back to the proof of our theorem we see that our assumptions on  $f$  ensure that  $f^{(n)}(x) = f_R^{(n)}(x)$  almost everywhere on  $(a, b)$  and, since  $f^{(n-1)}$  is absolutely continuous, we may appeal to Th. III in [6; p. 54] and infer from (2.2) that

$$(2.13) \quad DF = \frac{(\alpha - x)^n}{n!} - Df_R^{(n)}.$$

From (2.13) and Lemma B we infer that the following equivalences hold on  $(a, b)$  (to fix ideas let  $\alpha \geq b$ ):

$F$  increasing  $\Leftrightarrow DF \geq 0 \Leftrightarrow (\alpha - x)^n Df_R^{(n)} \geq 0 \Leftrightarrow Df_R^{(n)} \geq 0 \Leftrightarrow f_R^{(n)}$  increasing.

For strict monotonicity we must also take into account Lemma

A and that  $\text{supp } DF = \text{supp } [(\alpha - x)^n Df_R^{(n)}]$ . The second proof is so complete by appealing to Th. 1.1.

3. Some remarks. 1. The difference in the monotonicity character of the function  $F$  according as  $\alpha \leq a$  or  $\alpha \geq b$  parallels another characterization of an  $n$ th order convex function  $f$  based on the relative positions of the graph of  $f$  and that of its osculating parabola, namely that  $f(x) - F(x) \geq 0 \quad \forall x \in (a, b), x \geq \alpha$  and  $(-1)^n [f(x) - F(x)] \geq 0 \quad \forall x \in (a, b), x \leq \alpha$ . See [3; Chp. XI, Lemma 2.4] and [2; Chp. 4, pp. 173-176].

2. There is no analogous result if  $a < \alpha < b$ : indeed if  $f$  is convex of order  $n$  then  $F$  is increasing on  $(a, \alpha]$  while  $(-1)^n F$  is increasing on  $[\alpha, b)$ ; but generally speaking the converse is not true because a function which is convex on two adjacent intervals is not necessarily convex on their union. Hence we cannot give a characterization analogous to that of Th. 2.1 for functions convex on  $(-\infty, +\infty)$ .

3. Formula (2.9) can be used to give a characterization of  $n$ th order convex functions via an integral representation. For example we have the following

**THEOREM 3.1.** *If  $f: (a, b) \rightarrow \mathbf{R}$  with  $b < +\infty$  then  $f$  is [strictly] convex of order  $n$  on  $(a, b)$  iff there exist a [strictly] increasing function  $\tilde{F}$  on  $(a, b)$ , a point  $T \in (a, b)$ , a point  $\alpha \geq b$  and  $n$  constants  $c_1, \dots, c_n$  such that*

$$(3.1) \quad f(x) = c_1(\alpha - x) + \dots + c_n(\alpha - x)^n + n(\alpha - x) \int_T^x \frac{(x-t)^{n-1}}{(\alpha-t)^{n+1}} \tilde{F}(t) dt, \quad x \in (a, b).$$

Obviously the function  $\tilde{F}$  agrees with the function  $F$  given by (2.2) except possibly on a denumerable set. Formula (3.1) parallels the standard characterization through the relation

$$f(x) = \bar{c}_0 + \bar{c}_1 x + \dots + \bar{c}_{n-1} x^{n-1} + \frac{1}{(n-1)!} \int_T^x (x-t)^{n-1} \phi(t) dt$$

with a suitable increasing  $\phi$ , namely  $\phi = f_R^{(n)}$ .

4. In a further paper we shall highlight the meaning and usefulness of  $F$  (viewed as an operator in  $f$ ) in studying the asymptotic behavior of  $f$  with application to asymptotic expansions.

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