PACIFIC JOURNAL OF MATHEMATICS
Vol. 98, No. 1, 1982

ABSOLUTE C^*-EMBEDDING OF F-SPACES

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Let $\mathscr{U}$ be an open cover of a space X. We define $\mathscr{U}$ to be a P-cover if each element of $\mathscr{U}$ is a proper subset of X, $\mathscr{U}$ is closed under countable unions and for every $U \in \mathscr{U}$ there is a $V \in \mathscr{U}$ such that U and $X \backslash V$ are completely separated. We prove an F-space X is C^*-embedded in every F-space it is embedded in iff X has no P-covers or X is almost compact.

1. Introduction. In 1949, Hewitt [7] proved that a Tychonoff space is C^*-embedded in every Tychonoff space in which it is embedded iff X is almost compact. C. E. Aull [1] has shown that a P-space X is C^*-embedded in every P-space in which it is embedded iff X is almost Lindelöf (given disjoint zero sets of X at least one is Lindelöf). These two theorems are examples of absolute C^*-embedding theorems. In §3 of this paper we will provide the absolute C^*-embedding theorem for F-spaces. In §4 we obtain partial results concerning C^*-embeddings in basically disconnected spaces.

2. DEFINITIONS. All topological spaces will be assumed to be Tychonoff. The following theorem is useful when dealing with F-spaces and also provides a definition of F-spaces.

THEOREM 2.1 [6, 14.25]. *The following are equivalent*
(1) *X is an F-space.*
(2) *βX is an F-space.*
(3) *disjoint cozero subsets of X are completely separated.*
(4) *cozero subsets of X are C^*-embedded.*
(5) *disjoint cozero subsets of βX have disjoint closures.*

X is *basically disconnected* if the closure of every cozero set is clopen. X is a *P-space* if every zero set of X is open. The reader is referred to [6] for background on P-spaces, F-spaces and basically disconnected spaces. X is *weakly Lindelöf* if every open cover of X contains a countable subcollection whose union is dense in X [2]. If X is a subspace of Y and $\mathscr{C}$ is a collection of subsets of Y, we define $\mathscr{C}|_X = \{C \cap X : C \in \mathscr{C}\}$.

The cardinality of a set K is denoted by $|K|$ and the immediate successor of a cardinal α is denoted by α^+. The cofinality of a nonsuccessor ordinal α, denoted by $cf(\alpha)$, is the smallest cardinal κ such that $\alpha = \sup\{\delta_\gamma : \gamma < \kappa\}$, where $\delta_\gamma < \alpha$. Our notation and terminology follows that of the Gillman-Jerison text [6].

3. Absolute C^*-embedding of F-spaces.

DEFINITION 3.1. An open cover $\mathscr{C}$ of X is called a *P-cover* if each $U \in \mathscr{C}$ is a proper subset of X, $\mathscr{C}$ is closed under countable unions and for each $U \in \mathscr{C}$ there is a V in $\mathscr{C}$ such that U and $X \backslash V$ are completely separated in X.

It is immediate from the definition that a weakly Lindelöf space has no P-covers. In this paper we will find similarities between weakly Lindelöf F-spaces and F-spaces without P-covers, but in §5 we will give an example of an F-space without P-covers and which is not weakly Lindelöf.

DEFINITION 3.2. We will call $A \subset X$ a *P-set* of X if A is compact and any disjoint cozero set of X is completely separated from A. If $A = \{p\}$ is a P-set, then p (as usual) is called a P-point.

The following result motivates the use of the term "P-cover".

LEMMA 3.3. *There exists a P-set of βX contained in $\beta X \backslash X$* iff *X has a P-cover.*

Proof. Let P be a P-set of βX which is contained in $\beta X \backslash X$. Let $\mathscr{C} = \{C: C$ is a cozero subset of βX and $C \cap P = \phi\}$. We will show that $\mathscr{C}|_X = \{C \cap X: C \in \mathscr{C}\}$ is a P-cover of X. It is immediate that $\mathscr{C}|_X$ is closed under countable unions. If $U \in \mathscr{C}$, then P and U are completely separated by the definition of a P-set. Hence there is a zero set Z of βX containing P such that U and Z are completely separated in βX. Let $V = \beta X \backslash Z$; then $V \in \mathscr{C}$, and $U \cap X$ is completely separated from $Z \cap X = X \backslash V$. Also if $C \in \mathscr{C}$ then $\mathrm{cl}_{\beta X}(C \cap X) \cap P = \phi$ so $C \cap X$ is a proper subset of X. Therefore $\mathscr{C}|_X$ is a P-cover of X.

For the converse, assume $\mathscr{C}$ is a P-cover of X. Define P to be $\bigcap\{\mathrm{cl}_{\beta X}(X \backslash C): C \in \mathscr{C}\}$. We will show that P is the required P-set. P is compact and nonempty since $\mathscr{C}$ is closed under finite unions and therefore $\{\mathrm{cl}_{\beta X}(X \backslash C): C \in \mathscr{C}\}$ has the finite intersection property. Also P is contained in $\beta X \backslash X$ since $\mathscr{C}$ is a cover of X. Let U be a cozero subset of βX such that $U \cap P = \phi$. Then U is Lindelöf and $\bigcap\{\mathrm{cl}_{\beta X}(X \backslash C): C \in \mathscr{C}\} \cap U = \phi$, therefore there is a subset $\{C_n: n < \omega\}$ of $\mathscr{C}$ such that $\bigcap\{\mathrm{cl}_{\beta X}(X \backslash C_n): n < \omega\} \cap U = \phi$. In particular, $U \cap \mathrm{cl}_{\beta X}(X \backslash \bigcup\{C_n: n < \omega\}) = \phi$; so $U \subset \mathrm{cl}_{\beta X} \bigcup\{C_n: n < \omega\}$. Because $\mathscr{C}$ is a P-cover, there is a V in $\mathscr{C}$ such that $\bigcup\{C_n: n < \omega\}$ and $X \backslash V$ are completely separated. Since $P \subset \mathrm{cl}_{\beta X}(X \backslash V)$ and $U \subset \mathrm{cl}_{\beta X} \bigcup\{C_n: n < \omega\}$, we have P and U are completely separated in βX. □

LEMMA 3.4. *Let K be a compact F-space. If P is a P-set of K and q is a point of K, then the quotient space formed by collapsing $P \cup \{q\}$ to a point is an F-space.*

Proof. Let Y be the quotient space and f the quotient map. Since $P \cup \{q\}$ is compact, Y is Tychonoff. All that remains to be shown is that disjoint cozero sets C^0 and C^1 of Y can be completely separated. The cozero sets $f^{\leftarrow}(C^0)$ and $f^{\leftarrow}(C^1)$ of K are disjoint, so $\mathrm{cl}_K f^{\leftarrow}(C^0) \cap \mathrm{cl}_K f^{\leftarrow}(C^1) = \phi$. We can assume w.l.o.g. that $q \notin \mathrm{cl}_K f^{\leftarrow}(C^1)$. Since $q \notin \mathrm{cl}_K f^{\leftarrow}(C^1)$ implies $q \notin f^{\leftarrow}(C^1)$, we have $(P \cup \{q\}) \cap f^{\leftarrow}(C^1) = \phi$, and therefore $P \cap \mathrm{cl}_K f^{\leftarrow}(C^1) = \phi$. The function f is one-to-one on the set $K \backslash (P \cup \{q\})$ and $(P \cup \{q\}) \cap \mathrm{cl}_K f^{\leftarrow}(C^1) = \phi$, therefore the full preimage of $f(\mathrm{cl}_K f^{\leftarrow}(C^1))$ is $\mathrm{cl}_K f^{\leftarrow}(C^1)$. Thus $f(\mathrm{cl}_K f^{\leftarrow}(C^0))$ and $f(\mathrm{cl}_K f^{\leftarrow}(C^1))$ are disjoint compact sets of Y which contain C^0 and C^1 respectively, so C^0 is completely separated from C^1 in Y. □

It is known that the property "weakly Lindelöf" is inherited by regular closed subspaces. Though a regular closed subspace of an F-space without P-covers may have a P-cover [3, pg. 70], we do have the following result.

LEMMA 3.5. *If C is a cozero set of an F-space X and X has no P-covers then* $\mathrm{cl}_X C$ *has no P-covers.*

Proof. Assume $\mathrm{cl}_X C$ has a P-cover. Then, by Lemma 3.3, there exists a P-set P of $\beta(\mathrm{cl}_X C)$ contained in $\beta(\mathrm{cl}_X C) \backslash \mathrm{cl}_X C$. C, and therefore $\mathrm{cl}_X C$, are C^*-embedded in X, so $P \subset \beta(\mathrm{cl}_X C) = \mathrm{cl}_{\beta X} C \subset \beta X$. We will show that P is a P-set of βX. Let U be a cozero set of βX such that $U \cap P = \phi$. Then $U \cap \mathrm{cl}_{\beta X} C$ is a cozero set of $\mathrm{cl}_{\beta X} C$ which misses P, hence $\mathrm{cl}_{\beta X}(U \cap \mathrm{cl}_{\beta X} C) \cap P = \phi$. Since $\mathrm{cl}_{\beta X}(U \cap \mathrm{cl}_{\beta X} C)$ and P are disjoint compact sets of βX, there is a zero set Z of βX which contains $\mathrm{cl}_{\beta X}(U \cap \mathrm{cl}_{\beta X} C)$ and misses P. $(U \backslash Z) \cap X$ and C are disjoint cozero sets of X, and have disjoint closures in βX. But now we have $Z \cup \mathrm{cl}_{\beta X}[(U \backslash Z) \cap X]$ is a compact set containing U which misses P, so $P \cap (\mathrm{cl}_{\beta X} U) = \phi$. X has a P-cover since P is a P-set of βX and $P \subset \mathrm{cl}_{\beta X} C \backslash \mathrm{cl}_X C \subset \beta X \backslash X$. □

THEOREM 3.6. *Let A and B be subsets of an F-space X such* *that neither* A *nor* B *have P-covers and* $\mathrm{cl}_X A \cap B = A \cap \mathrm{cl}_X B = \phi$. *Then* A *and* B *are completely separated in* X.

Proof. Let $K = \mathrm{cl}_{\beta X}(A \cup B)$. The compact set K, as a C^*-embedded subset of an F-space, is an F-space [6, 14.26]. It will suffice to show A and B are completely separated in K.

Define $\mathscr{U} = \{U$: U is a cozero set of K and U and B are completely separated in $K\}$. Define $\mathscr{U}|_A = \{U \cap A: U \in \mathscr{U}\}$. By assumption, $A \cap \mathrm{cl}_K B = \phi$, so $\mathscr{U}|_A$ is an open over of A. If $A \in \mathscr{U}|_A$, then A and B are completely separated so we assume $A \notin \mathscr{U}|_A$ and we will arrive at a contradiction.

If $U \in \mathscr{U}$, then there exists a zero set Z of K containing U and completely separated from B. Choose a cozero set V containing Z and completely separated from B. So we now have $U \subset Z \subset V \subset K \backslash B$ and $V \in \mathscr{U}$. Since U is disjoint from the cozero set $K \backslash Z$, U is completely separated from $K \backslash V \subset K \backslash Z$. Since A has no P-covers, $\mathscr{U}|_A$ is not a P-cover, therefore there exist countably many cozero sets $\{U_i: i < \omega\} \subset \mathscr{U}$ such that $\cup\{U_i \cap A: i < \omega\} \notin \mathscr{U}|_A$. Let $W = \cup\{U_i: i < \omega\}$.

Define $\mathscr{V} = \{V$: V is a cozero set of K and $V \cap W = \phi\}$. $\mathscr{V}|_B$ is a cover of B since $B \cap \mathrm{cl}_K A = \phi$ and $\mathrm{cl}_K W \subset \mathrm{cl}_K A$. If $U \in \mathscr{V}$ then there exists a zero set Z of K containing W and completely separated from U. If $V = K \backslash Z$ then $V \in \mathscr{V}$. U is completely separated from $K \backslash V = Z$, so $U \cap B$ is completely separated from $B \backslash (V \cap B)$. $\mathscr{V}|_B$ is obviously closed under countable unions. But $\mathscr{V}|_B$ cannot be a P-cover of B, so $B \in \mathscr{V}|_B$, therefore there exists a cozero set V of K such that $B \subset V \in \mathscr{V}$ and $V \cap W = \phi$. Therefore B and W are completely separated, which is a contradiction to $W \notin \mathscr{U}$. □

We now state and prove the main theorem of this paper.

THEOREM 3.7. *An F-space* X *is* C^**-embedded in every F-space it is embedded in* iff X *has no P-covers or* X *is almost compact.*

Proof. Assume that X is an F-space with no P-covers and X is embedded in an F-space Y. It will suffice to show that disjoint cozero sets of X are completely separated in Y. Let C^0 and C^1 be disjoint cozero sets of X. By Lemma 3.5, $\mathrm{cl}_X C^0$ and $\mathrm{cl}_X C^1$ have no P-covers. We note that $\mathrm{cl}_Y(\mathrm{cl}_X C^0) \cap \mathrm{cl}_X C^1 = \phi$ and $\mathrm{cl}_X C^0 \cap \mathrm{cl}_Y(\mathrm{cl}_X C^1) = \phi$, so by Theorem 3.6, they are completely separated in Y.

For the converse assume X is not almost compact and X has a P-cover. By Lemma 3.3 there is a P-set P of βX contained in $\beta X \backslash X$. Choose a point $q \in \beta X \backslash X$ such that $|P \cup \{q\}| > 1$. Then by Lemma 3.4, the quotient space $\beta X/(P \cup \{q\})$ obtained by collapsing $P \cup \{q\}$ to a point is an F-space in which X is densely embedded but not C^*-embedded. □

The next corollary uses a construction similiar to one given in [10, pg. 96]. We will show that for every space X which is embedded in an F-space Y, there is an F-space W in which

(1) X is embedded as a closed set and

(2) X is C^*-embedded in W iff X is C^*-embedded in Y.

COROLLARY 3.8. *An F-space X is C^*-embedded in every F-space it is embedded in as a closed set* iff *X has no P-covers or X is almost compact.*

Proof. Suppose X is embedded in an F-space Y. Let λ be the least ordinal of cardinality $|\beta Y|^+$. Define $A = (\lambda + 1)\backslash\{\alpha: cf(\alpha) = \omega\}$. Negrepontis [8] has shown that the product of a P-space with a compact F-space is an F-space. A is a P-space, so $A \times \beta Y$ is an F-space. Let $W = (A \times \beta Y)\backslash(\{\lambda\} \times \beta Y \backslash X)$. W is a dense C^*-embedded subspace of $A \times \beta Y$ (see Example 5.1 or [10, pg. 96]), so W is an F-space. X is homeomorphic to the closed subspace $\{\lambda\} \times X$ of W. For every continuous real-valued function f defined on W, there exists an $\alpha < \lambda$ such that for all $x \in X$, $f(\alpha, x) = f(\lambda, x)$. As a consequence, $\{\lambda\} \times X$ is C^*-embedded in W iff X is C^*-embedded in Y. This will show that Corollary 3.8 is equivalent to Theorem 3.7. □

Note that if X is C-embedded in βX then X is pseudocompact; and a pseudocompact space is C-embedded iff it is C^*-embedded. This, along with Theorem 3.7, proves the next corollary.

COROLLARY 3.9. *An F-space X is C-embedded in every F-space it is embedded in* iff *X is almost compact or X is pseudocompact and has no P-covers.*

4. Absolute C^*-embedding in basically disconnected spaces. Let $\mathscr{C}$ be a cover by cozero sets of a basically disconnected space X, and assume the union of every countable subcollection of $\mathscr{C}$ is not dense. The set of unions of every countable subset of the open cover $\{\mathrm{cl}_X \cup \{C_n: n < \omega\}: \{C_n: n < \omega\} \subset \mathscr{C}\}$ is easily seen to be a P-cover of X. Therefore, for a basically disconnected space X, X has a P-cover iff X is not weakly Lindelöf. By Theorem 3.7 and this remark we have the following corollary.

COROLLARY 4.1. *A basically disconnected space X is C^*-embedded in every F-space it is embedded in* iff *X is weakly Lindelöf or X is almost compact.*

DEFINITION 4.2. A space X is *almost weakly Lindelöf* if given two disjoint cozero sets of X, at least one is weakly Lindelöf.

The next lemma is similar to Lemma 3.4.

LEMMA 4.3. *Let K be a compact basically disconnected space. If P is a P-set of K, then the quotient space formed by collapsing P to a point is basically disconnected.*

Proof. Let Y be the quotient space and $f: K \to Y$ the quotient map. Since P is compact, Y is Tychonoff. Let C be a cozero set of Y. $\mathrm{cl}_K f^{\leftarrow}(C)$ is open and f is a quotient map so we will prove that $\mathrm{cl}_Y C$ is open by showing $f^{\leftarrow}(\mathrm{cl}_Y C) = \mathrm{cl}_K f^{\leftarrow}(C)$. It is obvious that $\mathrm{cl}_K f^{\leftarrow}(C) \subset f^{\leftarrow}(\mathrm{cl}_Y C)$, so let $x \in K$ such that $f(x) \in \mathrm{cl}_Y C = f(\mathrm{cl}_K f^{\leftarrow}(C))$. We wish to prove $x \in \mathrm{cl}_K f^{\leftarrow}(C)$. There is a $y \in \mathrm{cl}_K f^{\leftarrow}(C)$ such that $f(x) = f(y)$. If $x = y$, we are done so assume $x \neq y$. Then $\{x, y\} \subset P$. We now have $y \in P \cap \mathrm{cl}_K f^{\leftarrow}(C) \neq \varnothing$ and since P is a P-set and $f^{\leftarrow}(C)$ is a cozero set, $P \cap f^{\leftarrow}(C) \neq \varnothing$. Therefore $x \in P \subset f^{\leftarrow}(C) \subset \mathrm{cl}_K f^{\leftarrow}(C)$. □

We now prove the main result in this section.

THEOREM 4.4. *If a basically disconnected space X is C^*-embedded is every basically disconnected space it is embedded in, then X is almost weakly Lindelöf.*

Proof. Let X be a basically disconnected space which is not almost weakly Lindelöf. Let C^0 and C^1 be disjoint cozero subsets of X neither of which is weakly Lindelöf. A cozero set of a weakly Lindelöf space is weakly Lindelöf [2, Lemma 1.2(c)], therefore $\mathrm{cl}_X C^0$ and $\mathrm{cl}_X C^1$ are not weakly Lindelöf, and since they are basically disconnected spaces, they both have P-covers. By the proof of Lemma 3.5 there are two disjoint P-sets, P^0 and P^1, of βX contained in $\mathrm{cl}_{\beta X} C^0 \backslash \mathrm{cl}_X C^0$ and $\mathrm{cl}_{\beta X} C^1 \backslash \mathrm{cl}_X C^1$ respectively. Then $P^0 \cup P^1$ is a P-set and the quotient space obtained by collapsing $P^0 \cup P^1$ to a point is basically disconnected by Lemma 4.3. X is a dense subspace of the quotient space, but it is not C^*-embedded since $|P^0 \cup P^1| > 1$. □

Unfortunately, an example in §5 will show that the property almost weakly Lindelöf is not a sufficient condition for C^*-embedding. It remains an open question to characterize the basically disconnected spaces which are C^*-embedded in every basically disconnected space in which they are embedded. But we do have the following theorem.

THEOREM 4.5. *If a basically disconnected space X is embedded as an open or dense subspace of a basically disconnected space Y, then X is C^*-embedded in Y* iff *X is almost weakly Lindelöf.*

Proof. Assume X is almost weakly Lindelöf and is embedded in a basically disconnected space Y. If C^0 and C^1 are disjoint cozero sets of X, then we can assume that one of them, say C^0, is weakly Lindelöf. Define $\mathscr{V} = \{V: V$ is a cozero set of Y, $V \cap \mathrm{cl}_Y C^1 = \varnothing\}$. $\mathscr{V}|_{C^0}$ is a cover of C^0, so there is a countable subcollection $\{V_n: n < \omega\}$ of $\mathscr{V}$ such that, if $W = \cup\{V_n: n < \omega\}$, then $\mathrm{cl}_Y W \supset C^0$. But if X is dense or open in Y, $\mathrm{cl}_Y W \cap C^1 = \varnothing$. $\mathrm{cl}_Y W$ is a clopen subset of Y and it is easily seen C^0 is completely separated from C^1. The other part of the proof is provided by Theorem 4.4. □

5. Some further remarks and examples.

EXAMPLE 5.1. We construct a non-weakly Lindelöf F-space which has no P-covers. Let $K = \beta\omega\backslash\omega$. Let λ be the initial ordinal of cardinality $|K|^+$. Define $D = (\lambda + 1)\backslash\{\alpha < \lambda: cf(\alpha) = \omega\}$ where $\lambda + 1$ has the order topology. D is a P-space and K is a compact F-space, so $D \times K$ is an F-space [8]. Choose a non-clopen cozero set C^0 of K [6, 6W], and let $B^0 = \mathrm{cl}_K C^0\backslash C^0$. Our example will be $X = \beta(D \times K)\backslash(\{\lambda\} \times B^0)$. To show X is an F-space we will first show that $Y = D \times K\backslash(\{\lambda\} \times K)$ is C^*-embedded in $D \times K$. Let f be a continuous real-valued function on Y. Modifying the arguments in [6, 9L] one has for every $k \in K$ an interval $[\alpha_k, \lambda]$ of $\lambda + 1$ such that f is constant on $([\alpha_k, \lambda] \cap D) \times \{k\}$. Let $\beta = \sup\{\alpha_k: k \in K\}$. Since $cf(\lambda) > |K|$, we have $\beta < \lambda$ and $[\beta + 1, \lambda] \cap D = V$ is a clopen neighborhood of λ in D. Define $g: K \to \mathscr{R}$, where $\mathscr{R}$ is the real line, by declaring $g(k) = f(\beta, k)$. Obviously g is continuous and for all $(\delta, k) \in (V\backslash\{\lambda\}) \times K$, $f(\delta, k) = g(k)$, so f can be continuously extended to $V \times K$ and hence to $D \times K$. We now have Y is a dense C^*-embedded subspace of the F-space $D \times K$, so Y is an F-space and $Y \subset X \subset \beta Y = \beta(D \times K)$, so X is also an F-space.

Choose a cozero set C' of $\beta X = \beta(D \times K)$ such that $C' \cap (D \times K) = D \times C^0$. Then we have $C' \cap (\{\lambda\} \times B^0) = \varnothing$ and $\mathrm{cl}_{\beta X} C' \supset (\{\lambda\} \times B^0) = \beta X\backslash X$, so there is no P-set of βX contained in $\beta X\backslash X$. By Lemma

3.3, X has no P-covers.

We now show X is not weakly Lindelöf. Let $\mathscr{U} = \{C: C$ is a cozero set of βX, $C \cap (\{\lambda\} \times B^0) = \emptyset\}$. $\mathscr{U}$ is an open cover of X. If $C \in \mathscr{U}$ choose a continuous function $f: \beta X \to \mathscr{R}$ such that $C = \operatorname{coz}(f)$. There is a clopen neighborhood V of λ in D and a continuous function $g: K \to \mathscr{R}$ such that $f(\delta, k) = g(k)$ for all $(\delta, k) \in (V\backslash\{\lambda\}) \times K$. Since nonempty zero sets of K have nonempty interior [5], $N = \operatorname{int}_K g^{\leftarrow}(0)$ is not empty. Thus $(V\backslash\{\lambda\}) \times N$ is an open set of the dense subspace $(D\backslash\{\lambda\}) \times K$ of X and it is disjoint from C, so C cannot be dense in X. Since a countable union of elements of $\mathscr{U}$ is again an element of $\mathscr{U}$ we have shown no union of a countable subcollection of $\mathscr{U}$ is dense in X.

EXAMPLE 5.2. The next example shows that an almost weakly Lindelöf basically disconnected space need not be C^*-embedded in every basically disconnected splce it is embedded in.

Let A be $(\omega_2 + 1)\backslash\{\alpha: cf(\alpha) = \omega\}$ where $\omega_2 + 1$ has the order topology. The space A is basically disconnected, in fact a P-space [6, 9L]. Let X be the free union of $A\backslash\{\omega_2\}$ with the countable discrete space ω. This space is almost weakly Lindelöf (see [9]) but we will construct a basically disconnected space in which it is embedded but not C^*-embedded. The product $Y = A \times \beta\omega$ is a basically disconnected space [8, Theorem 6.3]. Let q be any point of $\beta\omega\backslash\omega$. The subspace $((A\backslash\{\omega_2\}) \times \{q\}) \cup (\{\omega_2\} \times \omega)$ of Y is homeomorphic to X. The closures in Y of the sets $(A\backslash\{\omega_2\}) \times \{q\}$ and $\{\omega_2\} \times \omega$ have the point (ω_2, q) in common, so this copy of X is not C^*-embedded in Y.

Example 5.2 suggests a proof for the following theorem.

THEOREM 5.3. *A P-space X is C^*-embedded in every basically disconnected space it is embedded in* iff *X is Lindelöf.*

Proof. Suppose X is a P-space which is not Lindelöf. Then X is infinite and therefore not pseudocompact [6, 4K.2]. This also means that X is not almost compact. Zero sets of X are clopen so let A and B be complementary clopen subsets of X neither of which is compact. As X is not Lindelöf we can assume that A is not Lindelöf. A non-Lindelöf P-space also fails to be weakly Lindelöf and if a basically disconnected space is not weakly Lindelöf, it has a P-cover. Therefore there is a P-set P of βA contained in $\beta A\backslash A$. If we let $Y = A \cup \{P\}$ be the quotient space of $A \cup P$ obtained by collapsing P to a point then Y is also a P-space. Since B is not compact we can choose $q \in \beta B \backslash B$. The space $Y \times \beta B$ is basically disconnected [8, Theorem 6.3] and $(A \times \{q\}) \cup (\{P\} \times B)$ is homeomorphic to X but it is not C^*-embedded in $Y \times \beta B$. The converse follows from Corollary 4.1. $\square$

Recall that X is an *extremally disconnected* space if the closure of every open set of X is open. The class of extremally disconnected spaces is contained in the class of basically disconnected spaces, and though the absolute C^*-embedding theorem for basically disconnected spaces is not known, the first author has proven,

THEOREM 5.4. [4] *An extremally disconnected space X is C^*-embedded in every extremally disconnected space it is embedded in iff X is weakly Lindelöf or almost compact.*

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Received January 28, 1980 and in revised form September 2, 1980.

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