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NOTE ON THE SPACES $P(S)$ OF REGULAR PROBABILITY MEASURES WHOSE TOPOLOGY IS DETERMINED BY COUNTABLE SUBSETS

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Two closely connected topics are discussed: countable tightness in the spaces $P(S)$ of regular probability measures with the weak topology and a convex analogue to Lindelöf property of the weak topology of the function spaces $C(S)$ defined by H. H. Corson. The main result of this note exhibits a rather wide class of compact spaces stable under standard operations including the operation $P(S)$, such that within this class both of the properties we deal with are dual each other and they behave in a regular way. Some related open problems are stated.

1. Introduction. In this note we consider two closely connected topics: countable tightness in the spaces $P(S)$ of regular probability measures on compact spaces endowed with the weak* topology and property (C)—a convex analogue to Lindelöf property of the weak topology of function spaces $C(S)$ defined by H. H. Corson [6] (for the terminology and definitions see §§ 2 and 3).

Our results are related to the following two problems:

(A) *Property* (C) *of* $C(S)$ *is equivalent to a property of* $P(S)$ *which is a convex analogue to countable tightness* (*see Lemma* 3.2). *This property is a priori weaker than countable tightness but no example known to us shows that this is realy the case. So, for what compact spaces* S *countable tightness of* $P(S)$ *is equivalent to property* (C) *of* $C(S)$, *or putting this another way, when property* (C) *and countable tightness are dual each other?*

(B) *Does the function space* $C(S \times S)$ *or* $C(P(S))$ *have property* (C) *provided that the space* $C(S)$ *has this property? Does countable tightness of the space* $P(S \times S)$ *or* $P(P(S))$ *follow from countable tightness of the space* $P(S)$?

It should be mentioned here that the only examples we know of compact spaces S with countable tightness for which $C(S)$ fails to have property (C) or $P(S)$ fails to have countable tightness, due to Haydon [14] and to van Douwen and Fleissner [7], are constructed under additional set theoretic hypotheses. This yields yet another problem, whether in such examples some extra axioms for set theory are necessary (the results of this note, however, have no connection to this question).

We shall consider the class of compact spaces S such that each regular measure on S is determined by its values on a countable collection of compact sets (Definition 3.3). Our main result (see Theorem 4.1 and Corollary 4.2) is that this class of spaces is closed under the operation $P(S)$ and some other standard topological operations, and that in the realm of this class the questions stated in (A) and (B) have always a positive answer.

In this context the question arises how wide is the class of spaces we deal with and to what extent the countable determinantness of regular measures on S is connected to property (C) of $C(S)$ or to countable tightness of $P(S)$? The class we consider includes many "classical" compact spaces (cf. Example 3.7). In fact, no example is known to us of a compact space S outside of this class for which $C(S)$ has property (C) or $P(S)$ has countable tightness (but we see also no reason why such spaces should not exist). On the other hand, let us point out that we do not know whether or not each pointwise compact subspace of the space of the first Baire class functions on the irrationals belong to this class, while it was proved recently by Godefroy [11] that for each such a space S the space $P(S)$ has countable tightness; this is probably one of the most interesting questions about the class we consider in this paper.

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2. Terminology and notation. Our topological terminology follows [9] and the terminology related to functional analysis follows [23]. Given a compact space S we denote by $C(S)$ the Banach space of continuous real-valued functions on S with the sup norm and by $P(S)$ the space of regular probability measures on S endowed with the weak* topology $\Big($i.e., basic neighborhoods of a measure $\mu \in P(S)$ are of the form $\left\{\nu \in P(S): \left|\int f_i d\mu - \int f_i d\nu\right| < \varepsilon \text{ for } i \leqq n\right\}$, where $f_i \in C(S)$ and $\varepsilon > 0\Big)$.

We shall denote by ω the set of natural numbers, I will denote the unit interval $[0, 1]$, and S^ω is the countable product of a space S. If A is a subset of a linear space, then $convA$ is the convex hull of the set A. We shall also stick to a convention that the families of sets will be denoted by capitol script types or bold-face types, and the subsets of the spaces $P(S)$ will be denoted by capitol greek types.

DEFINITION 2.1. We say that a topological spaces S has countable tightness at a point $s \in S$ (or that S is determined at the point s by countable subsets) if for each subset A of S containing the point s in its closure, there exists a countable set $C \subset A$ such that s is in the closure of C; if S has countable tightness at each point we say that S has countable tightness (or that S is determined by countable subsets), cf. [9], [16] and [18, Def. 8.2]. If for each $s \in \bar{A} \subset S$ there exists a sequence $(a_n) \subset A$ converging to s, we say that S is a Fréchet space [9].

REMARK 2.2. In the sequel we often identify the points s of a compact space S with Dirac measures δ_s at these points [23, 18.1.1] which allows one to consider the space S as the set of extreme points of the convex set $P(S)$ [23, 4.5.4 and 7.4.7].

REMARK 2.3. If $\mu \in P(S \times T)$ is defined on the product $S \times T$ of two compact spaces then we define the marginal measures $\mu_S \in P(S)$ and $\mu_T \in P(T)$ by the formulas $\mu_S(E) = \mu(E \times T)$ and $\mu_T(E) = \mu(S \times E)$, E being a Borel set in S or T, respectively. Notice that the map $\mu \to (\mu_S, \mu_T)$, which maps the space $P(S \times T)$ onto the product $P(S) \times P(T)$, is continuous.

3. Property (C) and countably determined regular measures. The following property was defined by H. H. Corson [6] (cf. [19], [8] and [20]).

DEFINITION 3.1. *The function space $C(S)$ has property (C) provided that each family of closed convex sets in $C(S)$ with empty intersection contains a countable subfamily with empty intersection.*

The following dual characterization of property (C) was given in [19, 5.1 (t_3)].

LEMMA 3.2. *The function space $C(S)$ has property (C) if and only if $P(S)$ has the following property (C^*): for each measure μ in the closure of a set $\Lambda \subset P(S)$ there is a countable set $\Gamma \subset \Lambda$ such that $\mu \in \overline{\operatorname{conv} \Gamma}$.*

Notice that Lemma 3.2 shows in particular that property (C) of $C(S)$ implies that S has countable tightness (cf. Remark 2.1).

DEFINITION 3.3. *A regular probability measure μ on a compact space S is said to be countably determined provided that there exists a countable family $\mathscr{A}$ of compact or, equivalently, Borel subsets of S such that for each open set $U \subset S$, $\mu(U) = \sup\{\mu(A): A \in \mathscr{A}$ and $A \subset U\}$. We shall say that the family $\mathscr{A}$ determines the measure μ.*

REMARK 3.4. (a) Babiker [4] considered a stronger property of regular measures obtained by the requirement that the family $\mathscr{A}$ in Definition 3.3 consists of compact G_δ-sets (see also Sec. 3.4); Babiker called such measures "uniformly regular".

(b) In the first version of this note we used the property stated in Lemma 3.5 rather than Definition 3.3 (and we called the property "metrizable-like support", see [21, § 4]). The change was made after Professor D. H. Fremlin had sent us his manuscript [10] where the information about Babiker's paper and various characterizations illuminating this property were contained. Let us mention here one of the characterizations of countable determinantness of regular measures given by Fremlin in [10]:

A measure $\mu \in P(S)$ is countably determined if and only if there exists a compact metric space T and a Borel map $f: S \to T$ such that for each compact set $F \subset S$, $\mu(f^{-1}(f(F)) \backslash F) = 0$.

LEMMA 3.5. *A countable family $\mathscr{A}$ of compact or Borel sets in a compact space S determines a measure $\mu \in P(S)$ if and only if for each open cover $\mathscr{U}$ of S and each $\varepsilon > 0$ there exists a finite disjoint family $\mathscr{E} \subset \mathscr{A}$ such that $\mathscr{E}$ refines $\mathscr{U}$ and $\mu(S \backslash \bigcup \mathscr{E}) < \varepsilon$.*

We omit a fairly standard proof of this statement.

LEMMA 3.6. *If S is a compact space such that each measure $\mu \in P(S)$ is countably determined then the countable product S^ω and each compact continuous image of S have the same property.*

Proof. (a) Let $\mu \in P(S^\omega)$ and let $\mathscr{A}$ be a countable family of compact sets which determines the marginal measures (see Remark 2.3). Then one verifies easily that the family of rectangles $A_1 \times \cdots \times A_k \times S \times S \times \cdots$ where $A_i \in \mathscr{A}$, determines the measure μ.

(b) Let $f: S \to T$ be a continuous map and let $\mu \in P(T)$. There exists a measure $\nu \in P(S)$ such that $\nu(f^{-1}(E)) = \mu(E)$ for each Borel set $E \subset T$, see [23, 23.2.2]. Now, if a countable family $\mathscr{A}$ of compact subsets of S determines the measure ν, then the family $\{f(A): A \in \mathscr{A}\}$ determines the measure ν.

EXAMPLES 3.7.

(A) Let S be a compact scattered space. Then each measure $\mu \in P(S)$ is purely atomic and the countable family of the atoms of μ determines the measure μ [23, 19.7.6]. Moreover, if S has also countable tightness then the function space $C(S)$ has property (C) [19, Corollary 4.1.1].

(B) Let S be a weakly compact subspace of a Banach space. Then each measure $\mu \in P(S)$ has metrizable support [17, Theorem 4.3] and hence μ is countably determined. The space $P(S)$ is also homeomorphic to a weakly compact set in a Banach space [17, Theorem 3.3] and therefore $P(S)$ is a Fréchet space.

We refer also the reader to the papers of Talagrand [25], [26], where a class of compact spaces S such that each $\mu \in P(S)$ has metrizable support and $P(S)$ is a Fréchet space is described, which is essentially wider than the class of weakly compact sets.

(C) Let S be a compact subset of the space $B_1(\omega^\omega)$ of the first Baire class functions on irrationals ω^ω, endowed with the pointwise topology. Godefroy [11, Proposition 7] proved that $P(S)$ embeds also in the space $B_1(\omega^\omega)$ and hence $P(S)$ is a Fréchet space by a theorem of Bourgain Fremlin and Talagrand [5, Theorem 3 F]. We do not know whether or not each measure $\mu \in P(S)$ is in this situation countably determined; it seems to us that of special interest is the particular case of S being the unit ball B^{**} in the double dual E^{**} of a separable Banach space E which does not contain l_1 isomorphically, endowed with the weak* topology, see Rosenthal [22, Theorem 3].

(D) Let H be the Helly space of all nondecreasing functions from the interval I into itself considered with the pointwise topology [9, 3.2.E]. The space H embeds in the space $B_1(\omega^\omega)$ defined in (C) and thus the result of Godefroy quoted there shows that $P(H)$ is a Fréchet space (a more direct proof of this fact one obtains by combining the reasoning below with Example 3 in § 8.4).

We shall prove now that each measure $\mu \in P(H)$ is countably determined.

First, let us consider the classical double arrow space A [9, Exercise 3.10.C]. Let $f: A \to I$ be the natural map which makes from the split interval A the ordinary interval I again. Given a measure $\mu \in P(A)$, the set $J = \{t \in I: \mu(f^{-1}(t)) > 0\}$ is at most countable and it is easy to see that the family of Borel sets $\mathscr{A} = \{f^{-1}([a, b]\setminus J): a < b \text{ are rational}\} \cup \{\{a\}: a \in f^{-1}(J)\}$ satisfies the conditions in Lemma 3.5. Thus each measure $\mu \in P(A)$ is countably determined and the same is true for each $\mu \in P(A^\omega)$ by Lemma 3.6. At this point one can end the proof applying a general result stated in § 8.6, as the space A can be identified with the set of extreme points of the compact convex set H, which consists of the functions taking only the values 0 or 1. However, we shall also complete the proof in an elementary way by showing that the Helly space is a continuous image of a closed subset of the product A^ω (cf. Lemma 3.6).

Let C be the Cantor set in the interval I and let $M=\{a_1, a_2, \cdots\}$ be a countable subset of the complement $I\backslash C$, dense in the interval I. Let $L = \{f \in H: f(I) \subset C\}$; then H is a continuous image of L under the map $f \to f \circ u$, where $u: C \to I$ is any continuous monotone surjection. To show that L embeds in A^ω define functions $u_i: L \to A$, for $i = 1, 2, \cdots$, assuming that $u_i(f)$ is the characteristic function of the interval $f^{-1}[0, a_i]$; then the map $(u_i)_i: L \to A^\omega$ separates the points of L and hence it is an embedding.

(E) Let $\Sigma(\alpha) = \{x \in I^\alpha: x(\xi) = 0$ for all but countably many $\xi < \alpha\}$, α being an ordinal, be the Σ-product of α copies of I [9, 2.7.13]. It follows from a result of Arhangelskii [2] that under Martin's Axiom and the negation of the Continuum Hypothesis [15], if $S \subset \Sigma(\alpha)$ is a compact set then each measure $\mu \in P(S)$ has metrizable support and hence it is countably determined; moreover, the last property allows one to prove without any difficulty that, under these assumptions, $P(S)$ embeds in $\Sigma(\alpha)$ for each compact set $S \subset \Sigma(\alpha)$, and thus it is a Fréchet space (cf. [3], [1], [12]). However, under the Continuum Hypothesis the situation is quite different: an example of Haydon [14, Theorem 3.1] and a theorem of Šapirovskii [24, Corollary 9], [16, 3.22], yield that the Continuum Hypothesis implies an existence of a compact nonseparable space $S \subset \Sigma(\omega_1)$ which supports a regular measure $\mu \in P(S)$ and hence $P(S)$ fails to have countable tightness (cf. [21]).

Let us mention also a deep result of Gulko [12], [13] that each member of the class considered by Talagrand, mentioned in (B), embeds in some $\Sigma(\alpha)$.

4. **Results.** The following theorem is the main result of this note.

THEOREM 4.1. *Let* $\mathbf{S}$ *be the class of all compact spaces* S *such that the function space* $C(S)$ *has property* (C) *and each regular measure on* S *is countably determined. Then*:

(i) *the countable products, closed subspaces and compact continuous images of the spaces from the class* $\mathbf{S}$ *belong to this class, and*

(ii) *if* $S \in \mathbf{S}$ *then* $P(S) \in \mathbf{S}$.

COROLLARY 4.2. *Let* S *be a compact space such that each regular measure on* S *is countably determined. Then the following conditions are equivalent*:

(i) *$C(S)$ has property (C);*
(ii) *$C(P(S))$ has property (C);*
(iii) *$P(S)$ has countable tightness;*
(iv) *$P(S^{\omega})$ has countable tightness.*

Theorem 4.1 is based on the following three lemmas.

LEMMA 4.3. *Let S and T be compact spaces such that both function spaces $C(S)$ and $C(T)$ have property (C). If $\lambda \in P(S \times T)$ is a measure such that both marginal measures λ_S and λ_T (see Remark 2.3) are countably determined then the space $P(S \times T)$ has countable tightness at the point λ.*

LEMMA 4.4. *Let S be a compact space and let measures $\sigma \in P(S)$ and $\lambda \in P(P(S))$ be connected by the formula*

$$(*) \qquad \int \Phi_f d\lambda = \int f d\sigma\,, \quad \text{where } \Phi_f(\mu) = \int f d\mu \quad \text{for } \mu \in P(S) \text{ and } f \in C(S)\,.$$

Then, if the measure σ is countably determined, so is the measure λ.

LEMMA 4.5. *For any compact space S the following conditions are equivalent:*
(i) *$C(S^{\omega})$ has property (C);*
(ii) *$C(P(S^{\omega})^{\omega})$ has property (C);*
(iii) *$P(S^{\omega})$ has countable tightness.*

The proofs of these statements are given in §§ 5, 6 and 7, respectively. Here, let us show how Theorem 4.1 follows from Lemmas 4.3–5.

Proof of Theorem 4.1. (a) Let S and T be in the class $\boldsymbol{S}$. Lemma 4.3 shows that the space $P(S \times T)$ has countable tightness and hence the function space $C(S \times T)$ has property (C) (cf. Lemma 3.2). This and Lemma 3.6 show that $S \times T \in \boldsymbol{S}$.

(b) Let $S_1, S_2, \cdots \in \boldsymbol{S}$. Then (a) yields by induction that $S_1 \times \cdots \times S_k \in \boldsymbol{S}$ for each k and $S_1 \times S_2 \times \cdots \in \boldsymbol{S}$ by [19, Proposition 2] and Lemma 3.6.

(c) Let us prove assertion (ii). If $S \in \boldsymbol{S}$ then $C(S^{\omega})$ has property (C) by (b) and it follows by Lemma 4.5 that $C(P(S))$ has property (C). Hence it remains to check that each measure $\lambda \in P(P(S))$ is countably determined. There exists a measure $\sigma \in P(S)$ such that the formula $(*)$ in Lemma 4.4 holds [23, 23.2.2] and the assertion of this lemma is just what we need.

5. Proof of Lemma 4.3. Let the compact spaces S, T and the measure $\lambda \in P(S \times T)$ be as in Lemma 4.3, and let $\Lambda \subset P(S \times T)$ be a collection of measures such that

(1) $$\lambda \text{ is in the closure of } \Lambda .$$

We have to show that there exists a countable set $\Gamma \subset \Lambda$ containing λ in the closure.

We shall split the proof into three steps: the first one is a simple general fact (certainly well-known) about the weak* topology in the space of measures, needed in the next step; in the second step we use property (C) to show that, roughly speaking, given a finite family of disjoint compact rectangles in $S \times T$, one can choose a countable collection of measures from Λ which approximate the behavior of λ on this family of rectangles—this step is an extension of [19, 5.1(t_2)]; finally, in the last step we use countable determinantness of the measures λ_S and λ_T and the preceding step to choose a countable set Γ we are looking for.

I. *Let K be a compact space, and let $A \subset K$ be a compact set, $W \supset A$ a neighborhood of A, $\mu \in P(K)$ and $\varepsilon > 0$. Then there exists a neighborhood V of A contained in W and a neighborhood Ω of μ in $P(K)$ such that $|\nu(V) - \mu(V)| < \varepsilon$ for each $\nu \in \Omega$.*

Proof. Let $\varphi: K \to I$ be a continuous function such that $\varphi | A \equiv 1$ and $\varphi | K \backslash W \equiv 0$, and let $A(s, t) = \varphi^{-1}[s, t]$ for each $s, t \in I$ with $s < t$. Then there is an $r \in (0, 1)$ such that $\mu(\varphi^{-1}(r)) = 0$ and therefore one can find $0 < s_0 < r < t_0 < 1$ such that $\mu(A(s_0, t_0)) < \varepsilon/6$. Let $s_0 < s_1 < r < t_1 < t_0$ and let $f, g: K \to I$ be two continuous functions such that $f | A(s_1, t_1) \equiv 1$, $f | A(0, s_0) \cup A(t_0, 1) \equiv 0$ and $g | A(t_1, 1) \equiv 0$ and $g | A(0, s_1) \equiv 1$. Let $\Omega = \left\{\nu \in P(K): \left|\int f d\nu - \int f d\mu\right| < \varepsilon/6 \text{ and } \left|\int g d\nu - \int g d\mu\right| < \varepsilon/6\right\}$, and let $V = \operatorname{Int} A(0, r)$. To see that Ω and V have the desired property, let us take any $\nu \in \Omega$. Then, since $\int f d\mu < \varepsilon/6$, $\int f d\nu < \varepsilon/3$ and hence $\nu(A(s_1, t_1)) < \varepsilon/3$. Thus $\left|\int g d\nu - \nu(V)\right| \leqq \int_{A(s_1, t_1)} g d\nu < \varepsilon/3$, and similarly, $\left|\int g d\mu - \mu(V)\right| \leqq \int_{A(s_1, t_1)} g d\nu < \varepsilon/3$, hence $|\nu(V) - \mu(V)| < \varepsilon$.

II. *Let $\mathscr{S} = \{S_1, \cdots, S_n\}$ and $\mathscr{T} = \{T_1, \cdots, T_m\}$ be disjoint families of compact subsets of S and T respectively such that $\lambda_S(S \backslash \bigcup \mathscr{S}) + \lambda_T(T \backslash \bigcup \mathscr{T}) < \varepsilon$. Then there exists a countable set $\Gamma(\mathscr{S} \times \mathscr{T}, \varepsilon) \subset \Lambda$ with the following property: for arbitrary pairwise disjoint open neighborhoods G_{ij} of the rectangles $S_i \times T_j$ there exists a measure $\mu \in \Gamma(\mathscr{S} \times \mathscr{T}, \varepsilon)$ such that $\sum_{i,j} |\mu(G_{ij}) - \lambda(S_i \times T_j)| < 3\varepsilon$.*

Proof. The result proved in step I allows one to choose disjoint open neighborhoods V_{ij} of the rectangles $S_i \times T_j$ and a neighborhood Ω of the measure λ such that

$$(2) \qquad |\mu(V_{ij}) - \lambda(S_i \times T_j)| < \varepsilon/n\cdot m \text{ for } \mu \in \Omega .$$

We shall use property (C) to choose a countable set $\Gamma \subset \Lambda \cap \Omega$ such that

$$(3) \quad \text{if } G \supset \bigcup_{i,j} S_i \times T_j \text{ is open, then } \mu(S \times T \backslash G) < \varepsilon \text{ for some } \mu \in \Gamma .$$

Before we define Γ, let us check that any set Γ with property (3) can serve as the set $\Gamma(\mathscr{S} \times \mathscr{T}, \varepsilon)$ we are looking for. Let G_{ij} be as in II and let $G = \bigcup_{i,j} (G_{ij} \cap V_{ij})$. Take any $\mu \in \Gamma$ such that $\mu(S \times T\backslash G) < \varepsilon$. Then (2), (3) and the equality $\mu(G_{ij}) = \mu(G_{ij}\backslash V_{ij}) + \mu(V_{ij}) - \mu(V_{ij}\backslash G_{ij})$ yield the inequality $\sum_{i,j} |\mu(G_{ij}) - \lambda(S_i \times T_j)| \leqq \sum_{i,j} \mu(G_{ij}\backslash V_{ij}) + \sum_{i,j} \mu(V_{ij}\backslash G_{ij}) + \sum_{i,j} |\mu(V_{ij}) - \lambda(S_i \times T_j)| < \varepsilon + \varepsilon + n\cdot m\ \varepsilon/n\cdot m = 3\varepsilon$.

Hence it remains to define the set Γ. For each $\mu \in \Lambda$, let $C(\mu)$ be the set of all pairs $(f, g) \in C(S) \times C(T)$ satisfying the following conditions:

$$(4) \qquad 0 \leqq f \leqq 1/2,\ 0 \leqq g \leqq 1/2,\ f \,|\, \cup \mathscr{S} \equiv 0,\ g \,|\, \cup \mathscr{T} \equiv 0 ,$$

$$(5) \qquad \int f d\mu_S + \int g d\mu_T \geqq \varepsilon/2 .$$

The sets $C(\mu)$ are closed and convex, and the intersection $\cap \{C(\mu): \mu \in \Lambda \cap \Omega\}$ is empty; the last assertion follows from the fact that given a pair $(f, g) \in C(S) \times C(T)$ satisfying (4), the inequality $\int f d\lambda_S + \int g d\lambda_T < \varepsilon/2$ holds, and since the function $\mu \to \int f d\mu_S + \int g d\mu_T$ is continuous and $\lambda \in \overline{\Lambda \cap \Omega}$, the inequality (5) fails for some $\mu \in \Lambda \cap \Omega$, i.e., $(f, g) \notin C(u)$. Now, the product $C(S) \times C(T)$ has property (C) [19, 3.5], and therefore there exists a countable set $\Gamma \subset \Lambda \cap \Omega$ such that $\cap \{C(\mu): \mu \in \Gamma\} = \varnothing$. Let us check that Γ satisfies (3). Given an open set $G \supset \bigcup_{i,j} S_i \times T_j$, define a pair $(f, g) \in C(S) \times C(T)$ satisfying (4) and such that $f^{-1}[0, 1/2) \times g^{-1}[0, 1/2) \subset G$, and choose a measure $\mu \in \Gamma$ such that $(f, g) \notin C(\mu)$. Hence (5) fails for the functions f, g and the measure μ, which yields $\mu(S \times T\backslash G) < \varepsilon$.

III. In this step we shall complete the proof of Lemma 4.3. Let $\mathscr{A}_S$ and $\mathscr{A}_T$ be countable families of compact sets in S and T, respectively, which determine the measures λ_S and λ_T, respectively. For each pair of finite disjoint families $\mathscr{S} \subset \mathscr{A}_S$ and $\mathscr{T} \subset \mathscr{A}_T$ and a natural number p such that

$$(6) \qquad \lambda_S(S\setminus \cup \mathscr{S}) + \lambda_T(T\setminus \cup \mathscr{T}) < 1/p ,$$

let $\Gamma(\mathscr{S} \times \mathscr{T}, 1/p)$ be the family defined in step II. We shall verify that the union Γ of all such families $\Gamma(\mathscr{S} \times \mathscr{T}, 1/p)$ is a countable subset of Λ whose closure contains λ. We have to check that if $u_1, \cdots, u_k \in C(S \times T)$, with $\|u_i\| \leqq 1$, and $\varepsilon > 0$, then there exists a measure $\mu \in \Gamma$ such that

$$(7) \qquad \left|\int u_l d\mu - \int u_l d\lambda\right| < 11\varepsilon \qquad \text{for } l \leqq k .$$

Let $\mathscr{U}$ be an open cover of $S \times T$ such that each set $u_l(U)$ has diameter less than ε, for $U \in \mathscr{U}$ and $l \leqq k$. A standard reasoning allows one to choose finite disjoint families $\mathscr{S} = \{S_1, \cdots, S_n\} \subset \mathscr{A}_S$ and $\mathscr{T} = \{T_1, \cdots, T_m\} \subset \mathscr{A}_T$ such that the family $\mathscr{S} \times \mathscr{T} = \{S_i \times T_j: i \leqq n, j \leqq m\}$ refines $\mathscr{U}$ and (6) holds with $1/p < \varepsilon$. Let G_{ij} be pairwise disjoint neighborhoods of the rectangles $S_i \times T_j$ such that the family $\{G_{ij}: i \leqq n, j \leqq m\}$ refines $\mathscr{U}$. There exists a measure $\mu \in \Gamma(\mathscr{S} \times \mathscr{T}, 1/p) \subset \Gamma$ such that (see step II)

$$(8) \qquad \sum_{i,j} |\mu(G_{ij}) - \lambda(S_i \times T_j)| < 3\varepsilon .$$

Let us check that μ satisfies also (7). Put $F = S \times T\setminus \bigcup_{i,j} G_{ij}$. Then (6) and (8) yield the inequalities (recall that $1/p < \varepsilon)\lambda(F) < \varepsilon$ and $\mu(F) < 4\varepsilon$, and hence

$$(9) \qquad \left|\int_F u_l d\mu\right| + \left|\int_F u_l d\lambda\right| < 5\varepsilon .$$

The oscillation of u_1 on each G_{ij} being less than ε, we have also the inequality

$$(10) \qquad \sum_{i,j} \left|\int_{G_{ij}} u_l d\mu - \int_{G_{ij}} u_l d\lambda\right| \leqq 2\varepsilon + \sum_{i,j} |\mu(G_{ij}) - \lambda(G_{ij})| ,$$

and (6) and (8) yield the inequality

$$(11) \qquad \sum_{i,j} |\mu(G_{ij}) - \lambda(G_{ij})| \leqq \sum_{i,j} |\mu(G_{ij}) - \lambda(S_i \times T_j)| + \varepsilon < 4\varepsilon .$$

Now, (9), (10) and (11) together show that $\left|\int u_l d\mu - \int u_l d\lambda\right| < 5\varepsilon + 2\varepsilon + 4\varepsilon = 11\varepsilon$, as claimed in (7).

6. Proof of Lemma 4.4.[1] Let the measures $\sigma \in P(S)$ and $\lambda \in P(P(S))$ be as in Lemma 4.4. The proof will go, briefly, as follows.

[1] D. H. Fremlin [**10**] gave another proof of Lemma 4.4 based on a strengthening of the result of step I (see Lemma 8.5) and on the characterization of countable determinantness quoted in Remark 3.4 (b).

In the first step we assign to each compact set $A \subset S$ a compact convex set $\Gamma(A) \subset P(S)$ in such a way that if A covers a "large" part of S with respect to σ, then $\Gamma(A)$ covers a "large" part of $P(S)$ with respect to λ. Then, in the next step, we define in a natural way, for a given finite partition of A into compact sets, a finite partition of $\Gamma(A)$ into Borel sets and we check, in the third step, that if the partition of A is "small" with respect to the uniform structure, then so is the corresponding partition of $\Gamma(A)$. In the final fourth step we put together these results and we assign to a countable family of compact sets in S which determines σ a countable family of Borel sets in $P(S)$ which determines λ.

I. *Let $A \subset S$ be a compact set. Put $r = \sigma(S \backslash A)^{1/2}$ and let $\Gamma(A) = \{\mu \in P(S): \mu(S\backslash A) \leqq r\}$. Then $\lambda(P(S)\backslash\Gamma(A)) \leqq r$.*[2]

Proof. Let $\Gamma \subset P(S)\backslash\Gamma(A)$ be an arbitrary compact set. For each $\mu \in \Gamma$ there exists a compact set $C \subset S\backslash A$ such that $\mu(C) > r$ and if $f: S \to I$ is any continuous function such that $f|A \equiv 0$ and $f|C \equiv 1$, then the inequality $\Phi_f(\mu) = \int f d\mu > r$ holds. This observation and a standard compactness argument allow one to choose continuous functions $f_1, \cdots, f_m: S \to I$ vanishing on A and such that $\max_{i \leqq m} \Phi_{f_i}|\Gamma > r$. For $f = \max_{i \leqq m} f_i$ we have then $f|A \equiv 0$, $\Phi_f|\Gamma > r$ and $0 \leqq \Phi_f \leqq 1$. Now, the formula (*) in Lemma 4.4 yields the inequality $\int \Phi_f d\lambda = \int f d\sigma \leqq \sigma(S\backslash A) = r^2$ and, on the other hand, we have the inequality $\int \Phi_f d\lambda > r \cdot \lambda(\Gamma)$, provided that $\lambda(\Gamma) > 0$. Both of the inequalities show that if $r > 0$ then $\lambda(\Gamma) < r$, and if $r = 0$ then the second inequality fails, which means that $\lambda(\Gamma) = 0$. This ends the proof by the regularity of λ.

II. *Let $\mathscr{E} = \{A_1, \cdots, A_n\}$, $A = \bigcup \mathscr{E}$, and let $\Gamma(A)$ be as in step I. Let $\mathscr{K}$ be a finite partition of the $(n-1)$-dimensional simplex $\Delta_{n-1} = \{(t_1, \cdots, t_n): 0 \leqq t_i \leqq 1, \sum_i t_i = 1\}$ into Borel sets. Then for each $K \in \mathscr{K}$ the set $\Gamma(K) = \{\mu \in \Gamma(A): 1/\mu(A)(\mu(A_1), \cdots, \mu(A_n)) \in K\}$ is a Borel set and the family $\Gamma(\mathscr{E}, \mathscr{K}) = \{\Gamma(K): K \in \mathscr{K}\}$ is a finite partition of the set $\Gamma(A)$.*

Proof. Since for each compact set $C \subset A$, the map $\mu \to \mu(C)$ is upper semicontinuous [23, 19.5], the set $\{\mu \in \Gamma(A): \mu(A) > 0\}$ and the map $\mu \to 1/\mu(A)(\mu(A_1), \cdots, \mu(A_n))$ defined on this set are Borel, and hence the set $\Gamma(K)$ is Borel, K being a Borel set in Δ_{n-1}.

III. *Let $\mathscr{U}$ be an open cover of $P(S)$. Then there exists an*

[2] D. H. Fremlin [**10**] obtained a better result in this direction, see Lemma 8.5.

open cover $\mathscr{V}$ of S and a number $c > 0$ with the following property, where $\mathscr{E}$ and $\Gamma(\mathscr{E}, \mathscr{K})$ are as in step II: *if $\mathscr{E}$ refines $\mathscr{V}$ and $\sigma(S\setminus \bigcup \mathscr{E}) \leqq c^2$, then there exists a $\delta > 0$ such that the family $\Gamma(\mathscr{E}, \mathscr{K})$ refines $\mathscr{U}$ provided that $\mathscr{K}$ consists of sets of diameter less than δ.*

Proof. Choose a convex symmetric neighborhood W of the origin in the dual space $C(S)^*$ with the weak* topology such that for each $\Lambda \subset P(S)$

(1) $$\text{if } \Lambda - \Lambda \subset 3\cdot W, \text{ then } \Lambda \subset \Omega \text{ for some } \Omega \in \mathscr{U}\ .$$

In the sequel we shall consider S as the subspace of $P(S)$, see Remark 2.2. Let $\mathscr{V}$ be an open cover of S such that

(2) $$V - V \subset W \text{ for each } V \in \mathscr{V}\ ,$$

and let $c > 0$ be such that

(3) $$B = \{\mu \in C(S)^*:\ \|\mu\| \leqq c\} \subset 1/2\cdot W\ .$$

We claim that $\mathscr{V}$ and c have the required properties. Let $\mathscr{E}$ be a family as in step II which refines $\mathscr{V}$ and satisfies the inequality $\sigma(S\setminus \bigcup \mathscr{E}) = r^2 \leqq c^2$; for simplicity, let $\bigcup \mathscr{E} = S$.

For each $A_i \in \mathscr{E}$ put $P(A_i) = \{\mu \in P(S): \mu(S\setminus A_i) = 0\} = \overline{\operatorname{conv} A_i}$ (cf. [23, 7.1.24]). It follows by (2) that

(4) $$P(A_i) - P(A_i) \subset 2\cdot W\ .$$

Let us check that for each $t = (t_1, \cdots, t_n) \in \varDelta_{n-1}$ we have (cf. step II)

(5) $$\Gamma(t) = \{\mu \in \Gamma(A): \mu(A_i) = t_i\} \subset \sum_i t_i\cdot(P(A_i) + B) = \Lambda(t)\ .$$

Take any $\mu \in \Gamma(t)$, put $A_0 = S\setminus A$ and let $\mu_i(E) = \mu(E \cap A_i)$ for $i = 0, 1, \cdots, n$ and $E \subset S$ Borel. Since $\mu(A_0) \leqq c$, $1/t_i\cdot\mu_i + \mu_0 \in P(A_i) + B$, for $i = 1, \cdots, n$, and therefore $\mu = \sum_i t_i\cdot[1/t_i\cdot\mu_i + \mu_0] \in \Lambda(t)$. Now, by (4) and (3) we obtain the inclusion $\Lambda(t) - \Lambda(t) \subset 3\cdot W$, and since the correspondence $t \to \Lambda(t)$ is upper semicontinuous, there exists a positive δ such that $\Lambda(t') - \Lambda(t'') \subset 3\cdot W$, whenever $t', t'' \in \varDelta_{n-1}$ and $\|t' - t''\| < \delta$. Inclusion (5) and condition (1) show that δ is the number we were looking for.

IV. We can now complete the proof easily. Let a countable family $\mathscr{A}$ of compact sets determines the measure μ and let $\mathscr{B}$ be a countable collection of Borel subsets of the simplexes $\varDelta_m$ such that for each m and $\delta > 0$ there exists a finite partition $\mathscr{K} \subset \mathscr{B}$ of $\varDelta_m$ consisting of the sets of diameter less than δ. Let us show that the union $\boldsymbol{\Gamma}$ of all families $\Gamma(\mathscr{E}, \mathscr{K})$ defined in step II, where $\mathscr{E} \subset \mathscr{A}$ and $\mathscr{K} \subset \mathscr{B}$ is a countable collection of Borel sets which determines the measure λ.

Let an open cover $\mathscr{U}$ of $P(S)$ and an $\varepsilon > 0$ be given. Let $\mathscr{V}$ and $c > 0$ be as in step III, and let a disjoint finite collection $\mathscr{E} \subset \mathscr{A}$ refines $\mathscr{V}$ and satisfies the inequality $\sigma(S \setminus \bigcup \mathscr{E}) \leqq \min \{\varepsilon^2, c^2\}$. Take the number δ defined in step III and choose a finite partition $\mathscr{K} \subset \mathscr{B}$ of the simplex Δ_m, m being the cardinality of $\mathscr{E}$, consisting of sets of diameter less than δ. Then the family $\Gamma(\mathscr{E}, \mathscr{K}) \subset \boldsymbol{\Gamma}$ refines $\mathscr{U}$ (see step III) and since it covers the set $\Gamma(\bigcup \mathscr{E})$ defined in step I, it follows by the assertion proved in this step that $\lambda(P(S) \setminus \bigcup \boldsymbol{\Gamma}) \leqq \varepsilon$. This ends the proof by Lemma 3.5.

7. **Proof of Lemma 4.5.** We shall split the proof into two steps, the first one being a fairly general result which belongs probably to the folklor related to function spaces.

I. *Let K be a compact convex set in a locally convex topological vector space and let T be the closure of the set of extreme points of K. Then there exists a compact convex set $\Gamma \subset P(T^\omega)$ and a continuous affine onto mapping $\varphi: \Gamma \to P(K)$.*

Proof. Let $A \subset C(K)$ be the space of continuous affine functions on K [23, 23.1] and let E be the closed linear span of the set of all functions $u \in C(K^\omega)$ of the form

$$(1) \qquad u(x_1, x_2, \cdots) = f_1(x_1) \cdots f_m(x_m), \text{ where } f_i \in A ,$$

i.e., E is the weak tensor product $A \hat{\hat{\otimes}} A \hat{\hat{\otimes}} \cdots$ [23, 20.5.8]. Since the restriction operator maps A isometrically onto the space $A | T = \{f | T: f \in A\}$ [23, 23.1.18], E can also be considered as the weak tensor product $A | T \hat{\hat{\otimes}} A | T \hat{\hat{\otimes}} \cdots$ which we identify again with a subspace of $C(T^\omega)$. This way we identify E with a subspace of $C(T^\omega)$ containing constants and moreover, this identification takes the unit of $C(K^\omega)$ to the unit of $C(T^\omega)$, so we can also identify the space $K(E)$ of states on E with $K(F)$, the space of states on F [23, 23.2.1]. In effect, $K(F)$ being the image of $P(T^\omega)$ under the restriction map [23, proof of 23.2.27], we obtain a continuous affine onto mapping $\psi: P(T^\omega) \to K(E)$. Let $T: E \to C(K)$ be the continuous linear mapping defined by the formula $T\,u(x) = u(x, x, \cdots)$. Since for each u of the form (1) we have $T\,u = f_1 \cdot \cdots \cdot f_m$, the range of u contains the algebra generated by A and hence it is dense in $C(K)$ by Stone-Weierstass theorem. This yields that the adjoint map $T^*: C(K)^* \to E^*$ is injective, and since T preserves the units, T^* embeds $P(K)$ affinely into $K(E)$. So, finally, one can take

$$\Gamma = \psi^{-1} \circ T^* P(K) \text{ and } \varphi = (T^*)^{-1} \circ \psi | \Gamma .$$

II. The implications (iii) $\rightarrow$ (i) and (ii) $\rightarrow$ (iii) in Lemma 4.5 follow by Lemma 3.2 and a remark below this lemma, respectively. Thus it remains to demonstrate the implication (i) $\rightarrow$ (ii).

Let S be a compact space such that the space $C(S^\omega)$ has property (C). The result proved in step I applied to the compact convex set $K = P(S^\omega)^\omega$ yields an existence of an affine continuous onto mapping $\varphi: \Gamma \rightarrow P(K)$ defined on a compact convex subset of $P(T)$, T being the closure of the set of extreme points of K. Since T is homeomorphic to $(S^\omega)^\omega = S^\omega$ (see Remark 2.2), one can assume that $\Gamma \subset P(S^\omega)$ and hence Γ has property (C^*), see Lemma 3.2. Now, property (C^*) being an invariant of affine continuous mappings between compact spaces, $P(P(S^\omega))$ has property (C^*) and hence $C(P(S^\omega)^\omega)$ has property (C), again by Lemma 3.2.

8. Comments.

5.2. There is a natural approach to the problem whether property (C) of $C(S)$ implies countable tightness of $P(S)$ which is different from the one discussed in this note. Given a Banach space E, let $\mathscr{K}_m E$ be the collection of all nonempty sets which are unions of m closed convex subsets of E, and let us say that E has property (C_m) if each collection of sets from $\mathscr{K}_m(E)$ with empty intersection contains a countable subcollection with empty intersection (cf. Definition 3.1). Then one can easily checks that if E^m has property (C_{2m}) for each m, then the unit ball B^* of the dual space E^* endowed with the weak* topology has countable tightness (indeed: let $x^* \in \bar{A} \subset B^*$; since $C(y^*) = \{(x_1, \cdots, x_m) \in E^m : \max |x^*(x_i) - y^*(x_i)| \geqq 1/n\} \in \mathscr{K}_{2m}(E^m)$ and since $\cap \{C(y^*): y^* \in A\} = \emptyset$, there exists a countable set $C_{mn} \subset A$ with $\cap \{C(y^*): y^* \in C_{mn}\} = \emptyset$ and then $x^* \in \bigcup_{m,n} C_{mn}$). I do not know whether property (C) always implies property (C_m), or even property (C_2). Does the function space $C(A)$, where A is the double arrow space, have property (C_2)?

5.3. If a Banach space E is Lindelöf in the weak topology, then E has property (C_m) defined in 5.2 for each m, as each set from $\mathscr{K}_m(E)$ is closed in the weak topology. However, it is an open problem whether if $C(S)$ is Lindelöf in the weak topology, then so is $C(S) \times C(S)$, cf. [20]. Let us notice the following simple result in this direction.

PROPOSITION. *If E is a Banach space which is Lindelöf in the weak topology and M is a complete separable metric space* (*or a regular image of such a space*) *then $E_w \times M$ is Lindelöf, E_w being the space E endowed with the weak topology.*[3]

This can be verified by a reasoning similar to that given in [19, p. 145]; let us indicate briefly the argument: if $E_w \times M$ were not Lindelöf then there would exist a collection $\mathscr{K}$ of closed non-empty subsets of $E_w \times M$ closed under countable intersections, and an $\varepsilon > 0$, such that for each $x \in M$ there is a $C_x \in \mathscr{K}$ with $\operatorname{dist}(E \times \{x\}, C_x) \geqq \varepsilon$, and this would yield a contradiction when one considers the set $E \times A$, where A is a countable set dense in M.

8.3. The example of Haydon [14] mentioned in the Introduction shows that $P(S)$ need not have countable tightness whenever S does, at least under the Continuum Hypothesis. However, the example does not answer the question whether if S has countable tightness and $\mu \in P(S)$ is purely atomic then $P(S)$ has countable tightness at the point μ, cf. [20, property (t_3) on p. 968], [21].

8.4. Let us consider the following notion.

DEFINITION 1. *A regular probability measure μ on a compact space S is said to be strongly countably determined provided that there exists a countable family $\mathscr{A}$ of compact G_δ-sets or, equivalently, Baire subsets of S such that for each open set $U \subset S$, $\mu(U) =$* $\sup\{\mu(A)\colon A \in \mathscr{A}$ *and* $A \subset U\}$.

Such measures were discussed by Babiker [4] who called them "uniformly regular", see Remark 3.4 (the terminology we have chosen seems to us more adequate to the context we deal with).

PROPOSITION 2. *Let S be a compact space such that each measure $\mu \in P(S)$ is strongly countably determined. Then the countable product S^ω and the space of measures $P(S)$ have the same property and the space $P(S)$ is first-countable* (i.e., *each point $\mu \in P(S)$ has a countable base of neighborhoods*).

Proof. The nontrivial part of Proposition 2 consists of the assertions that each $\lambda \in P(P(S))$ is strongly countably determined and that $P(S)$ is first-countable.

The proof of the first statement is almost the same as the proof of Lemma 4.4 given in § 6; the only minor modification is that

[3] If E is only a normed space this need not to be true in general, cf. [20].

in step II it should be observed that if the sets A_i are in addition G_δ-sets, then the sets $\Gamma(K)$ are actually Baire sets.

Let us give now a brief proof of the second statement. Let $\lambda \in P(S)$ and let a countable collection $A_1, A_2, \cdots$ of compact G_δ-sets determines the measure λ. The observation made in step I of the proof of Lemma 4.3 allows one to choose open neighborhoods $V(i, j)$ of the sets A_i satisfying the conditions $\overline{V(i, j+1)} \subset V(i, j)$, $\bigcap_j V(i, j) = A_i$ and such that the set $\Omega_n = \{\mu \in P(S)\colon \max_{i,j \leq n} |\mu(V(i, j)) - \lambda(V(i, j))| < 1/n\}$ is open for each $n = 1, 2, \cdots$. Then one checks easily that the collection $\{\Omega_n\}_n$ forms a countable base at the point λ in $P(S)$.

EXAMPLE 3. Let A be the double arrow space (cf. 3.7 (D)). Then the reasoning given in Example 3.7(D) shows actually that each regular measure on A is strongly countably determined. In particular each space $P(A^\omega)$, $P(P(A^\omega)^\omega)$, $\cdots$ is first-countable and it carries only strongly countably determined regular measures, by Proposition 2; cf. Edgar [8, Example 5.7].

8.5. Fremlin [10] proved the following strengthening of the result of step I of the proof of Lemma 4.4.

LEMMA (*Fremlin*). *Let $\sigma \in P(S)$ and $\lambda \in P(P(S))$ be as in Lemma 4.4. Then for any σ-measurable set $E \subset S$ the function $\mu \to \mu(E)$ is defined λ-almost everywhere on $P(S)$ and, for any $\varepsilon > 0$,*

$$\lambda\{\mu \in P(S)\colon \mu(E) > \varepsilon\} \leqq 1/\varepsilon \cdot \sigma(E) .$$

8.6. *Let K be a compact convex set in a locally convex topological vector space and let T be the closure of the set of extreme points of K. If each regular probability measure on T is countably determined, or if T belongs to the class $\mathbf{S}$ defined in Theorem* 4.1, *then the same is true for the space K.*

This follows immediately from the results of this note, because under the assumptions we deal with, there is a continuous affine map of the space $P(T)$ onto K [23, 23.7.1].

REFERENCES

1. K. Alster and R. Pol, *On function spaces of compact subspaces of Σ-products of the real line*, Fund. Math., **107** (1980), 135-143.

2. A. V. Arhangelskii, *There is no "naive" example of non-separable sequential bicompactum with the Suslin property*, DAN SSSR, **203** (1972), 983-985 (in Russian).

3. ———, *On spaces of continuous functions in the topology of pointwise convergence*, Soviet Math. Dokl., **19** (1978), 605-609.

4. A. G. Babiker, *On uniformly regular topological measure spaces*, Duke Math. J., **43** (1976), 775-789.
5. J. Bourgain, D. H. Fremlin and M. Talagrand, *Pointwise compact sets of Baire-measurable functions*, Amer. J. Math., **100** (1978), 845-886.
6. H. H. Corson, *The weak topology of a Banach space*, Trans. Amer. Math. Soc., **101** (1961), 1-15.
7. E. K. van Douwen and W. G. Fleissner, *The definable forcing axiom: an alternate to Martin's axiom*, (preprint).
8. G. A. Edgar, *Measurability in a Banach space II*, Indiana University Math. J., **28** (1979), 559-579.
9. R. Engelking, *General Topology*, Warszawa, 1977.
10. D. H. Fremlin, *Notes on a paper by R. Pol*, manuscript dated July 4, 80.
11. G. Godefroy, *Compacts de rosenthal*, Pacific J. Math., **91** (1980), 293-306.
12. S. P. Gulko, *On the properties of some function spaces*, DAN SSSR, **243** (1978), 839-842.
13. ———, *The structure of a class of function spaces*, Uspehi Mat. Nauk,, **210** (1979), 33-40.
14. R. Haydon, *On dual L^1-spaces and injective bidual Banach spaces*, Israel J. Math., **31** (1978), 142-152.
15. T. Jech, *Set Theory*, New York, 1978.
16. I. Juhász, *Cardinal functions in topology-ten years later*, Math. Centre Tracts 123, Amsterdam, 1980.
17. J. Lindenstrauss, *Weakly compact sets, their topological properties and Banach spaces they generate*, Proc. Symp. Infinite Dim. Topology 1967, Annals of Math. Studies, Princeton Univ. Press, 1972.
18. E. Michael, *A quintuple quotient quest*, General Topology and Appl., **2** (1972), 91-138.
19. R. Pol, *On a question of H. H. Corson and some related problems*, Fund. Math., **59** (1980), 143-153.
20. ———, *The Lindelöf property and its convex analogue in function spaces with the weak topology*, Coll. Math. Soc. Janos Bolyai 23, Topology, Budapest, (1978), 965-969.
21. ———, *The spaces P(S) of regular probability measures whose topology determined by countable subsets*, Proceedings of the Conference in honor of Professor F. B. Jones, Riverside, 1980.
22. H. Rosenthal, *Pointwise compact subsets of the first Baire class*, Amer. J. Math., **99** (1977), 362-378.
23. Z. Semadeni, *Banach spaces of continuous functions*, Warszawa, 1971.
24. B. E. Šapirovskii, *Special types of embeddings in Tyhonoff cubes. Subspaces of Σ-products and cardinal invariants*, Coll. Math. Soc. Janos Bolyai 23, Topology, Budapest, (1978), 1055-1086.
25. M. Talagrand, *Sur les espaces de Banach faiblement K-analytiques*, C. R. Acad. Sci. Paris, **284** (1977), A-745-748.
26. ———, *Espaces de Banach faiblement K-analytiques*, Annals of Math., **110** (1979), 407-438.

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