PACIFIC JOURNAL OF MATHEMATICS
Vol. 100, No. 1, 1982

ON THE REPRESENTATION THEORY OF RINGS IN MATRIX FORM

EDWARD L. GREEN

We study the category of modules for rings of the form

$$\begin{pmatrix} \Lambda_1 & N \\ M & \Lambda_2 \end{pmatrix}$$

where M is a $\Lambda_2 - \Lambda_1$-bimodule and N is a $\Lambda_1 - \Lambda_2$-bimodule. We first obtain a structural result and then study special cases of such rings. The goal is to reduce the study of modules over such rings to modules over generalized lower triangular rings.

Rings of the form

$$(0.1) \qquad \Gamma = \begin{pmatrix} \Lambda_1 & 0 \\ M & \Lambda_2 \end{pmatrix}$$

where Λ_1, Λ_2 are rings and M is a $\Lambda_2 - \Lambda_1$-bimodule have appeared often in the study of the representation theory of Artin rings and algebras. We list just a few references ([2, §4], [3], [4], [5, §9 and appendix], [6], [7, §2], [9] and [10]). Such rings appear naturally in the study of homomorphic images of hereditary Artin algebras. For, if Ω is such an algebra, since Ω must have a simple injective left module it follows that Ω can be put in the form (0.1) with Λ_1 a semisimple Artin ring. One of the main reasons rings of the form (0.1) are so useful is that their modules can be studied by knowing the Λ_1 and Λ_2 modules together with certain homomorphisms. In particular, we have

(0.2) THEOREM. [4] *Let Γ be the ring*

$$\begin{pmatrix} \Lambda_1 & 0 \\ M & \Lambda_2 \end{pmatrix}.$$

The category of left Γ-modules is equivalent to the following category: the objects are triples (X, Y, f) where X is a left Λ_1-module, Y is a left Λ_2-module and $f \in \mathrm{Hom}_{\Lambda_2}(M \otimes_{\Lambda_1} X, Y)$. The morphisms $\alpha: (X, Y, f) \to (X', Y', f')$ are pairs $\alpha = (\alpha_1, \alpha_2)$ where $\alpha_1 \in \mathrm{Hom}_{\Lambda_1}(X, X')$, $\alpha_2 \in \mathrm{Hom}_{\Lambda_2}(Y, Y')$ such that the following diagram commutes

$$\begin{array}{ccc} M \otimes_{\Lambda_1} X & \xrightarrow{f} & Y \\ {\scriptstyle 1_M \otimes \alpha_1}\downarrow & & \downarrow{\scriptstyle \alpha_2} \\ M \otimes_{\Lambda_1} X' & \xrightarrow{f'} & Y' . \end{array} \qquad \square$$

In this paper we study rings of the form

$$\begin{pmatrix} \Lambda_1 & N \\ M & \Lambda_2 \end{pmatrix} \tag{0.3}$$

where M is a $\Lambda_2 - \Lambda_1$-bimodule and N is a $\Lambda_1 - \Lambda_2$-bimodule. I. Palmer developed results on the homological and projective dimensions of such rings in [8]. The first section of this paper contains notation and a structure result which is analogous to Theorem (0.2). The remainder of the paper studies rings of form (0.3) in special cases. The goal is to reduce the study of modules over such rings to modules over generalized lower triangular rings. Under certain hypothesis this is done in the case where Λ_1 is a semisimple Artin ring or a left Artin ring whose Jacobson radical is square zero. A study of more general cases is conducted in §3. In §4, we give a useful application of the results of §2 and a number of examples.

1. Notation and a structure result. Throughout this paper we keep the following notation. Let Λ_1, Λ_2 be rings. Let M be a $\Lambda_2 - \Lambda_1$-bimodule and let N be a $\Lambda_1 - \Lambda_2$-bimodule. Let $\varphi: M \otimes_{\Lambda_1} N \to \Lambda_2$ be a $\Lambda_2 - \Lambda_2$-bimodule homomorphism and let $\psi: N \otimes_{\Lambda_2} M \to \Lambda_1$ be a $\Lambda_1 - \Lambda_1$-bimodule homomorphism. Let

$$\Gamma = \begin{pmatrix} \Lambda_1 & N \\ M & \Lambda_2 \end{pmatrix} = \left\{ \begin{pmatrix} x_1 & n \\ m & x_2 \end{pmatrix} : x_i \in \Lambda_i,\ m \in M,\ n \in N \right\} .$$

We define addition of elements of Γ componentwise and multiplication by

$$\begin{pmatrix} x_1 & n \\ m & x_2 \end{pmatrix} \cdot \begin{pmatrix} x_1' & n' \\ m' & x_2' \end{pmatrix} = \begin{pmatrix} x_1x_1' + \psi(n \otimes m') & x_1n' + nx_2' \\ mx_1' + x_2m' & x_2x_2' + \varphi(m \otimes n) \end{pmatrix} . \tag{1.1}$$

For Γ to be an associative ring we must have

$$\varphi(m \otimes n)m' = m\psi(n \otimes m') \quad \text{and} \quad n\varphi(m \otimes n') = \psi(n \otimes m)n' \tag{1.2}$$

for all $m, m' \in M$ and $n, n' \in N$.

We henceforth assume (1.2) is always satisfied by φ and ψ. With the above definitions Γ is an (associative) ring.

If R is ring, we let Mod(R) denote the category of left R-modules. We now introduce a category which we will show is equivalent to the category $\operatorname{Mod}(\Gamma)$. Let $\mathscr{A}(\Gamma)$ be the category whose objects are tuples (X, Y, f, g) where $X \in \operatorname{Mod}(\Lambda_1)$, $Y \in \operatorname{Mod}(\Lambda_2)$, $f \in \operatorname{Hom}_{\Lambda_2}(M \otimes_{\Lambda_1} X, Y)$ and $g \in \operatorname{Hom}_{\Lambda_1}(M \otimes_{\Lambda_2} Y, X)$ so that the following diagrams commute:

$$\begin{array}{ccccc}
N \otimes_{\Lambda_2} M \otimes_{\Lambda_1} X & \xrightarrow{1_N \otimes f} & N \otimes_{\Lambda_2} Y & \xrightarrow{g} & X \\
& {\scriptstyle \psi \otimes 1_X} \searrow & & \nearrow {\scriptstyle \sim \ \text{canonical}} & \\
& & \Lambda_1 \otimes_{\Lambda_1} X & &
\end{array}$$

(1.3)

$$\begin{array}{ccccc}
M \otimes_{\Lambda_1} N \otimes_{\Lambda_2} Y & \xrightarrow{1_M \otimes g} & M \otimes_{\Lambda_1} X & \xrightarrow{f} & Y \\
& {\scriptstyle \varphi \otimes 1_Y} \searrow & & \nearrow {\scriptstyle \sim \ \text{canonical}} & \\
& & \Lambda_2 \otimes_{\Lambda_2} Y & &
\end{array} .$$

The morphisms of $\mathscr{A}(\Gamma)$, $\alpha\colon (X, Y, f, g) \to (X', Y', f', g')$ are pairs of homomorphisms $\alpha = (\alpha_1, \alpha_2)$ where $\alpha_1 \in \operatorname{Hom}_{\Lambda_1}(X, X')$ and $\alpha_2 \in \operatorname{Hom}_{\Lambda_2}(Y, Y')$ so that the following diagrams commute:

(1.4)
$$\begin{array}{ccc}
M \otimes_{\Lambda_1} X & \xrightarrow{f'} & Y \\
{\scriptstyle 1_M \otimes \alpha_1} \downarrow & & \downarrow {\scriptstyle \alpha_2} \\
N \otimes_{\Lambda_1} X' & \xrightarrow{f'} & Y'
\end{array} , \qquad
\begin{array}{ccc}
N \otimes_{\Lambda_2} Y & \xrightarrow{g} & X \\
{\scriptstyle 1_N \otimes \alpha_2} \downarrow & & \downarrow {\scriptstyle \alpha_1} \\
N \otimes_{\Lambda_2} Y' & \xrightarrow{g'} & X'
\end{array}$$

Although the proof of the following result is just a generalization of the proof of Theorem (0.2) and implicitly contained in [8], since it is of central importance to the remainder of the paper, we include a sketch of the proof for completeness.

(1.5) THEOREM. *The category* $\operatorname{Mod}(\Gamma)$ *is equivalent to the category* $\mathscr{A}(\Gamma)$.

Proof. We first define a functor $F\colon \operatorname{Mod}(\Gamma) \to \mathscr{A}(\Gamma)$. Let $A \in \operatorname{Mod}(\Gamma)$. Define $F(A)$ to be (e_1A, e_2A, f_A, g_A) where $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $f_A\colon M \otimes_{\Lambda_1} e_1A \to e_2A$ is induced from the multiplication map

$$\begin{pmatrix} 0 & 0 \\ M & 0 \end{pmatrix} \otimes_\Gamma A \longrightarrow A$$

and $g_A\colon N \otimes_{\Lambda_2} e_2A \longrightarrow e_1A$ is induced from the multiplication map

$$\begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix} \otimes_\Gamma A \longrightarrow A .$$

Note that f_A and g_A satisfy (1.3) by (1.2). If $A, B \in \mathrm{Mod}\,(\Gamma)$ and $\delta \in \mathrm{Hom}_\Gamma\,(A, B)$ define $F(\delta)$ to be (δ_1, δ_2) where δ_i is induced from δ noting that $\delta(e_iA) = e_i\delta(e_iA) \subseteqq e_iB$. Condition (1.4) is satisfied by $F(\delta)$ since δ is a Γ-homomorphism.

We now define $G\colon \mathscr{A}(\Gamma) \to \mathrm{Mod}\,(\Gamma)$ as follows. Let $(X, Y, f, g) \in \mathscr{A}(\Gamma)$. Define $G(X, Y, f, g)$ to be $\left\{\begin{pmatrix} x \\ y \end{pmatrix} : x \in X, y \in Y\right\}$. Let Γ act on $G(X, Y, f, g)$ by

$$\begin{pmatrix} \lambda_1 & n \\ m & \lambda_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda_1 x + g(n \otimes y) \\ f(m \otimes x) + \lambda_2 y \end{pmatrix}.$$

The reader may easily verify that $G(X, Y, f, g)$ is in fact a left Γ-module. If $(\alpha_1, \alpha_2)\colon (X, Y, f, g) \to (X', Y', f', g')$ is an $\mathscr{A}(\Gamma)$-morphism we define $G(\alpha_1, \alpha_2)$ by $G(\alpha_1, \alpha_2)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha_1(x) \\ \alpha_2(y) \end{pmatrix}$ for $x \in X$, $y \in Y$. Again it is easy to verify that $G(\alpha_1, \alpha_2)$ is a Γ-homomorphism.

Finally we leave it to the reader to check that GF is naturally equivalent to $1_{\mathrm{Mod}(\Gamma)}$ and FG is naturally equivalent $1_{A(\Gamma)}$. □

Note that if either $M = 0$ or $N = 0$ then (1.2) and (1.3) are vacuously satisfied. Thus, in either of these cases, Theorem (1.5) reduces to Theorem (0.2). We will need another structure result. Consider the generalized lower triangular ring

$$\Sigma = \begin{pmatrix} R_1 & 0 & 0 \\ U & R_2 & 0 \\ W & V & R_3 \end{pmatrix} \tag{1.6}$$

where each R_i is a ring, U is an $R_2 - R_1$-bimodule, V is an $R_3 - R_2$-bimodule and W is an $R_3 - R_1$-bimodule. Addition in Σ is defined componentwise and multiplication is defined by

$$\begin{pmatrix} r_1 & 0 & 0 \\ u & r_2 & 0 \\ w & v & r_3 \end{pmatrix}\begin{pmatrix} r'_1 & 0 & 0 \\ u' & r'_2 & 0 \\ w' & v' & r'_3 \end{pmatrix} = \begin{pmatrix} r_1 r'_1 & 0 & 0 \\ ur'_1 + r_2 u' & r_2 r'_2 & 0 \\ wr'_1 + \bar{\psi}(v \otimes u') + r_3 w' & vr'_2 + r_3 v' & r_3 r'_3 \end{pmatrix} \tag{1.7}$$

where $\bar{\psi}\colon V \otimes_{R_2} U \to W$ is an $R_3 - R_1$-bimodule homomorphism.

If Σ is of form (1.6) we define $\mathscr{B}(\Sigma)$ to be the category whose objects are tuples (X, Y, Z, f, g, h) where $X \in \mathrm{Mod}\,(R_1)$, $Y \in \mathrm{Mod}\,(R_2)$, $Z \in \mathrm{Mod}\,(R_3)$, $f \in \mathrm{Hom}_{R_2}\,(U \otimes_{R_1} X, Y)$, $g \in \mathrm{Hom}_{R_3}\,(V \otimes_{R_2} Y, Z)$ and $h \in \mathrm{Hom}_{R_3}\,(W \otimes_{R_1} X, Z)$ so that the following diagram commutes

$$\begin{array}{ccccc} V \otimes_{R_2} U \otimes_{R_1} X & \xrightarrow{1_V \otimes f} & V \otimes_{R_2} Y & \xrightarrow{g} & Z \\ & \searrow^{\bar{\psi} \otimes 1_X} & & \nearrow_{h} & \\ & & W \otimes_{R_1} X & & \end{array}$$

The morphisms of $\mathscr{B}(\Sigma)$ are triples $(\alpha_1, \alpha_2, \alpha_3)\colon (X, Y, Z, f, g, h) \to (X', Y', Z', f', g', h')$ where $\alpha_1 \in \mathrm{Hom}_{R_1}(X, X')$, $\alpha_2 \in \mathrm{Hom}_{R_2}(Y, Y')$ and $\alpha_3 \in \mathrm{Hom}_{R_3}(Z, Z')$ so that the following diagrams commute:

$$(1.9)\qquad \begin{array}{ccc} U \otimes_{R_1} X & \xrightarrow{f} & Y \\ {\scriptstyle 1_U \otimes \alpha_1}\downarrow & & \downarrow{\scriptstyle \alpha_2} \\ U \otimes_{R_1} X' & \xrightarrow{f'} & Y' \end{array}, \quad \begin{array}{ccc} V \otimes_{R_2} Y & \xrightarrow{g} & Z \\ {\scriptstyle 1_V \otimes \alpha_2}\downarrow & & \downarrow{\scriptstyle \alpha_3} \\ V \otimes_{R_2} Y' & \xrightarrow{g'} & Z' \end{array}, \quad \begin{array}{ccc} W \otimes_{R_1} X & \xrightarrow{h} & Z \\ {\scriptstyle 1_W \otimes \alpha_1}\downarrow & & \downarrow{\scriptstyle \alpha_3} \\ W \otimes_{R_1} X' & \xrightarrow{h'} & Z' \end{array}.$$

We get

(1.10) THEOREM. *Let Σ be a ring of the form* (1.6). *The categories* Mod (Σ) *and* $\mathscr{B}(\Sigma)$ *are equivalent.*

Proof. The reader may either apply Theorem (0.2) twice, to $\begin{pmatrix} R_1 & 0 \\ U & R_2 \end{pmatrix}$ and then to $\begin{pmatrix} \begin{pmatrix} R_1 & 0 \\ U & R_2 \end{pmatrix} & 0 \\ (W, V) & R_3 \end{pmatrix}$ or directly prove the result in fashion analogous to Theorem (1.5). □

The remainder of this paper is devoted to showing that under suitable hypothesis, the study of Mod (Γ) where Γ is of form (0.3) may be reduced to the study of Mod (Σ) for some "simpler" ring [of the form (1.6)]. We call such results "separation" theorems for obvious reasons.

2. The main separation result. (2.1) Let Γ be a ring of form (0.3). For the remainder of the paper we study Mod (Γ) under the assumption that $\varphi\colon M \otimes_{\Lambda_1} N \to \Lambda_2$ is the zero map. Note that if $\varphi = 0$ then property (1.2) is equivalent to the image of ψ is contained in $(\mathrm{ann}\, M_{\Lambda_1}) \cap (\mathrm{ann}_{\Lambda_1} N)$ where $\mathrm{ann}\, M_{\Lambda_1}$(resp. $\mathrm{ann}_{\Lambda_1} N$) denotes the right (resp. left) annihilator of M(resp. N) in Λ_1.

In this section we will be concerned with the case where Λ_1 is a left Artin ring with Jacobson radical r where $r^2 = 0$. Before proceeding we state the following basic result:

(2.2) THEOREM. [1, §4]. *Let Λ be a left Artin ring with Jacobson radical r so that $r^2 = 0$. Let $Gr(\Lambda/r, r)$ be the category whose objects are triples (X, Y, f) where $X, Y \in$ Mod (Λ, r) and $f \in \mathrm{Hom}_{\Lambda/r}(r \otimes_{\Lambda/r} X, Y)$ is a surjection. The morphisms of $Gr(\Lambda/r/r)$ are pairs of maps $(\alpha_1, \alpha_2)\colon (X, Y, f) \to (X', Y', f')$ where $\alpha_1 \in \mathrm{Hom}_{/r}(X, X')$ and $\alpha_2 \in \mathrm{Hom}_{\Lambda/r}(Y, Y')$ so that the following diagram commutes*

$$\begin{array}{ccc} r \otimes_{\Lambda/r} X & \xrightarrow{f} & Y \\ {\scriptstyle 1_r \otimes \alpha_1}\downarrow & & \downarrow{\scriptstyle \alpha_2} \\ r \otimes_{\Lambda/r} X' & \xrightarrow{f'} & Y' \end{array}.$$

Let $H\colon \mathrm{Mod}(\Lambda) \to Gr(\Lambda/r, r)$ *be the functor defined by* $H(A) = (A/rA, rA, f_A)$ *where* $f_A\colon r \otimes_{\Lambda/r} A/rA \to rA$ *is induced from the multiplication map* $r \otimes_\Lambda A \to rA$. *Then* H *is representation equivalence; that is,* H *is full and dense and* $A \in \mathrm{Mod}(\Lambda)$ *is indecomposable if and only if* $H(A)$ *is indecomposable in* $Gr(\Lambda/r, r)$. □

For the remainder of this section we assume that

(2.3) (a) Λ_1 is a left Artin ring with Jacobson radical denoted r.
(b) $r^2 = 0$
(c) M_{Λ_1} is a right semisimple Λ_1-module
(d) N_{Λ_1} is a left semisimple Λ_1-module
(e) the image of ψ is contained in r.

Note that under the above conditions (2.1) is satisfied. Let T be the ring

$$(2.4)\qquad \begin{pmatrix} \Delta & 0 & 0 \\ M & \Lambda_2 & 0 \\ r & N & \Delta \end{pmatrix}$$

where $\Delta = \Lambda_1/r$ and T has multiplication given by

$$\begin{pmatrix} d_1 & 0 & 0 \\ m & \lambda & 0 \\ x & n & d_2 \end{pmatrix} \cdot \begin{pmatrix} d_1' & 0 & 0 \\ m' & \lambda' & 0 \\ x' & n' & d_2' \end{pmatrix} = \begin{pmatrix} d_1 d_1' & 0 & 0 \\ m d_1' + \lambda m' & \lambda\lambda' & 0 \\ x d_1' + \psi(n \otimes m') + d_2 x' & n\lambda' + d_2 n' & d_2 d_2' \end{pmatrix}.$$

We are now in a position to state and prove the main result of this section.

(2.5) THEOREM. *Let* Γ *be a ring of the form* (0.3) *and suppose* φ *is the zero map. Suppose further that conditions* (2.3) *are satisfied. Let* T *be the ring given by* (2.4). *There is a canonical additive functor* $F\colon \mathrm{Mod}(\Gamma) \to \mathrm{Mod}(T)$ *which has the following properties:*

(1) *if* $A \in \mathrm{Mod}(\Gamma)$ *then* $F(A) = 0 \Leftrightarrow A = 0$.

(2) *for all but a finite number of nonisomorphic indecomposable* T*-modules, if* B *is an indecomposable* T*-module then there exists a* Γ*-module* A *so that* $F(A) \cong B$ *as* T*-modules.*

(3) *if* $A, A' \in \mathrm{Mod}(\Gamma)$ *then* $F(A) \cong F(A')$ *as* T*-modules if and only if* $A \cong A'$ *as* Γ*-modules.*

(4) *if* $A \in \mathrm{Mod}(\Gamma)$ *then* $F(A)$ *is indecomposable as a* T*-module if and only if* A *is indecomposable as a* Γ*-module.*

Proof. We identify $\mathrm{Mod}(\Gamma)$ with $\mathscr{A}(\Gamma)$ and $\mathrm{Mod}(T)$ with $\mathscr{B}(T)$. Define the functor $F\colon \mathrm{Mod}(\Gamma) \to \mathrm{Mod}(T)$ as follows. Let $(X, Y, f, g) \in \mathrm{Mod}(\Gamma)$. Set $X' = X/rX$ and $X'' =$ image of g. Let $p\colon X \to X'$ be the canonical surjection. Let $X''' = p(X'')$ and $X_1 = X'/X'''$. Let $X_2 =$ (image of g) $+ rX$, (viewing image of g and rX as submodules of X). First we note that $X_1 \oplus X_2 = 0 \Leftrightarrow X = 0$. Next we show that f induces a Λ_2-homomorphism $f_1: M \otimes_\Delta X_1 \to Y$. Since M_{Λ_1} is semisimple, f induces a Λ_2-homomorphism $f': M \otimes_\Delta X' \to Y$. Now, since $\varphi = 0$ it follows by (1.3) that $f(M \otimes_{\Lambda_1}$ (image of g)$) = 0$. We get the following commutative diagram

$$\begin{array}{ccccccc} M \otimes_{\Lambda_1} (\text{image of } g) & \xrightarrow{1_M \otimes \text{incl.}} & M \otimes_{\Lambda_1} X & \xrightarrow{1_M \otimes p} & M \otimes_\Delta X' & \xrightarrow{epi} & M \otimes_\Delta X_1 \\ & \searrow & \downarrow f & \swarrow f' & & \swarrow f_1 & \\ & & Y & & & & \end{array} .$$

Thus there exists a unique map $f_1: M \otimes_\Delta X_1 \to Y$ making the above diagram commute. Next, it is clear that g induces a map: $g_1: N \otimes_{\Lambda_2} Y \to X_2$ since (image of g) $\subseteq X_2$. Now we have a Λ_1-homomorphism given by multiplication $r \otimes_{\Lambda_1} X \to rX$. This induces a Δ-homomorphism $h': r \otimes_\Delta X' \to rX$. We claim $h'(r \otimes_\Delta X''') = 0$. We have a split exact sequence

$$0 \longrightarrow rX \cap \operatorname{im} g \longrightarrow \operatorname{im} g \longrightarrow X''' \longrightarrow 0 , \quad \text{where } \operatorname{im} g = \text{image of } g .$$

This sequence is split since $\operatorname{im} g \subseteq$ left socle of X which follows from the assumption that ${}_{\Lambda_1}N$ is a semisimple Λ_1-module. Thus the multiplication map $r \otimes_{\Lambda_1} \operatorname{im} g \to rX$ is zero. From this it follows that $h'(r \otimes_\Delta X''') = 0$. Since $0 \to X''' \to X' \to X_1 \to 0$ is a split exact sequence (X' is a semisimple Λ_1-module), $h'(r \otimes_\Delta X''') = 0$ and $rX \subseteq X_2$, we get a canonically induced map $h_1: r \otimes_\Delta X_1 \to X_2$. It can be verified, using (1.3) that the following diagram commutes

$$\begin{array}{ccccc} N \otimes_{\Lambda_2} M \otimes_{\Lambda_1} X_1 & \xrightarrow{1_N \otimes f_1} & N \otimes_{\Lambda_2} Y & \xrightarrow{g_1} & X_2 \\ & {\scriptstyle \psi \otimes 1_{X_1}} \searrow & & \nearrow {\scriptstyle f_1} & \\ & & r \otimes_\Delta X_1 & & \end{array}$$

We define $F(X, Y, f, g) = (X_1, Y, X_2, f_1, g_1, h_1)$.

We define F on morphisms as follows. If $(\alpha_1, \alpha_2): (X, Y, f, g) \to (X', Y', f', g')$ is a Γ-morphism then $\alpha_1(\operatorname{im} g) \subseteq \operatorname{im} g'$ by (1.4), $\alpha_1(rX) \subseteq rX'$ and hence $\alpha_1(x_2) \subseteq X_2'$. Thus we get an induced map $\delta_2: X_2 \to X_2'$. Since α_1 induces a map $\bar{\alpha}_1: X/rX \to X'/rX'$ so that $\bar{\alpha}_1(\text{in } g/rX \cap \operatorname{im} g) \subseteq \operatorname{im} g/rX' \cap \operatorname{im} g'$ we get an induced map $\delta_1: X_1 \to X_1'$. We set $F(\alpha_1, \alpha_2) = (\delta_1, \alpha_2, \delta_2)$. Again the reader can verify that $(\delta_1, \alpha_2, \delta_2)$ is a T-morphism.

We now verify properties (1)–(4). Property (1) holds since

$X_1 \oplus X_2 = 0 \Leftrightarrow X = 0$. We now verify (2). Let $B = (X, Y, Z, f, g, h) \in \mathrm{Mod}(T)$. If $\mathrm{im}(g) + \mathrm{im}(h) \neq Z$, since Δ is semisimple we have $B \cong (X, Y, \mathrm{im}\, g + \mathrm{im}\, h, f, g', h') \oplus (0, 0, Z/\mathrm{im}\, g + \mathrm{im}\, h, 0, 0, 0)$ where g' and h' are induced from g and h. The semisimplicity of Δ implies there are only a finite number of nonisomorphic indecomposable T-modules of the form $(0, 0, Z, 0, 0, 0)$. Hence we may assume B has the property that $Z = \mathrm{im}(g) + \mathrm{im}(h)$. We write $Z = Z_1 \oplus Z_2$ where $Z_1 = \mathrm{im}\, h$. We construct $A = (U, V, u, v) \in \mathrm{Mod}(\Gamma)$ as follows. Let $\pi: P \to X$ be the Λ_1-projective cover of X. Then since $r \otimes_\Delta X \cong r \otimes_\Delta P/rP \cong r \otimes_{\Lambda_1} P \cong rP$ we have the following exact sequence of Λ-modules:

$$0 \longrightarrow r \underset{\Delta}{\otimes} X \longrightarrow P \xrightarrow{\ \pi\ } X \longrightarrow 0 .$$

Let U' be the pushout of

$$\begin{array}{ccc} r \underset{\Delta}{\otimes} X & \longrightarrow & P \\ {\scriptstyle h}\big\downarrow & & \\ Z_1 = \mathrm{im}(h) . & & \end{array}$$

Finally set $U = U' \oplus Z_2$ and $V = Y$. Note that $rU = rU' \cong Z_1$ and $U/rU \cong X \oplus Z_2$. We define $u: M \otimes_\Delta U \to V$ to be the composition $M \otimes_{\Lambda_1} U \simeq M \otimes_\Delta U/rU \to M \otimes_\Delta X \to V$. We define $v: N \otimes_{\Lambda_2} V \to U$ by $N \otimes_{\Lambda_2} V = N \otimes_{\Lambda_2} Y \xrightarrow{g} \mathrm{im}(g) \subseteq Z_1 \oplus Z_2 \subseteq U' \oplus Z_2 = U$. Using (4.3) one immediately verifies that $A = (U, V, u, v) \in \mathrm{Mod}(\Gamma)$. We now claim $F(A) \cong B$ as T-modules. But $\mathrm{im}(v) + rU = Z_1 \oplus Z_2$ and $Z_1 = rU$. Thus $\mathrm{im}(v) \to U \to U/rU$ has image Z_2 and so $F(A) = (X, V, Z_1 \oplus Z_2, u_1, v_1, u_2)$. It is a straightforward calculation to show that $F(A) \cong B$ as T-modules.

Next we prove property (4). Let $A = (X, Y, f, g) \in \mathrm{Mod}(\Gamma)$. We let $F(A) = (X_1, Y, X_2, f_1, g_1, h_1)$ as above. We assme $F(A)$ decomposes:

$$(2.6) \qquad F(A) = (X_1^1, Y^1, X_2^1, f_1^1, g_1^1, h_1^1) \oplus (X_1^2, Y^2, X_2^2, f_1^2, g_1^2, h_1^2) ,$$

where $X_i = X_i^1 \otimes X_i^1$, for $i = 1, 2$, $Y = Y^1 \oplus Y^2$, $f_i = \begin{pmatrix} f_i^1 & 0 \\ 0 & f_i^2 \end{pmatrix}$ for $i = 1, 2$ and $g_1 = \begin{pmatrix} g_1^1 & 0 \\ 0 & g_1^2 \end{pmatrix}$. Set $V_i = \mathrm{im}\, h_1^i$ for $i = 1, 2$. Now for $i = 1, 2$, $g_1^i: N \otimes_{\Lambda_2} Y^i \to X_2^i$ and $V_i \subseteq X_2^i$. Since $\mathrm{im}\, g_1 + \mathrm{im}\, h_1 = X_2$, for $i = 1, 2$, $\mathrm{im}\, g_1^i + \mathrm{im}\, h_1^i = X_2^i$. Let $Z_i \subseteq X_2^i$ so that $X_2^i = V_i \oplus Z_2$ and $\mathrm{im}\, g_1^i = (\mathrm{im}\, g_1^i \cap V_i) \oplus (\mathrm{im}\, g_1^i \cap Z_i)$. This is possible since X_2^i is a semisimple Δ-module. Finally, let $V = V_1 \oplus V_2$ and $Z = Z_1 \oplus Z_2$. Then $V = \mathrm{im}\, h_1$. Now $X_2 \subseteq X$. In particular, $Z \subseteq X$. By choice of Z, we have that Z is a semisimple Λ-module and $Z \not\subseteq \mathrm{im}(h_1)$. But $\mathrm{im}(h_1) = rX$, by construction of h_1. Thus $X = U \oplus Z$ and we have the following easily verified properties:

(a) $rU = \operatorname{im} h_1 = V$

(b) $U/rU = X_1$

(c) the map $h_1: r \otimes_\Delta X_1 \to V$ is induced from the multiplication map $r \otimes_{\Lambda_1} U \to rU$. We now apply Theorem (2.2). We see that $H(U) = (X_1, V, h_1)$. The decomposition (2.6) yields a decomposition

$$H(U) = (X_1^1, V_1, h_1^1) \oplus (X_1^2, V_2, h_1^2) .$$

By Theorem (2.2), we get a decomposition $U = U_1 \oplus U_2$ so that $H(U_i) = (X_1^i, V_i, h_1^i)$ for $i = 1, 2$. We now leave it to the reader to check that $(U_1 \oplus Z_1, Y_1, f^1, g^1) \oplus (U_2 \oplus Z_2, Y_2, f^2, g^2)$ is a decomposition of (X, Y, f, g) where $f^i: M \oplus_{\Lambda_1} (U_i \oplus Z_i) \to Y_i$ is defined by the composition

$$M \underset{\Lambda_1}{\otimes} (U_i \oplus Z_i) \xrightarrow{1_M \otimes \text{projection}} M \underset{\Lambda_1}{\otimes} U_i \xrightarrow{\sim} M \underset{\Delta}{\otimes} X_1^i \xrightarrow{f_1^i} Y_i$$

and $g^i: N \otimes_{\Lambda_2} Y_i \to U_i \oplus Z_i$ is defined by the composition

$$N \underset{\Lambda_2}{\otimes} Y_i \xrightarrow{g_1^i} V_i \oplus Z_i = rU_i \oplus Z_i \xrightarrow{\text{inclusion}} U_i \oplus Z_i .$$

Finally, we sketch a proof of property (3). Let $A, A' \in \operatorname{Mod}(\Gamma)$ so that $F(A) \cong F(A')$ as T-modules. Let $A = (X, Y, f, g)$, $A' = (X', Y', f', g')$, $F(A) = (X_1, Y, X_2, f_1, g_1, f_2)$ and $F(A') = (X_1', Y', X_2', f_1', g_1', f_2')$. Let $(\alpha_1, \alpha_2, \alpha_3): F(A) \to F(A')$ be a T-isomorphism. As in the proof of property (4), one may write $X = U \oplus Z$ where Z is a semisimple Λ_1-module, $f(M \otimes_{\Lambda_1} Z) = 0$ and $Z \subseteq \operatorname{im} g$. Similarly $X' = U' \oplus Z'$. Using the functor $H: \operatorname{Mod}(\Lambda_1) \to Gr(\Delta, r)$, as in (4), it is not difficult to show $(\alpha_1, \alpha_2, \alpha_3)$ induces an isomorphism $H(U) \to H(U')$. This lifts to an isomorphism $U \to U'$ by (2.2). Furthermore, by appropriate choice of Z', $(\alpha_1, \alpha_2, \alpha_3)$ induces an isomorphism $Z \to Z'$. Thus we get a Λ_1-isomorphism $\alpha: U \oplus Z \to U' \oplus Z'$. Lastly, one verifies that $(\alpha, \alpha_2): A \to A'$ is a Γ-isomorphism. □

We say a ring R is of (left) *finite representation type* if there are only a finite number of nonisomorphic indecomposable finitely generated left R-modules. If R is not of finite representation type we say R is of *infinite representation type*. As an immediate consequence of Theorem (2.5) we get

(2.7) THEOREM. *Let* $\Gamma = \begin{pmatrix} \Lambda_1 & N \\ M & \Lambda_2 \end{pmatrix}$ *be such that* $\varphi = 0$ *and* (2.3) *holds. Then* Γ *is of finite representation type if and only if* $T = \begin{pmatrix} \Lambda_1/r & 0 & 0 \\ M & \Lambda_2 & 0 \\ r & N & \Lambda_1/r \end{pmatrix}$ *is of finite representation type, where* $r =$ *Jacobson radical of* Λ_1. □

This result has many useful applications. For, assume the hypothesis of (2.5) hold. In general, Γ need not have finite left global dimension. But by [9, §4], T has finite left global dimension if Λ_2 does. See example (5.3). If, for example, Λ_2 is a semisimple ring then T is a factor ring of a left hereditary ring.

For the remainder of this section, we deal with the case where Λ_1 is semisimple and $\varphi = 0$. We show that we automatically get a separation result.

(2.8) LEMMA. *Under the above hypothesis, the map* $\psi\colon N \otimes_{\Lambda_2} M \to \Lambda_1$ *is the zero map.*

Proof. By (2.1), the image of $\psi = I$ is contained in $\operatorname{ann} M_{\Lambda_1} \cap \operatorname{ann}_{\Lambda_1} N$. But $I^2 = I \operatorname{im} \psi$. Thus $I^2 = I\psi(N \otimes_{\Lambda_2} M) = 0$. Since I is a two-sided ideal in a semisimple ring we conclude $I = 0$. □

It now follows that properties (a)-(e) of (2.3) are automatically satisfied. Furthermore, since the Jacobson radical of Λ_1 is 0, T is the ring

$$\begin{pmatrix} \Lambda_1 & 0 & 0 \\ M & \Lambda_2 & 0 \\ 0 & N & \Lambda_1 \end{pmatrix}$$

with the obvious multiplication.

Thus, Theorem (2.5) becomes:

(2.9) THEOREM. *Let Γ be the ring* $\begin{pmatrix} \Lambda_1 & N \\ M & \Lambda_2 \end{pmatrix}$ *with* $\varphi = 0$ *and* Λ_1 *a semisimple ring. Let T be the ring* $\begin{pmatrix} \Lambda_1 & 0 & 0 \\ M & \Lambda_2 & 0 \\ 0 & N & \Lambda_1 \end{pmatrix}$. *There is an additive functor* $F\colon \operatorname{Mod}(\Gamma) \to \operatorname{Mod}(T)$ *so that the following properties hold:*

(1) *if* $A \in \operatorname{Mod}(\Gamma)$ *then* $F(A) = 0 \Leftrightarrow A = 0$

(2) *for all but a finite number of nonisomorphic indecomposable T-modules, if B is an indecomposable T-module, there is an Γ-module A so that* $B \simeq F(A)$ *as T-modules.*

(3) *if* $A, A' \in \operatorname{Mod}(\Gamma)$ *then* $F(A) \cong F(A')$ *as T-modules if and only if* $A \simeq A'$ *as Γ-modules.*

(4) *if* $A \in \operatorname{Mod}(\Gamma)$ *then $F(A)$ is an indecomposable T-module if and only if A is an indecomposable Γ-module.* □

(2.10) COROLLARY. *Keeping the hypothesis of* (2.9). *Γ is of finite representation type* $\Leftrightarrow$ *T is of finite representation type.* □

3. The general case with $\varphi = 0$. For the remainder of this section we keep the following notations:

$\Gamma = \begin{pmatrix} \Lambda_1 & N \\ M & \Lambda_2 \end{pmatrix}$ with $\varphi = 0$

$I =$ image of ψ

$m = \operatorname{ann} M_{\Lambda_1}$

$n = \operatorname{ann}_{\Lambda_1} N$.

(3.1) LEMMA. *We have that*

(a) $Im = 0$

(b) $nI = 0$.

Proof. We only prove (a).

$$Im = \psi(N \otimes_{\Lambda_2} M)m = 0 . \qquad \square$$

Let

$$T_1 = \begin{pmatrix} \Lambda_1 & 0 & 0 \\ M & \Lambda_2 & 0 \\ I & N & \Lambda_1 \end{pmatrix}$$

and

$$T_2 = \begin{pmatrix} \Lambda_1/m & 0 & 0 \\ M & \Lambda_2 & 0 \\ I & N & \Lambda_1/n \end{pmatrix}$$

with the obvious ring structures (see §2). We get

(3.2) PROPOSITION. *For $i = 1, 2$ there are canonical additive functors G_i:* $\operatorname{Mod}(\Gamma) \to \operatorname{Mod}(T_i)$.

Proof. We identify $\operatorname{Mod}(\Gamma)$ with $\mathscr{A}(\Gamma)$ and for $i = 1, 2$, $\operatorname{Mod}(T_i)$ with $\mathscr{B}(T_i)$. Since the proof is analogous with the first part of Theorem (2.5) we only define the functors on objects and leave the rest to the reader. For $i = 1$ define G_1: $\operatorname{Mod}(\Gamma) \to \operatorname{Mod}(T_1)$ as follows: if $A = (X, Y, f, g) \in \operatorname{Mod}(\Gamma)$ let $G_1(A) = (X, Y, \operatorname{im}(g), f, g, h)$ where h is induced from the multiplication map $I \otimes_{\Lambda_1} X \to X$. For $i = 2$, define G_2: $\operatorname{Mod}(\Gamma) \to \operatorname{Mod}(T_2)$ as follows: if $A = (X, Y, f, g) \in \operatorname{Mod}(\Gamma)$ let $G_2(A) = (X/mX, Y, \operatorname{im}(g), \bar{f}, g, h)$, where $\bar{f}$ is induced from f and h is induced from the multiplication map $I \otimes_{\Lambda_1} X \to X$. Note $n \operatorname{im}(g) = 0$ since $n \operatorname{im}(g) = ng(N \otimes_{\Lambda_2} Y)$. $\square$

Unfortunately, neither of these functors seem to yield much information in general. But the following result shows that when

$\psi = 0$ one can say something. Note when $\psi = 0$, T_1 is the ring

$$\begin{pmatrix} \Lambda_1 & 0 & 0 \\ M & \Lambda_2 & 0 \\ 0 & N & \Lambda_1 \end{pmatrix}.$$

(3.3) THEOREM. *Let* $\Gamma = \begin{pmatrix} \Lambda_1 & N \\ M & \Lambda_2 \end{pmatrix}$ *with* $\varphi = 0$ *and* $\psi = 0$. *Let* $T_1 = \begin{pmatrix} \Lambda_1 & 0 & 0 \\ M & \Lambda_2 & 0 \\ 0 & N & \Lambda_1 \end{pmatrix}$. *There is a functor* $K\colon \mathrm{Mod}\,(T_1) \to \mathrm{Mod}\,(\Gamma)$ *so that if* $B \in \mathrm{Mod}\,(T_1)$, B *is a direct summand of* $G_1K(B)$. *In particular, if* Γ *is of finite representation type then so is* T_1.

Proof. We identify $\mathrm{Mod}\,(\Gamma)$ with $\mathscr{A}(\Gamma)$ and $\mathrm{Mod}\,(T_1)$ with $\mathscr{B}(T_1)$. Define $K\colon \mathrm{Mod}\,(T_1) \to \mathrm{Mod}\,(\Gamma)$ as follows: if $B = (X, Y, Z, f, g) \in \mathrm{Mod}\,(T_1)$ let $K(B) = (X \oplus Z,\ Y,\ f^*,\ g^*)$ where $f^*\colon M \otimes_{\Lambda_1} (X \oplus Z) \to Y$ by $f^*(m \otimes (x, z)) = f(m \otimes x)$ and $g^*\colon N \otimes_{\Lambda_2} Y \to X \oplus Z$ by $g^*(n \otimes y) = (0, g(n \otimes y))$. Now let $\alpha = (\alpha_1, \alpha_2, \alpha_3)\colon B \to G_1K(B)$ be defined by $\alpha_1\colon X \xrightarrow{\binom{1}{0}} X \oplus Z$, $\alpha_2\colon Y \overset{id}{\to} Y$, and $\alpha_3\colon Z \xrightarrow{\binom{0}{1}} X \oplus Z$. Then α is a T_1-monomorphism which is split by $\beta = (\beta_1, \beta_2, \beta_3)\colon G_1K(B) \to B$ where $\beta_1\colon X \oplus Z \xrightarrow{(1,0)} X$, $\beta_2\colon Y \overset{id}{\to} Y$ and $\beta_3\colon X \oplus Z \xrightarrow{(1,0)} Z$. Thus B is a direct summand of $G_1K(B)$. The last part of the theorem follows immediately. □

By example (4.2) we see that Γ may be of infinite representation type and yet T_1 is of finite representation. We conclude this section by giving an application of Theorem (3.3).

(3.4) THEOREM. *Let* $\Gamma = \begin{pmatrix} \Lambda_1 & N \\ M & \Lambda_2 \end{pmatrix}$ *be such that* $\varphi = 0$. *Then if* Γ *is of finite representation type so is the ring*

$$T_3 = \begin{pmatrix} \Lambda_1/I & 0 & 0 \\ M & \Lambda_2 & 0 \\ 0 & N & \Lambda_1/I \end{pmatrix}.$$

Proof. Since $I \subseteq m \cap n$ by (2.1) M is a right Λ_1/I-module and N is a left Λ_1/I-module. Let J be the two-sided ideal in Γ given by $J = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$. Note $J^2 = 0$ since $I^2 = I\psi(N \otimes M) = 0$. Let $\bar{\Gamma} = \Gamma/J$. Then if Γ is of finite representation type so is $\bar{\Gamma}$. Furthermore, $\bar{\Gamma} = \begin{pmatrix} \Lambda_1/I & N \\ M & \Lambda_2 \end{pmatrix}$ with $\bar{\varphi}$, $\bar{\psi}$ the maps induced from φ and ψ in Γ. But $\varphi = 0 \Rightarrow \bar{\varphi} = 0$. Since $I = \mathrm{im}\,(\psi)$, $\bar{\psi} = 0$. The result now follows from Theorem (3.3). □

Example (4.2) shows that the converse is not true.

4. Applications and examples. We begin with an application of Theorem (2.9).

(4.1) Let Ω be a left Artin ring and let P be a left projective ideal in Ω such that

(a) $\mathrm{End}_\Omega(P)$ is a semisimple ring

(b) if Q and Q' are indecomposable projective left Ω-modules and $f: Q \to Q'$ is a nonzero Ω-morphism then f factors through $P \Leftrightarrow$ either Q or Q' is isomorphic to a direct summand of P.

Then Ω is of left finite representation type if and only if the ring

$$\Sigma = \begin{pmatrix} \mathrm{End}_\Omega(P) & 0 & 0 \\ \mathrm{Hom}_\Omega(P, P^*) & \mathrm{End}_\Omega(P^*) & 0 \\ 0 & \mathrm{Hom}_\Omega(P^*, P) & \mathrm{End}_\Omega(P) \end{pmatrix}$$

is of finite representation type where $P \oplus P^*$ is a finitely generated projective generator for Ω such that no direct summand of P^* is isomorphic to a direct summand of P.

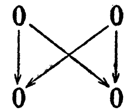
Proof. By the Morita theorems, $\mathrm{Mod}(\Omega)$ is equivalent to $\mathrm{Mod}(\mathrm{End}_\Omega(P \oplus P^*)^{OP})$ where R^{OP} denotes the opposite ring of R. Now $\mathrm{End}_\Omega(P \oplus P^*) = \begin{pmatrix} \mathrm{End}_\Omega(P) & \mathrm{Hom}_\Omega(P, P^*) \\ \mathrm{Hom}_\Omega(P^*, P) & \mathrm{End}_\Omega(P^*) \end{pmatrix}$. Writing this as $\begin{pmatrix} \Lambda_1 & N \\ M & \Lambda_2 \end{pmatrix}$ we see that property (a) implies Λ_1 is a semisimple ring and property (b) implies that the multiplication map $M \otimes_{\Lambda_1} N \to \Lambda_1$ is zero. Finally noting that a left Artin ring is of finite representation type implies that the ring is right Artin and of finite representation type, we can see the result follows from Theorem (2.9). □

(4.2) We now give an example which verifies a number of remarks made in earlier sections.

Let $\Gamma = \begin{pmatrix} \Lambda_1 & N \\ M & \Lambda_2 \end{pmatrix}$ where $\Lambda_1 = \Lambda_2 = k[X]/(X^2)$, k is a field. Let $M = N = k$ where k is the simple $k[X]/(X^2)$-bimodule. We write elements in $k[X]/(X^2)$ as $a + bX$ where $a, b \in k$. We define multiplication in Γ by

$$\begin{pmatrix} a+bx & e \\ f & c+dx \end{pmatrix} \cdot \begin{pmatrix} a'+b'x & e' \\ f' & c'+d'x \end{pmatrix} = \begin{pmatrix} aa'+(ab'+a'b)x & ae'+ec' \\ fa'+cf' & cc'+(cd'+c'd)X \end{pmatrix}.$$

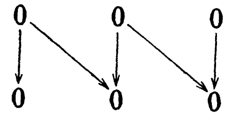
Note that the maps $\varphi: M \otimes N \to \Lambda_2$ and $\psi: N \otimes M \to \Lambda_1$ are both zero. Now Γ has Jacobson radical $\begin{pmatrix} xk & k \\ k & xk \end{pmatrix}$ and its square is zero. Thus applying [5, appendix], since the separated quiver of Γ is



Γ is of infinite representation type. We now consider T_1 of §4. Namely

$$T_1 = \begin{pmatrix} \Lambda & 0 & 0 \\ M & \Lambda & 0 \\ 0 & N & \Lambda \end{pmatrix}.$$

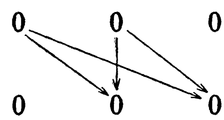
The ring T_1 has radical square zero but its separated quiver is



Hence T_1 is of finite representation type [5, appendix] and we conclude that the converse of Theorem (3.3) does not hold. We now apply the results of §2. Let T be the ring

$$\begin{pmatrix} k & 0 & 0 \\ k & k[X]/X^2 & 0 \\ k & k & k \end{pmatrix}$$

as in §2. Its separated quiver is



and hence is of infinite representation type (as is also shown by Theorem (2.5)).

(4.3) We conclude with a final example. Let $\Lambda_1 = \begin{pmatrix} k & k \\ k & k \end{pmatrix}$ with $\varphi = 0$ and $\psi = 0$. Let $\Lambda_2 = k$. For $i = 1, 2$ let S_i be the simple left Λ_1-module given by $S_1 = \begin{pmatrix} k \\ 0 \end{pmatrix}$ and $S_2 = \begin{pmatrix} 0 \\ k \end{pmatrix}$. [Note that by Theorem (2.9), the study of $\mathrm{Mod}\,(\Lambda_1)$ reduces to the study of $\mathrm{Mod}\,(T)$ where

$$T = \begin{pmatrix} k & 0 & 0 \\ k & k & 0 \\ 0 & k & k \end{pmatrix}$$

with the obvious multiplication.] Let $M = N = k$ where $M_{\Lambda_1} \cong (0, k)$

and ${}_{\Lambda_1}N \cong \begin{pmatrix} k \\ 0 \end{pmatrix}$: Now consider the ring $\Gamma = \begin{pmatrix} \Lambda_1 & N \\ M & \Lambda_2 \end{pmatrix}$; i.e.,

$$\begin{pmatrix} k & k & k \\ k & k & 0 \\ 0 & k & k \end{pmatrix}$$

with multiplication given by

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & 0 \\ 0 & c_2 & c_3 \end{pmatrix}\begin{pmatrix} a_1' & a_2' & a_3' \\ b_1' & b_2' & 0 \\ 0 & c_2' & c_3' \end{pmatrix} = \begin{pmatrix} a_1a_1' & a_1a_2' + a_2b_2' + a_3c_2' & a_1a_3' + a_3c_3' \\ b_1a_1' + b_2b_1' & b_1a_2' + b_2b_2' & 0 \\ 0 & c_1b_2' + c_3c_2' & c_3c_3' \end{pmatrix}.$$

We note that $\varphi: M \otimes_{\Lambda_1} N \to \Lambda_2$ is the zero map. One easily verifies that Λ_1 has radical square zero and that conditions (2.3) are satisfied. Thus Theorem (2.5) implies that the study of Mod (Γ) can be achieved by studying Mod (T) where T is the ring

$$\begin{pmatrix} k & 0 & 0 & 0 & 0 \\ 0 & k & 0 & 0 & 0 \\ 0 & k & k & 0 & 0 \\ 0 & k & k & k & 0 \\ k & 0 & 0 & 0 & k \end{pmatrix}.$$

We show

(i) Γ has infinite left global dimension

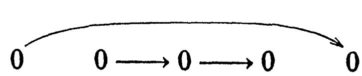
(ii) Γ is of radical cubed zero

(iii) T is an hereditary Artin algebra (i.e., gl. dim. $T = 1$)

(iv) T is of finite representation type

(v) Γ is of finite representation type.

First we show (i). Consider the simple Γ-module $\left(\begin{pmatrix} k \\ 0 \end{pmatrix} \atop 0\right) = S$. (That is, $S \cong S_1$ as Λ_1-modules and $\begin{pmatrix} 0 & N \\ M & \Lambda_2 \end{pmatrix} S = 0$.) Let $0 \to K \to P \to S \to 0$ be an exact sequence of Γ-modules with $P \to S$ the Γ-projective cover of S. Then $K \cong S' = \left(\begin{pmatrix} 0 \\ k \end{pmatrix} \atop 0\right)$ where S' is the simple Γ-module which as a Λ-module is isomorphic to S_2 and $\begin{pmatrix} 0 & N \\ M & \Lambda_2 \end{pmatrix} S' = 0$. Now let $0 \to L \to Q \to S' \to 0$ be an exact sequence of Γ-modules so that $Q \to S'$ is the Γ-projective cover of S'. Then S is a direct summand of L and we conclude that the left projective dimension of S is infinite. Properties (ii) and (iii) are immediate. Applying [5, appendix], since the quiver of T is



we conclude that T is of finite representation type. Property (v) follows from (2.7). It is worthwhile noting that without Theorems (2.5) and (2.7) there is no known method of attacking the question of whether or not Artin rings of radical cubed zero are of finite representation type.

REFERENCES

1. M. Auslander, *Representation Dimension of Artin Algebras*, Queen Mary College Mathematics Notes, London, 1971.
2. M. Auslander, M. I. Platzeck, and I. Reiten, *Coxeter functors without diagrams*, Trans. Amer. Math. Soc., **250** (1979), 1-46.
3. V. Dlab and C. M. Ringel, *Representations of graphs and algebras*, Memoirs of the A.M.S. No. 173, 1976.
4. R. M. Fossum, P. A. Griffith, and I. Reiten, *Trivial Extensions of Abelian Categories*, Lecture Notes in Math., No. 456, Springer-Verlag, Berlin, Heidelberg, New York, 1975.
5. R. Gordon and E. L. Green, *Modules with cores and amalgamations of indecomposable modules*, Memoir of the A.M.S., No. 187, 1978.
6. E. L. Green, *The representation theory of tensor algebras*, J. Algebra, **34** (1975), 136-171.
7. E. L. Green and I. Reiner, *Integral representations and diagrams*, Michigan Math. J., **25** (1978), 53-84.
8. I. Palmer, *The global homological dimension of semi-trivial extension of rings*, Math. Scand., **37** (1975), 223-256.
9. I. Reiten, *Stable equivalence of self-injective algebras*, J. Algebra, **40** (1976), 63-74.
10. C. M. Ringel, *Representations of k-species and bimodules*, J. Algebra, **41** (1976), 269-302.

Received April 11, 1979. This research was partially supported by a grant from the National Science Foudation.

VIRGINIA POLYTECHNIC INSTITUTE AND STATE UNIVERSITY
BLACKSBURG, VA 24061