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LOCALLY COMPACT GROUPS ACTING ON TREES

ROGER ALPERIN

Following Serre's original description of groups having the fixed point property for actions on trees, Bass has introduced the notion of a group of type FA′. Groups of type FA′ can not be nontrivial free products with amalgamation. We show that a locally compact (hausdorff) topological group with a compact set of connected components is of type FA′. Furthermore, any locally compact group which is a nontrivial free product with amalgamation has an open amalgamated subgroup.

1. A group G is called an amalgam if it is a free product with amalgamation of subgroups A and B along C, i.e., $G = A *_C B$, so that $C \neq A$, $C \neq B$.

If a group G acts without inversions on a tree so that it has a fixed vertex we say G has property FA on X. Serre has introduced the notion of a group of type FA. We say that G is of type FA if G has property FA whenever it acts on a tree. The following theorem characterizes G group theoretically.

THEOREM 1 (Serre). *A group G is of type* FA *if and only if it satisfies the following conditions*:

(1) *G has no infinite cyclic quotient.*

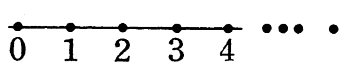
(2) *G is not an amalgam.*

(3) *G is not the union of any sequence*

$$G_0 \subsetneqq G_1 \subsetneqq G_2 \cdots \subsetneqq G_n \subsetneqq \cdots$$

of its proper subgroups. □

This theorem was originally formulated by Serre for countable groups [6, Theorem 15; 2, Theorem 3.2]. Bass has introduced the notion of a group of type FA′. In order to formulate this we introduce the ends of a tree X. Consider the collection $\mathscr{L}$ of half-lines of X: $L \in \mathscr{L}$ is isometric to the standard half-line



The ends of X is the set of equivalence classes $\mathscr{E}$ of $\mathscr{L}$ under the equivalence relation $\sim$:

$$L \sim M \quad \text{iff} \quad L \cap M \quad \text{is a half-line} .$$

Notice that if $e, f \in \mathscr{E}$, $e \neq f$, we can choose representatives $L \in e$, $M \in f$ so that $L \cup M$ is a doubly infinite line of X denoted (e, f). If G acts as a group of isometries on X then it also acts on the set $\mathscr{L}$ of half-lines of X and the set $\mathscr{E}$ of ends of X. If $L \in \mathscr{L}$, $g \in G$ we say L is neutral, repulsing or attracting for g if $gL = L$ (i.e., pointwise fixed), $gL \supsetneqq L$, or $gL \subsetneqq L$ respectively. If L contains a half-line $L'(L - L'$ is finite) which is neutral, repulsing or attracting for g then we say L is almost neutral, repulsing or attracting for g. An end $e \in \mathscr{E}$ is neutral, repulsing or attracting for $g \in G$ if it possesses a representative half-line which is so for g. Denote the ends which are fixed by $G(ge = e, \forall g \in G)$ by $\mathscr{E}^G$.

We can now formulate the property FA′.

THEOREM 2. *Suppose G acts without inversion on the tree X. The following conditions are equivalent.*

(i) *Each element of G has a fixed vertex.*

(ii) *Each finitely generated subgroup of G has a fixed vertex.*

(iii) *There is either a fixed vertex for G or a neutral fixed end.*

Proof. The implication (i) ⇒ (ii) is proved by Serre [6, Corollary 3 to Proposition 26]. The implication (i) ⇒ (iii) is proved by Tits [8, Corollary 3.4]. The implications (ii) ⇒ (i) and (iii) ⇒ (i) are obvious. □

If G satisfies the equivalent conditions of Theorem 2 for a given action without inversions on a tree X then we shall say G has property FA′ on X. This property has been further analyzed by Bass [2, Propositions 1.6, 3.7]. In case G has property FA′ on X and has no fixed vertex then there is a half-line L with vertices (v_n), $n \geqq 0$, so that $G_{v_n} \subset G_{v_{n+1}}$, $n \geqq 0$, and

$$G = \bigcup_{n \geqq 0} G_{v_n} .$$

We say that G is of type FA′ if G has property FA′ whenever it acts on a tree.

THEOREM 3 (Bass). *A* group *G is of type* FA′ *if and only if it satisfies the following conditions*:

(1) *G has no infinite cyclic quotient.*

(2) *G is not an amalgam.* □

One obtains information about homomorphisms from a group G of type FA or FA′ to amalgams using the next propositions.

PROPOSITION 1 (Serre [6, Proposition 21]). *If G is a group of* *type* FA *and* $\varphi: G \to A *_C B$ *is a homomorphism to an amalgam then* $\varphi(G)$ *is contained in a conjugate of* A *or* B. □

PROPOSITION 2. *If* G *is a group of type* FA′ *and* $\varphi: G \to A*B$ *is a homomorphism then* $\varphi(G)$ *is contained in a conjugate of* A *or* B.

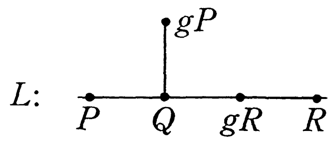
Proof. First notice that a homomorphic image of type FA′ is also of type FA′. Thus $\varphi(G)$ acts without inversions on the tree X for $A*B$. Using condition iii) of Theorem 2, $\varphi(G)$ has a fixed point and consequently $\varphi(G)$ is contained in a conjugate of A or B, or there is a neutral fixed end for $\varphi(G)$. However, the edge stabilizers for this fixed end are trivial since $A*B$ has no amalgamation; this is impossible and consequently $\varphi(G)$ has a fixed point. □

2. If H is a normal subgroup of type FA of a group G and G/H is of type FA′(FA) then G is of type FA′(FA). To see this notice that if G acts on a tree X and K is a finitely generated subgroup of G then $L = K \cap H$ has a fixed tree X^L and thus the finitely generated subgroup $KH/H \cong K/L$ of G/H acts on X^L with a fixed vertex which is then fixed by K. Also, if G contains a subgroup of finite index H of type FA′ then G is also of type FA′. Indeed, if K is a finitely generated subgroup of G then $L = K \cap H$ is a finitely generated subgroup of H; without loss of generality we may assume H is normal and thus the finite group K/L has a fixed point for its action on X^L.

Based on some remarks of Tits [7; §2.3] we shall show that every extension of groups of type FA′ is again of type FA′. For this we shall need some further comments on ends. We suppose that a group G is acting without inversions on a tree X.

PROPOSITION 3. *Let* $e \in \mathscr{E}^G$. *Any half-line* $L \in e$ *is almost neutral, repulsing or attracting for* $g \in G$.

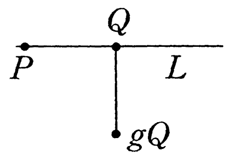
Proof. Given $g \in G$ and $L \in e$. Let P be the initial vertex of L. If $gP \in L$ then L is neutral or attracting for g. If $gP \notin L$ there are two possibilities: (1) The geodesic from P to gP meets L only at P or (2) The geodesic from P to gP meets L at a vertex $Q \neq P$. In the first case L is repulsing for g. In the second case let L' be the half-line contained in L starting at Q.



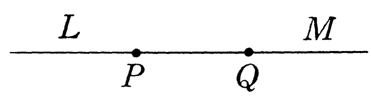
If $R \in L'$ then gR belongs to the geodesic from gP to Q for only finitely many R. Thus there is a half-line $L'' \subset L'$ so that $gL'' \subset L'$. Choose $R \in L''$ so that Q belongs to the geodesic from P to R and gQ belongs to the geodesic from gP to gR. If gQ belongs to the geodesic from gP to Q then L' is repulsing or neutral for g; otherwise, L' is attracting for g. It is now easy to see that if one half-line $L \in e$ is almost neutral, almost repulsing or almost attracting for g then so is every half-line in e; viz. if g has a fixed point on L in e then it must be almost neutral for g. □

PROPOSITION 4. *If G has a neutral fixed end e then either $\mathscr{E}^G = \{e\}$ or there is a doubly infinite line of fixed points for G.*

Proof. Suppose f is a repulsing or attracting end for $g \in G$. Let $P \in e$ so that $gP = P$ and choose $L \in f$ on which g is repulsing or attracting starting at Q.



This is impossible since the length of the geodesic from P to Q is different from that of gP to gQ. Thus any other fixed end f for G must be a neutral fixed end. Choose representative $L \in e$, $M \in f$ so that $L \cup M$ is a double infinite line.



Thus for each $g \in G$ there exists $P \in L$, $Q \in M$ fixed by g; hence the doubly infinite $L \cup M$ is fixed identically for all $g \in G$. □

From the above remarks we see that every half-line in $e \in \mathscr{E}^G$ is one of the mutually exclusive alternatives for a given $g \in G$. We can then define $v_e \colon G \to \boldsymbol{Z}$ for a fixed end e as follows

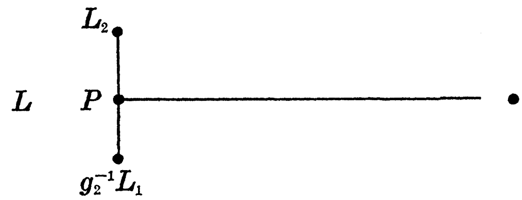
$$v_e(g) = \begin{cases} 0 & \text{if } e \text{ is neutral for } g \\ \min |L - (L \cap gL)| \\ L \in e, L \text{ attracting for } g & \text{if } e \text{ is attracting for } g \\ -\min |gL - (L \cap gL)| \\ L \in e, L \text{ repulsing for } g & \text{if } e \text{ is repulsing for } g\,. \end{cases}$$

THEOREM 4. *For each fixed end $e \in \mathscr{E}^G$ there is a canonical* *homomorphism*

$$v_e\colon G \longrightarrow Z$$

with the property that $v_e(g) = 0$ *if and only if* e *is neutral for* g *and* $L^g \neq \varnothing$ *for all* $L \in e$.

Proof. To see that v_e is a homomorphism let $g_1, g_2 \in G$ achieve their v_e value on L_1, L_2 respectively and let g_1g_2 achieve its v_e value on L. Consider the half-line $g_2^{-1}L_1 \cap L_2 \cap L$ starting at P.



Now $P \in L_2$ and thus $g_2P \subset L_1$ so P has moved $v_e(g_2)$ under the action of g_2. However $g_1(g_2P) \subset L_1$ so that g_2P has moved $v_e(g_1)$ under the action of g_1. Thus P has moved $v_e(g_1) + v_e(g_2)$ under the action of g_1g_2; however $P \in L$ so P moves $v_e(g_1g_2)$ under the action of g_1g_2, so that

$$v_e(g_1g_2) = v_e(g_1) + v_c(g_2) .$$

If $v_e(g) = 0$ then e is neutral for g and $L_g \neq \varnothing$ for some $L \in e$. Moreover, since g fixes identically a half-line $L' \subset L$ then it must have a fixed point on every half-line in e. Conversely if g has a fixed point on some $L \in e$ then L is almost neutral for g and e is neutral for g; thus $v_e(g) = 0$. □

COROLLARY 1. *If for a given action of* G *on a tree* X *a normal subgroup* H *has a unique neutral fixed end* e *then either* e *is a neutral fixed end for* G *or there is a nontrivial homomorphism* $v\colon G/H \to Z$.

Proof. It is easy to see that e is a fixed end for G; viz. suppose $ge = f \in \mathscr{E}$, then for $h \in H$

$$e = g^{-1}hge = g^{-1}hf .$$

Thus $f = ge = hf$ and by uniqueness $f = e$. Thus from the theorem above $v_e\colon G \to Z$ factors through a homomorphism $v\colon G/H \to Z$ since e is neutral for H. □

COROLLARY 2. *If* G *has a normal subgroup* H *so that* H *and*

G/H have property FA$'$ *then G has property* FA$'$.

Proof. Let G act on a tree X. If H has a fixed point then there is an action of G/H on X^H. Since G/H has property FA$'$ we can find a fixed point for $g \in G$ by finding a fixed point for gH on X^H. If H has no fixed points on X then it has a neutral fixed end and thus since G/H has no homomorphism to Z this neutral fixed end for H is also a neutral fixed end for G. □

COROLLARY 3. *Suppose that G acts without inversions on a tree X. If G is generated by a set S with $X^s \neq \emptyset$ for all $s \in S$ then either G has no fixed end or a fixed end is neutral.*

Proof. If G has a fixed end e then any $s \in S$ has a fixed point lying on some half-line $L \in e$; thus $v_e(s) = 0$. It follows immediately from the theorem then that v_e is trivial and consequently that e is neutral for G. □

A nonempty collection of subgroups $\mathscr{N} = \{N_\alpha \mid \alpha \in \mathscr{A}\}$ of a group G is called a normal filtering family if

(1) given $\alpha, \beta \in \mathscr{A}$, $\exists \gamma \in \mathscr{A}$ so that $N_\gamma \subset N_\alpha \cap N_\beta$ and

(2) given $\alpha \in \mathscr{A}$, $g \in G$, $\exists \beta \in \mathscr{A}$ so that $N_\beta \subset g N_\alpha g^{-1}$.

(These are the conditions that guarantee G is a topological group with $\mathscr{N}$ as a fundamental system of open subgroups.)

PROPOSITION 5. *Suppose that G acts without inversions on a tree X such that $\mathscr{E}^G = \emptyset$. If $\mathscr{N}$ is a normal filtering family of subgroups of G having property* FA$'$ *on X then some $N \in \mathscr{N}$ has a fixed point.*

Proof. Suppose by way of contradiction that no $N \in \mathscr{N}$ has a fixed point; it follows then from the FA$'$ property that each $N_\alpha \in \mathscr{N}$ has a unique neutral fixed end e_α. Given $N_\alpha, N_\beta \in \mathscr{N}$, choose $N_\gamma \subset N_\alpha \cap N_\beta$; we have then

$$\{e_\alpha\} = \mathscr{E}^{N_\alpha} = \mathscr{E}^{N_\gamma} = \mathscr{E}^{N_\beta} = \{e_\beta\} .$$

Thus there is a common neutral fixed end e for $\mathscr{N}$. Given $N_\alpha \in \mathscr{N}$, $g \in G$, choose $N_\beta \subset g N_\alpha g^{-1}$; it follows that

$$\{e\} = \mathscr{E}^{N_\beta} = \mathscr{E}^{g N_\alpha g^{-1}} = \{ge\}$$

and thus e is a fixed end for G. □

Since an amalgam has elements which have no fixed points on the tree corresponding to the amalgamation it follows from Corollary 3 and Theorem 2 that there can be no fixed end for this action. Similarly, for an HNN extension $A_C^*(C \neq A)$ acting on its corresponding tree there can be no fixed end. To see this, we may choose without loss of generality a representative half-line for this end with initial vertex and edge having stabilizers A and C respectively; then for $g \in A$ it follows from Theorem 4 that g is neutral on this half-line and thus $g \in C$, whence $C = A$. We shall use these remarks together with Proposition 5 to derive some important consequences for topological groups. Also, this proposition will provide useful information if the family consists of a single normal subgroup.

As a further remark on extensions of groups having property FA′ we have the following result.

THEOREM 5. *If H and K are subgroups of G having property* FA′ *and $G = HK$ then G has property* FA′.

Proof. Let $g \in G$ be written as $g = hk$, $h \in H$, $k \in K$; express now $kh = h'k'$, $h' \in H$, $k' \in K$. Thus we have

$$(h^{-1}h')k'(h^{-1}k^{-1}h) = 1 .$$

By results of Serre [6, Corollary 1 to Proposition 26] we can find a common fixed point $P \in X$ of the automorphisms $h^{-1}h'$, k', $h^{-1}k^{-1}h$ for an action of G on the tree X if each has a fixed point; this is so from the FA′ hypothesis for H and K. Consequently, we have the properties:

$$hP = h'P,\ k'P = P,\ k(hP) = hP .$$

Let X^k, $X^{k'}$ be the trees of fixed points of k and k'; $X^k \cap X^{k'} \neq \emptyset$ since K has property FA′ (condition (ii)). Since $P \in X^{k'}$, $hP \in X^k$, it follows that the midpoint Q of the geodesic from P to hP is fixed by h [6, Corollary 2 to Proposition 23] and also by h' since $hP = h'P$; thus $Q \in X^k$ or $Q \in X^{k'}$. If $Q \in X^k$ then $hkQ = Q$. If $Q \in X^{k'}$ then $h'k'Q = Q$; but $hk = h(kh)h^{-1} = h(h'k')h^{-1}$ so $hk(Q) = hk(hQ) = h(kh)h^{-1}(hQ) = hh'k'Q = hQ = Q$. Hence g has a fixed point for its action on X. □

3. We now derive consequences for topological groups from the results of the previous sections.

THEOREM 6. *If G is a connected locally compact topological group then G is of type* FA′.

Proof. As a first step we decompose G as

$$G = LCR$$

where L is a semisimple (connected) Lie subgroup, C is a compact connected semisimple subgroup and R is the radical of G (maximal solvable connected closed normal subgroup) and CR is a closed normal subgroup [5, Theorem 1]. [One uses the solution of Hilbert's fifth problem to see the equivalence of connected locally compact and Iwasawa's notion of (L) group [3].] Now using [4, Lemma 3.12] we decompose the group L as $L = HM$ where H is a connected solvable Lie group and M is either the maximal compact subgroup K of L or $M = K \times V$ where V is a vector group. It suffices then using Theorem 5 to verify that H, M, C, R are of type FA$'$. Compact groups are of type FA$'$ [1]; also any vector group being divisible and abelian is FA$'$. It remains to show that a connected solvable group S is of type FA$'$. Using Iwasawa's decomposition of a locally compact connected group as

$$G = H_1 H_2 \cdots H_r K$$

where K is maximal compact and $H_i \cong \boldsymbol{R}$ $1 \leqq i \leqq r$ [4; Theorem 13], we see that G has no nontrivial homomorphisms to $\boldsymbol{Z}$. Now if S is an amalgam then using Bass' result for solvable groups [2, Theorem 6.1] we obtain a surjective homomorphism $S \xrightarrow{\varphi} \boldsymbol{Z}_2 * \boldsymbol{Z}_2$. However using the Iwasawa decomposition above for $S(=G)$ we obtain $\varphi|_{H_i}$, $\varphi|_K$ are trivial homomorphisms. To see this notice that H_i, K are of type FA$'$ and hence by Proposition 2 each of the restrictions has image in a conjugate of one of the $\boldsymbol{Z}_2$ factors; the divisibility of H_i $1 \leqq i \leqq r$, K then forces each image to be trivial and thus also φ. □

COROLLARY 1. *If G is a locally compact topological group with G/G_0 compact then G is of type* FA$'$.

Proof. This follows immediately from the theorem above, Corollary 2 to Theorem 4 and main result of [1].

COROLLARY 2. *Suppose G is a locally compact topological group. If G is an amalgam, $G = A *_C B$, or an* HNN *extension, $G = A *_C$, then $G_0 \subset C$.*

Proof. The connected component of the identity G_0 is of type FA$'$. Using Proposition 5 and the remarks following it we see that G_0 has a fixed point for its action on the tree corresponding to the amalgam or the HNN extension if $C \neq A$. Since G_0 is normal we see immediately that $G_0 \subset C$. In case the HNN extension has $C = A$ the corresponding tree has a fixed end, say e; using Theorem 4 now, each element of G_0 has a fixed point so $G_0 \subset \ker v_e = A$. □

COROLLARY 3. *Suppose G is a locally compact topological group. If G is an amalgam, $G = A *_C B$ or an* HNN *extension, $G = A *_C$, then C is open in G.*

Proof. Without loss of generality we may replace G by G/G_0 using Corollary 2 above and assume then that G is a locally compact totally disconnected topological group. It is well known that G has a neighborhood basis of the identity given by compact open subgroups [Hewitt and Ross, *Abstract Harmonic Analysis*, p. 62]. Since compact groups are of type FA′ this is a normal filtering family of type FA′; by dint of Proposition 5 then some compact open subgroup U has a fixed point. Without loss of generality we may assume $U \subset A$. In case G is an amalgam choose $g \in B - C$ so that $U \cap gUg^{-1} \subset A \cap gAg^{-1} \subset C$; hence C is open. If G is an HNN extension it is generated by A together with an element t (which generates the fundamental group of X/G [6, p. 62]); thus

$$U \cap tUt^{-1} \subset A \cap tAt^{-1} \subset C .$$

For the HNN extension to which Proposition 5 doesn't apply, viz. $G = A *_A$ we notice as in the proof of Corollary 2 that there is a fixed end e and hence for any compact open subgroup U,

$$U \subset \ker v_e \subset A$$

since U is of type FA′. □

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THE UNIVERSITY OF OKLAHOMA
NORMAN, OK 73019