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PTOLEMY'S INEQUALITY, CHORDAL METRIC, MULTIPLICATIVE METRIC

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Ptolemy's inequality in R^2 states: If A, B, C, D are vertices of a quadrilateral, then

$$AB \cdot CD + BC \cdot AD \geqq AC \cdot BD$$

with equality only $ABCD$ is a convex cyclic quadrilateral. A real normed linear vector space is called *ptolemaic* if

$$\|x - y\| \, \|z\| + \|y - z\| \, \|x\| \geqq \|z - x\| \, \|y\|$$

for all x, y and z in the space and it is called *symmetric* if

$$\|\lambda x - y\| = \|x - \lambda y\|$$

for all unit vectors x, y and real λ. The equivalence of these two properties of a normed linear space is established and related results concerning distance functions in such spaces are proven.

Although Ptolemy's inequality is a useful tool and has often been applied (e.g., see [7]) it does not seem to be as widely known as would be desirable. Recently Apostol [1] gave an elegant proof of this inequality using complex numbers in the plane (see also [2], [4] and [5]) and extended the inequality to R^3 thereafter. Apostol used Ptolemy's inequality to show that the chordal distance

$$\chi(a, b) = \frac{|a - b|}{\sqrt{1 + |a|^2}\sqrt{1 + |b|^2}},$$

defined for pairs of complex numbers, satisfies the triangle inequality $\chi(a, b) + \chi(b, c) \geqq \chi(a, c)$. In an earlier paper, Schoenberg [9], answering a problem raised by Blumenthal, proved the following: If S is a real, seminormed space which is ptolemaic then the seminorm is a norm which springs from an inner product. In this note we wish to treat these results from a different point of view. We provide simpler proofs for some of the earlier results and extend a recent result of Schattschneider [6], [8].

2. DEFINITION 2. Let X be real normed linear space with norm $\|\cdot\|$.

(i) X is called *ptolemaic* if for every $x, y, z \in X$ we have

$$\|x - y\| \cdot \|z\| + \|y - z\| \cdot \|x\| \geqq \|x - z\| \cdot \|y\| . \tag{2.1}$$

(ii) X is called *symmetric* if for every $x, y \in X$ with $\|x\| =$

$\|y\| = 1$ and for all real λ we have

$$(2.2) \qquad \|\lambda x - y\| = \|x - \lambda y\| \,.$$

3. THEOREM 1. *Let* $(X, \|\cdot\|)$ *be normed linear space. Then* X *is ptolemaic if and only if* X *is symmetric.*

Proof. Suppose X is symmetric. Let $x, y, z \in X$; we wish to prove (2.1). Clearly we may assume without loss of generality that $\|x\| > 0$, $\|y\| > 0$, $\|z\| > 0$. Now, by (2.2),

$$(3.1) \qquad \|x - y\| = \left\|\frac{x}{\|x\|}\|y\| - \frac{y}{\|y\|}\|x\|\right\| = \|x\|\cdot\|y\|\left\|\frac{x}{\|x\|^2} - \frac{y}{\|y\|^2}\right\|$$

and similar relations hold for the pair of vectors x and z and for y and z. Thus (2.1) is equivalent to the triangle inequality for the vectors $x/\|x\|^2$, $y/\|y\|^2$ and $z/\|z\|^2$ in X. Conversely, if X is ptolemaic, then by [9], X is a real inner product space. (2.2) is then immediate, i.e., X is symmetric.

COROLLARIES. (i) $\boldsymbol{R}_n$ $(n = 1, 2, \cdots)$ *is ptolemaic, for, it is clearly symmetric.*

(ii) *If* X *is a symmetric normed linear space, then the distance function*

$$(3.2) \qquad d(x, y) = \frac{\|x - y\|}{\|x\| \cdot \|y\|}$$

defined for $\|x\|, \|y\| > 0$, satisfies the triangle inequality. For, by (3.1), the triangle inequality for $d(x, y)$ follows from the triangle inequality in X.

We note that the proof of Ptolemy's inequality using the symmetry condition is, in $\boldsymbol{R}^n$, equivalent to using inversion.

4. **The chordal metric.** We shall establish the following extension of Apostol's result mentioned in our introduction.

THEOREM 2. *Let* $(X, \|.\|)$ *be a normed linear space. If* X *is symmetric, then the chordal distance given by*

$$(4.1) \qquad \chi(x, y) = \frac{\|x - y\|}{(\alpha + \beta\|x\|^p)^{1/p} \cdot (\alpha + \beta\|y\|^p)^{1/p}}$$

is a metric for every $\alpha > 0$, $\beta \geqq 0$, $p \geqq 1$.

Proof. We only have to prove that χ satisfies the triangle inequality. Let x, y, z be arbitrary vectors in X. Then by the triangle inequality

$$\alpha\cdot(\|x-y\|+\|y-z\|)^p \geqq \alpha\cdot\|x-z\|^p\,, \tag{4.2}$$

and since X is ptolemaic,

$$\beta\cdot(\|z\|\cdot\|x-y\|+\|x\|\cdot\|y-z\|)^p \geqq \beta\cdot(\|y\|\cdot\|x-z\|)^p\,. \tag{4.3}$$

Adding (4.2) and (4.3) and using Minkowski's inequality, we get

$$\begin{aligned}\|x-y\|\cdot(\alpha+\beta\|z\|^p)^{1/p}+\|y-z\|(\alpha+\beta\|x\|^p)^{1/p}\\ \geqq \|x-z\|(\alpha+\beta\|y\|^p)^{1/p}\end{aligned}$$

which proves that χ in (4.1) satisfies the triangle inequality.

5. A multiplicative metric. We shall establish the following extension of Schattschneider's result [8].

THEOREM 3. *Let* $(X, \|\cdot\|)$ *be a normed linear vector space. If* X *is symmetric, then the distance function defined by*

$$\begin{aligned} d(x,y) &= \frac{\|x-y\|}{(\|x\|^p+\|y\|^p)^{1/p}}\,, \quad \textit{if} \quad \|x\|+\|y\|>0\\ &= 0 \qquad\qquad\qquad\quad , \quad \textit{if} \quad \|x\|+\|y\|=0 \end{aligned} \tag{5.1}$$

is a metric for every $p \geqq 1$.

Proof. Denote, for brevity, $\|x-y\|=a$, $(\|x\|^p+\|y\|^p)^{1/p}=a'$, $\|y-z\|=b$, $(\|y\|^p+\|z\|^p)^{1/p}=b'$ and $\|z-x\|=c$, $(\|z\|^p+\|x\|^p)^{1/p}=c'$. We only need to prove the triangle inequality for $d(x,y)$, i.e., with the above notation, that

$$\frac{a}{a'}+\frac{b}{b'} \geqq \frac{c}{c'}\,. \tag{5.2}$$

By the triangle inequality of the norm,

$$a+b \geqq c\,, \tag{5.3}$$

and by Ptolemy's inequality,

$$a\|z\|+b\|x\| \geqq c\|y\|\,. \tag{5.4}$$

If $c' \geqq a'$ and $c' \geqq b'$, then (5.2) follows from (5.3). If $c' \leqq a'$ and $c' \leqq b'$, then, one sees easily, $\|y\|c' \geqq \|z\|a'$ and $\|y\|c' \geqq \|x\|b'$. Hence, (5.2) follows from (5.4). In the remaining case, c' is between a' and b', say $a' < c' < b'$ or equivalently $\|x\| < \|y\| < \|z\|$. Now, using the inequality $u^p + v^p \geqq 2^{1-p}(u+v)^p$ and then (5.3) and (5.4), we obtain

$$ab'+ba' \geqq 2^{(1-p)/p}(a\|y\|+a\|z\|+b\|x\|+b\|y\|) \geqq 2^{1/p}\cdot c\cdot\|y\|\,.$$

A simple calculation shows that, because of $\|x\| < \|y\| < \|z\|$, we have

$$2^{1/p} \cdot \|y\| \geqq \frac{a'b'}{c'} .$$

Whence,

$$ab' + ba' \geqq a'b' \frac{c}{c'} .$$

This proves (5.1) in the last case.

COROLLARY. *The multiplicative distance defined by* (5.1) *is a metric in* $\boldsymbol{R}^n (n = 1, 2, \cdots)$ *and, in fact, in any inner product space. (Schattschneider's metric corresponds to the special case* $p = 1$ *in* $\boldsymbol{R}^n$.)

We do not know whether or not $d(x, y)$ of (5.1) is a metric for every $p \geqq 1/2$. We can prove that the triangle inequality holds if $p = 1/2$ and fails if $p = 1/4$.

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