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FIXED POINTS ON FLAG MANIFOLDS

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When K is $\boldsymbol{R}$, $\boldsymbol{C}$, or $\boldsymbol{H}$, let $U_K(n)$ denote the group of $n \times n$ orthogonal, unitary, or symplectic matrices, respectively. If G is a closed connected subgroup of $U_K(n)$ of maximal rank, then it is conjugate to a subgroup of the form $U_K(n_1) \times U_K(n_2) \times \cdots \times U_K(n_k)$. A simple condition on the integers n_i is shown to be necessary for $U_K(n)/G$ to have the fixed point property (that every self map has a fixed point). It is conjectured that this condition is also sufficient, and a proof is given for some cases.

For a partition $n = n_1 + n_2 + \cdots + n_k$ of a positive integer n and $K = \boldsymbol{R}$, $\boldsymbol{C}$, or $\boldsymbol{H}$, the corresponding generalized flag manifold $U_K(n)/(U_K(n_1) \times \cdots \times U_K(n_k))$ will be denoted $KM(n_1, \cdots, n_k)$. We conjecture that $KM(n_1, \cdots, n_k)$ has the fixed point property if and only if $n_1, \cdots, n_k$ are distinct integers and, when $K = \boldsymbol{R}$ or $\boldsymbol{C}$, at most one is odd. We prove that this condition is necessary and that it is sufficient, in addition to previously known cases, for the manifolds $KM(1, n_2, n_3)$ where n_3 is large relative to n_2 (and, when $K = \boldsymbol{R}$, in some other cases as well).

THEOREM 1. *If $KM(n_1, n_2, \cdots, n_k)$ has the fixed point property, then $n_1, \cdots, n_k$ are distinct integers and, if $K = \boldsymbol{R}$ or $\boldsymbol{C}$, at most one is odd.*

Proof. We can regard $M = \boldsymbol{C}M(n_1, \cdots, n_k)$ as the space of orthogonal direct sum decompositions $\boldsymbol{C}^n = V_1 \oplus \cdots \oplus V_k$, where V_m has dimension n_m over $\boldsymbol{C}$. If $n_r = n_s$, interchanging the rth and sth summands defines a fixed point free self map of M.

For the rest of the proof, we define a conjugate linear transformation J of $\boldsymbol{C}^n$ and consider the associated self map f of M, which takes $V_1 \oplus \cdots \oplus V_k$ to $JV_1 \oplus \cdots \oplus JV_k$. If $n = 2m$, we regard $\boldsymbol{C}^n$ as $\boldsymbol{H}^m$ and take J to be multiplication by the quaternion j. Any subspace of $\boldsymbol{C}^n$ invariant under J has the structure of a vector space over $\boldsymbol{H}$ and so has even dimension as a vector space over $\boldsymbol{C}$. Thus if at least one (and so necessarily at least two) of the integers $n_1, \cdots, n_k$ is odd, f has no fixed points.

If $n = 2m + 1$, we write $\boldsymbol{C}^n = \boldsymbol{H}^m \oplus \boldsymbol{C}$ and take J to be multiplication by j on the first summand and complex conjugation on the second. If W is a subspace of $\boldsymbol{C}^n$ which is invariant under J, then its projection onto the first summand is invariant under multiplication by j and so has even dimension over $\boldsymbol{C}$. Hence each odd dimensional subspace which is invariant under J must contain the second summand. If at least two of $n_1, \cdots, n_k$ are odd, it follows that f again has no fixed points.

The proof in the real case is analogous. In the quaternionic case we can construct only the self maps which interchange summands of equal dimensions.

Conjecture 2. The converse of 1 is also true: If $n_1, \cdots, n_k$ are distinct positive integers and, when $K = \boldsymbol{R}$ or $\boldsymbol{C}$, at most one is odd, then $KM(n_1, \cdots, n_k)$ has the fixed point property.

This is well known to be true for complex projective spaces ($k = 2$ and $n_1 = 1$) and has been proved for many Grassmann manifolds ($k = 2$ and either $n_1 \leqq 3$ or $n_2 \geqq 2n_1^2 - n_1 - 1$ [5, 3]). Here we verify the following additional cases.

THEOREM 3. *If n_2 and n_3 are distinct positive even integers and $n_2 \geqq 2n_2^2 - 1$, then $CM(1, n_2, n_3)$ has the fixed point property.*

Proof. Under these hypotheses, Theorem 1.4 of [4] states that every graded ring endomorphism of $H^*(CM(1, n_2, n_3); \boldsymbol{Z})$ takes one of two simple forms, termed grading and projective endomorphisms. It suffices to check that for neither type can the Lefschetz number be zero if both n_2 and n_3 are even. This will follow from the general results 4 and 5 below. □

Recall that the grading endomorphism φ of degree $\lambda \in \boldsymbol{Z}$ has the form $\varphi(x) = \lambda^m x$ for $x \in H^{2m}(CM(n_1, \cdots, n_k); \boldsymbol{Z})$.

PROPOSITION 4. *A grading endomorphism of $H^*(CM(n_1, \cdots, n_k); \boldsymbol{Z})$ has Lefschetz number zero if and only if its degree is -1 and at least two of $n_1, \cdots, n_k$ are odd.*

Proof. Since the Lefschetz number is congruent to 1 modulo the degree λ and is positive if $\lambda = 1$, it can be zero only if $\lambda = -1$. From [1], the Poincaré polynomial for $CM(n_1, \cdots, n_k)$ is

$$\prod_{j=1}^{n} (1 - t^{2j}) \Big/ \prod_{i=1}^{k} \prod_{j=1}^{n_i} (1 - t^{2j}) .$$

The Lefschetz number of the grading endomorphism of degree -1 is the value of this polynomial when $t^2 = -1$. This value is zero if and only if the number of factors of the form $1 - t^{4j}$ in the numerator exceeds the number of such factors in the denominator, which in turn happens precisely when at least two of $n_1, \cdots, n_k$ are odd. □

Recall that a projective endomorphism of $H^*(\boldsymbol{CM}(1, n_2, \cdots, n_k); \boldsymbol{Z})$ is an endomorphism which factors through the monomorphism induced by the canonical map $\pi: \boldsymbol{CM}(1, n_2, \cdots, n_k) \to \boldsymbol{CM}(1, n-1) = \boldsymbol{CP}(n-1)$.

$$\begin{array}{ccc} H^*(\boldsymbol{CM}(1, \cdots)) & \xrightarrow{\varphi} & H^*(\boldsymbol{CM}(1, \cdots)) \\ \uparrow{\scriptstyle \pi^*} & \searrow{\scriptstyle \psi} & \uparrow{\scriptstyle \pi^*} \\ H^*(\boldsymbol{CP}(n-1)) & \xrightarrow[\theta]{} & H^*(\boldsymbol{CP}(n-1)) \end{array}$$

If φ factors as $\pi^* \circ \psi$, we define its degree to be that of $\theta = \psi \circ \pi^*$. Since the Lefschetz number of φ equals that of θ, we have the following results.

PROPOSITION 5. *A projective endomorphism of* $H^*(\boldsymbol{CM}(1, n_2, \cdots, n_k); \boldsymbol{Z})$ *has Lefschetz number zero if and only if its degree is* -1 *and* $n - 1 = n_2 \cdots + n_k$ *is odd.*

The statements for the quaternionic and real flag manifolds are as follows.

THEOREM 6. *If* 1, n_2, *and* n_3 *are distinct positive integers and* $n_3 \geqq 2n_2^2 - 1$, *then* $\boldsymbol{HM}(1, n_2, n_3)$ *has the fixed point property.*

The proof of 6 is analogous to that for 3, with one additional observation. A degree -1 endomorphism (either grading or projective) of cohomology does not commute with reduced third power operations (cf. [2]) and so cannot be induced by a self map.

THEOREM 7. (i) *If* $n_2 < n_3$ *are even integers greater than* 1 *and either* $n_2 \leqq 6$ *or* $n_3 \geqq n_2^2 - 2n_2 - 2$, *then* $\boldsymbol{RM}(1, n_2, n_3)$ *has the fixed point property.*

(ii) *If* n_1, n_2, *and* n_3 *are positive integers such that at most one is odd,* $n_1 \leqq 3$, $n_3 \geqq n_2^2 - 1$, *and* $[n_1/2] < [n_2/2] < [n_3/2]$, *then* $\boldsymbol{RM}(n_1, n_2, n_3)$ *has the fixed point property. (Here* $[r]$ *denotes the greatest integer in* r.*)*

Proof. Since at most one of n_1, n_2 and n_3 is odd, $O(n_1) \times O(n_2) \times O(n_3)$ has maximal rank in $O(n)$. It follows that $H^*(\boldsymbol{RM}(n_1, n_2, n_3); \boldsymbol{Q})$ is generated as an algebra by the Pontryagin classes of the canonical n_i-plane bundles (cf. [1]). To simplify notation, let $\bar{m}$ denote $[m/2]$.

Ad (i): If $n_1 = 1$, then $H^*(\boldsymbol{RM}(1, n_2, n_3); \boldsymbol{Q})$ is isomorphic as a graded algebra (with a shift in the grading) to $H^*(\boldsymbol{CM}(\bar{n}_2, \bar{n}_3); \boldsymbol{Q})$. Since n_2 and n_3 are distinct even integers, $\bar{n}_2$ and $\bar{n}_3$ are distinct.

By Theorem 1 of [3] (with the slight improvement noted in [4]), $H^*(CM(\bar{n}_2, \bar{n}_3); \boldsymbol{Q})$ admits only grading endomorphisms when either $\bar{n}_2 \leqq 3$ or $\bar{n}_3 \geqq 2\bar{n}_2^2 - 2\bar{n}_2 - 1$, which are equivalent to the inequalities in the statement of (i). The grading endomorphism of degree -1 has Lefschetz number 0 when $\bar{n}_2$ and $\bar{n}_3$ are both odd. But suppose f is a self map which takes the first two rational Pontryagin classes p_1 and p_2 of the canonical n_2-plane bundle over $RM(1, n_2, n_3)$ to $-p_1$ and p_2 respectively. Then for the mod 3 Pontryagin classes we also have $f^*(\tilde{p}_1) = -\tilde{p}_1$ and $f^*(\tilde{p}_2) = \tilde{p}_2$. The splitting principle (Proposition 25.4 of [1]) and the Cartan formula imply that the reduced third power operation $\mathscr{P}^1$ takes $\tilde{p}_1$ to $2(\tilde{p}_1^2 + \tilde{p}_2)$, so we have

$$\mathscr{P}^1 \circ f^*(\tilde{p}_1) = \mathscr{P}^1(-\tilde{p}_1) = -2(\tilde{p}_1^2 + \tilde{p}_2)$$
$$f^* \circ \mathscr{P}^1(\tilde{p}_1) = 2f^*(\tilde{p}_1^2 + \tilde{p}_2) = 2(\tilde{p}_1^2 + \tilde{p}_2)\ .$$

Hence no such f exists, and so every self map of $RM(1, n_2, n_3)$ has nonzero Lefschetz number. (Note that this corrects the proof of Theorem 5 in [3], where $\tilde{p}_2$ was inadvertently omitted from the formula for $\mathscr{P}^1(\tilde{p}_1)$.)

Ad (ii): If $n_1 = 2$ or 3, then $H^*(RM(n_1, n_2, n_3); \boldsymbol{Q})$ is isomorphic (with a shift in grading) to $H^*(CM(1, \bar{n}_2, \bar{n}_3); \boldsymbol{Q})$. We proceed as in the proof of (i), with Theorem 1.4 of [4] restricting the possibilities for cohomology endomorphisms and a similar $\mathscr{P}^1$ argument to show that the projective endomorphism of degree -1 (which has Lefschetz number 0 if $\bar{n}_2 + \bar{n}_3$ is odd) is not realized by a self map.

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