MAPGERMS INFINITELY DETERMINED WITH RESPECT TO RIGHT-LEFT EQUIVALENCE

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Mather has given both algebraic and geometric characterizations of finitely determined germs. We conjecture analogous characterizations of infinitely determined germs and prove parts of this conjecture. Recall that two mapgerms f and g are (right-left) equivalent if there are germs of diffeomorphisms l and r such that $f = l \circ g \circ r$. A mapgerm f at x is finitely determined if there is a k such that every germ having the same k-jet as f at x is equivalent to f; f is infinitely determined if every germ having the same Taylor series at x as f is equivalent to f.

Let E_n denote the space of germs at 0 in R^n of C^∞ real valued functions and let m_n denote the unique maximal ideal in E_n. Let E_n^p denote the set of p-tuples of elements of E_n; m_n^k may denote k-tuples or may be the kth power of m_n—which should be clear from context. If f is analytic, let f_C denote its complexification.

THEOREM 1.1. (*Mather; see* [2] *and* [4]). *For f in m_n^p, the following are equivalent:*

(1) *f is finitely determined;*

(2) *$df E_n^n + f^* E_p^p \supset m_n^k E_n^p$ for some $k > 0$;*

(3) *(assuming f analytic) f_C is locally multistable in a deleted neighborhood of 0.*

Since f_C is a germ, (3) is to be interpreted as saying that, for each representative F of the germ f_C, there is a deleted neighborhood U of 0 on which F is locally multistable (i.e., for each finite $S \subset U$ the germ of F at S is stable).

Since $df m_n^n + f^* m_p^p$ represents the tangent space to the orbit of f, the motivation for (2) is clear (except for the replacement of m by E). Also clear is that the analogous condition for infinite determination is gotten by replacing k by ∞.

Gaffney (see [2]) explains the motivation for (3) as follows: "It is relatively easy to understand why a finitely determined germ should have this property. Any perturbation at zero, whose Taylor expansion vanishes to sufficiently high order there, can be removed by a coordinate change in source and target. However, by a suitable high order perturbation at zero, one can obtain any kind of a perturbation at some fixed x different from zero. This low order perturbation at x is also removed by the induced change in

source and target at the origin. Thus, one would expect the germ at x to be stable."

If the perturbation at zero is by polynomial terms, the effect is felt at x whether complex or real. If the perturbation at zero is by flat functions, the quote still applies for real x. Thus we are led to:

(3a) (assuming f is analytic) f is locally multistable in a deleted neighborhood of 0.

However, it is not known whether infinitely determined germs need be equivalent to analytic germs. Since there are flat functions which satisfy (3a)—aside from analyticity—we need to repace (3a) by a condition which guarantees that f becomes unstable at a finite rate as we approach 0.

We write $R^{pr} \times J^k(n, p)^r$ for the space of r-tuples of k-jets of elements of E_n^p. If f maps U into R^p, we define a map $(j^k f)^r$ into the above jet space in the obvious way. If the germ of f at a finite set S is unstable, then there is a subset of $p + 1$ or fewer points on which it is also unstable. Further, its stability on this set depends only upon its $p + 1$ jet. Thus, if f is unstable at $\{x_1, \cdots, x_r\}$, we call $(j^{p+1} f)^r(x_1, \cdots, x_r)$ an unstable multijet, and let Uns denote the set of unstable multijets. If f is unstable at 0, then $(j^{p+1} f)^r$ is necessarily unstable at D, the set of r-tuples satisfying:

(a)   at least one component is zero, or

(b)   two or more component points are the same and are critical points of f.

Conjecture 1.2. For f in m_n^p, the following are equivalent:
(1) f is infinitely determined;
(2) $df E_n^n + f^* E_p^p \supset m_n^\infty E_n^p$;
(3) $d((j^{p+1} f)^{p+1}(x), Uns) \geqq Cd(x, D)^r$, for some $C, r > 0$.

Suppose f is analytic and (3a) holds. Then $F = (j^{p+1} f)^{p+1}$ is analytic and $F^{-1}(Uns) = D$. We will show later that Uns is the zero set of some analytic map P. Since $P \circ F$ is analytic, it satisfies a Lojasiewicz inequality. By Lemma 1.3 of [11], (3) holds. Of course, (3) implies (3a) even if f isn't analytic.

In [11], we prove the corresponding conjectures to be true when left-right equivalence is replaced by C, R, K or L equivalence. We will henceforth assume the notation and terminology of that paper. The present conjecture is an order of magnitude more difficult than those proved in [11]. We prove the following partial results.

THEOREM 1.3.   (1) *implies* (3).

The proof, which is carried out in §2, is entirely different from that of "(1) implies (3)" in theorem 1.1, where powerful algebraic techniques are employed.

THEOREM 1.4. *If f is finitely K-determined, then* (2) *implies* (1).

The proof is carried out in §3. The requirement that f be finitely K-determined is a real restriction: $(x^2 + y^2)^2$ is not finitely K-determined, but is ∞-R-determined, hence infinitely A-determined. Nevertheless, the restriction is not too great, as finitely determined germs are plentiful (see [10] for a discussion of this point).

THEOREM 1.5. *If f is analytic and finitely K-determined, then* (3a) *implies* (2).

EXAMPLE 1.6. Let $f(x, y) = (x, y^4 + xy^2)$. The critical set $C(f)$ is $2y(2y^2 + x) = 0$; 0 is the only nonfold point of f in C^2. f is two-to-one on $2y^2 + x = 0$ and df has the same image at these double points, so f is not stable on $R^2 - \{0\}$. Thus f is not infinitely determined.

EXAMPLE 1.7. Let $f(x, y) = (x, y^4 + x^2y^2)$. $C(f)$ is $2y(2y^2 + x^2) = 0$ and 0 is the only nonfold point in C^2. Since $C(f) \cap R^2$ is the x-axis, on which f is an embedding, $f|(R^2 - \{0\})$ is stable. Note that the points $(x, \pm xi/2^{1/2}) \in C(f)$ have a common real image. The image of df at both these points is spanned by $e_1 + 2xy^2e_2$ (where e_1 and e_2 form the standard basis). Thus f_C is not stable on $f_C^{-1}(R^2 - \{0\})$. In particular, f is not finitely determined, but is infinitely determined.

Belickii, in [1], establishes a sufficient condition for a germ to be infinitely determined. However, his condition is very restrictive—it implies that, on a deleted neighborhood of 0, the germ has only fold singularities and is an embedding on its fold set. There is also a mistake in the statement of his sufficient condition; for a discussion of this see [11]. (One should add that his principal result concerns germs of diffeomorphisms infinitely determined with respect to conjugation.) He also gives examples of infinitely determined germs.

There is a useful reformulation of condition (3). First we prove that Uns is an algebraic variety. Given a mapgerm $f: (R^n, S) \to R^p$, define

$$A: (R^n \times J^p(n, n) \times R^p \times J^p(p, p))^r \longrightarrow R^{pr} \times J^p(n, p)^r$$

to be the real linear map induced by

$$(df, f^*): E_S^n \times E_{f(S)}^p \longrightarrow E_S^p .$$

A depends only upon $z = (j^{p+1}f)^r \in \boldsymbol{R}^{pr} \times J^{p+1}(n, p)^r$ and is surjective iff z is stable. Considered as a matrix, the coefficients of A are polynomials on $\boldsymbol{R}^{pr} \times J^{p+1}(n, p)^r$. Let I be the polynomial ideal generated by the determinants of the maximal minors of A. Then Uns is the zero set of I.

Let $I(f) = (((j^{p+1}f)^{p+1})^{-1}I)E_{(p+1)n}$. Then condition (3) is equivalent to requiring that $I(f)$ be a Lojasiewicz ideal at D, which is equivalent to $I(f) \supset m_D^\infty$ (see V. 4.3 of [10]).

2. Proof of Theorem 1.3.

We assume that f does not satisfy condition (3). We will prove that two representatives of the Taylor series Tf of f at 0 exist which are inequivalent.

We need a method to prove that two maps are inequivalent. If they are equivalent, their singularity sets of each type are diffeomorphic. The strategy then is to find one representative whose singularities of some type are very nice, and another whose singularities of that type are demonstrably bad.

The nice representative will be found using a consequence of the Multijet Transversal Extension Theorem (see [11]). First some terminology and notation. The generalized diagonal in $\boldsymbol{R}^{nr}$ is $\{(x_1, \cdots, x_r): x_i \neq x_j \text{ for some } i \neq j\}$. Let $(\boldsymbol{R}^n)^{(r)}$ denote $\boldsymbol{R}^{nr}$-(generalized diagonal) and let $_rj^k f$ denote $(j^k f)^r|(\boldsymbol{R}^n)^{(r)}$. A deleted neighborhood of 0 in $(\boldsymbol{R}^n)^{(r)}$ means a neighborhood of 0 less those points, at least one component of which is 0.

LEMMA 2.1. *Given a formal power series z and a countable collection of submanifolds of $\boldsymbol{R}^{pr} \times J^k(n, p)^r$, there is a residual set of representatives f of z for which $_rj^k f$ is transverse on a deleted neighborhood of 0 to each of the submanifolds.*

We have two devices for constructing bad representatives. The first was proved in [11], and is used to construct a representative having singular jets along a given sequence.

LEMMA 2.2. *Suppose there exist w_i in $\boldsymbol{R}^{pr} \times J^k(n, p)^r$ ($k \leq \infty$), $x_i = (x_i^1, \cdots, x_i^r)$ in $\boldsymbol{R}^{nr}$ converging to 0, and f in E_n^p such that $q_i = w_i - (j^k f)^r(x_i)$ is flat along $|x_i^s|$ for each s and along $|x_i^s - x_i^t|$ for each $s \neq t$. Then there is a g such that $Tg = Tf$ and $(j^k g)^r(x_i) = w_i$ holds for a subsequence of $\{x_i\}$.*

The second device is used to construct a representative which fails to be transverse to some given singularity.

LEMMA 2.3. *Let* $\{S_j\}$ *be a collection of manifolds whose union is a closed set* $S \subset J^k(n, p)$. *Let* f *be a* C^∞ *map restricted to a closed disk* D *centered at* 0. *Assume* $V = (j^k f)^{-1} S$ *contains* 0. *Suppose there is a sequence* x_i *converging to* 0 *such that* $d(j^k f(x_i), S)$ *is flat along* $d(x_i, V)$. *Let* D_i *be a closed disk centered at* x_i *such that* $V, D_1, D_2, \cdots$ *are mutually disjoint. Then there is a* y_i *in the open disk* D_i^0 *and a* g *such that:*

(i) $Tf = Tg$ *and, on* $D - \cup D_i^0$, $g = f$;

(ii) $j^k g(y_i) \in S_j$ *for some* j *and* $j^k g$ *is not transverse to* S_j *at* y_i.

Proof. By assumption, there exist $w_i \in J^k$, flat along $d(x_i, V)$, such that $j^k f(x_i) + w_i \in S$. Let p_i be the polynomial of degree k with $j^k p_i(x_i) = w_i$. For each $l \geq k$, the C^l norm $|p_i|_{D_i}^l = |p_i|_{D_i}^k$ is flat along $d(x_i, V) = d_i$.

Using Lemma IV. 3.3 of [10], there is, for each i, a C^∞ a_i such that $a_i = 1$ on a neighborhood of x_i, $a_i = 0$ off D_i^0 and, for each multi-index I, a constant C_I independent of i such that

$$|D^I a_i(x)| \leq C_I / d_i^{|I|}.$$

Thus, $|a_i p_i|_D^l$ is flat along d_i for each l. Thus, for any sequence of numbers t_i between 0 and 1, $f + \sum t_i a_i p_i$ converges in the C^∞ topology to some C^∞ map g. This g satisfies (i).

Fix an i. Let g_t denote the map g as defined above with $t_i = t$ (hold the other t_j fixed). $j^k g_t(x)$ depends smoothly on both x and t. Thus the set of t such that $j^k g_t(D_i) \cap S = \varnothing$ is open and contains 0. Hence there is an s, $0 < s \leq 1$, and a $y_i \in D_i$ such that

(iii) $j^k g_t(D_i) \cap S = \varnothing$ for all $t < s$ and $j^k g_s(y_i) \in S_j$ for some j. However, by (i), $j^k g_s(D_i - D_i^0)$ misses S. Thus y_i is in D_i^0. If $j^k g_s$ were transverse to S_j at y_i, then for each t near s there would be some x in D_i^0 with $j^k g_t(x) \in S_j$. But this contradicts (iii). $\square$

First we describe the stratification to which we'll apply Lemma 2.1. By a stratification, we mean a locally finite partition into embedded submanifolds. Let $A = \text{Diff}(\boldsymbol{R}^n, 0) \times \text{Diff}(\boldsymbol{R}^p)$ act on $\boldsymbol{R}^{pr} \times J^k(n, p)^r$ in the usual way.

From Mather's results in [6], it follows that there is a stratification $\mathscr{S}$ of $\boldsymbol{R}^{pr} \times J^k(n, p)^r$ such that each stratum S is A-invariant and semialgebraic, S/A is an analytic manifold each compoent of which has the same dimension c, and $\pi: S \to S/A$ is analytic. If $c > 0$ and Q is an A-orbit in S, then a chart on S/A is given by any sufficiently small direct transversal L to Q in S—in particular, each A-orbit sufficiently near to Q in S intersects L exactly once. If $c > 0$, we call S an A-moduli stratum.

Assume f does not satisfy condition (3). By Lemma 2.1, there

is an h with $Th = Tf$ such that, on a deleted neighborhood of 0, $_r j^{p+1} h$ is transverse to the above-defined $\mathscr{S}$ for $r = 1, \cdots, p + 1$ and $j^1 h$ is transverse to the stratification of $J^1(n, p)$ by rank.

Case A. h is not locally multistable on any deleted neighborhood of 0.

In this case, there is a smallest r between 1 and $p + 1$ such that $_r j^{p+1} h$ has unstable values on every deleted neighborhood of 0. Suppose $z = _r j^{p+1} h(x)$ is in *Uns*. Since multitransversality to A-orbits is equivalent to stability (of a germ on a finite set), and h is multitransverse to all A-orbits except those in the A-moduli strata, it follows that z lies in an A-moduli stratum S. Let $x = (x_1, \cdots, x_r)$. By the minimality property of r, if r is greater than 1, then h is stable at each x_i (in fact at each proper subset of $\{x_1, \cdots, x_r\}$) and $h(x_i)$ has the same value for each i. It follows from a result in [12] that the A-orbit Q of z has codimension larger than nr (when $r = 1$, this is easy: an orbit is stable iff it has a transverse representative iff it has codimension $\leq n$). At any point y such that $_r j^{p+1} h(y) \in S$, $_r j^{p+1} h$ is transverse at y to S; for a sufficiently small neighborhood U_y of y, $_r j^{p+1} h(U_y) \cap S$ is a manifold of dimension less then the codimension of Q in S. Thus $\pi(im_r j^{p+1} h \cap S)$ is nowhere dense in S/A. Thus we can choose a w in S arbitrarily near z which is inequivalent to every jet $_r j^{p+1} h(y)$.

There is a sequence x_i in a deleted neighborhood of 0 which converges to 0 such that $z_i = _r j^{p+1} h(x_i)$ is unstable. Then we can choose w_i inequivalent to every jet $_r j^{p+1} h(y)$ such that $w_i - z_i$ is flat along $|x_i^s|$ for all s and along $|x_i^s - x_i^t|$ for every $s \neq t$. By Lemma 2.2, there is a map g such that $Tg = Th$ and $_r j^{p+1} g(x_i) = w_i$. Clearly g is not equivalent to h.

If Case A doesn't hold, then h is locally multistable on some deleted neighborhood of 0. Since f doesn't satisfy (3), there is a sequence x_i converging to zero such that (letting $z_i = (j^{p+1} f)^{p+1}(x_i)$) $d(z_i, Uns)$ is flat along $d(x_i, D)$. This implies that $d(z_i, Uns)$ is flat along

$$(*) \qquad d_i = \min \{|x_i^1|, \cdots, |x_i^{p+1}|, e_i\}$$

where e_i is the minimum of $|x_i^s - c| + |x_i^t - c|$ over all $s \neq t$ and critical points c of f.

Let w_i be the nearest element of *Uns* to z_i. Form $w_i' \in R^{pr} \times J^{p+1}(n, p)^r$ by omitting from w_i all noncritical components. Form x_i' and z_i' by omitting from x_i and z_i the corresponding components. Define d_i' by omitting from $(*)$ all the omitted components of x_i. From now on, we write w_i for w_i', etc.

Case B. $|z_i - w_i|$ is also flat along $|x_i^s - x_i^t|$, for all $s \neq t$.

Then, by Lemma 2.2, there is a g such that $Tg = Tf$ and $(j^{p+1}g)^r(x_i) = w_i$. Thus g and h are inequivalent.

Case C. For some $s \neq t$, $|z_i - w_i|$ is not .flat along $|x_i^s - x_i^t|$.

Then (passing to a subsequence of x_i if necessary, and renumbering) $|x_i^s - x_i^t|$ is flat along $|x_i^s|$, $|x_i^t|$ and $|x_i^s - c_i| + |x_i^t - c_i|$, for some $c_i \in C(f)$. Then $|x_i^s - c_i| + |x_i^t - c_i|$ is approximately $2|x_i^s - c_i|$ for i large. So $|z_i^s - w_i^s|$ is flat along $d(x_i^s, C(f))$ and w_i^s is a critical jet. By Lemma 2.3, there is a g with $Tg = Tf$ such that g is not transverse to the stratification of $J^1(n, p)$ by rank on any deleted neighborhood of 0. Thus g and h are inequivalent. $\square$

3. Proof of Theorem 1.4. We are given that f is finitely K-determined and

$$(3.1) \qquad dfE_n^n + f^*E_p^p \supset m_n^\infty E_n^p .$$

Let $g(x, t) = f(x) + tu(x)$, u flat, let $F = (f, t)$ and $G = (g, t)$, and let f^a, g^a, F^a and G^a denote the germs of these maps at $(0, a)$ (where $f(x, t) = f(x)$ for all t). Via translation we identify the germs at $(0, a)$ with those at $(0, 0)$. Thus $g^a - f^a$ is in $m_n^\infty E_{n+1}^p$. We fix a and henceforth will suppress the superscript a, writing F for F^a, etc.

Let m_T^k denote the germs which vanish to order k along the t-axis in either R^{n+1} or R^{p+1}. Then $m_n^\infty E_{n+1} = m_T^\infty$ and $m_p^\infty E_{p+1} = m_T^\infty$.

It is clear that $d(g - f)E_{n+1}^n \subset (m_T^\infty)^p$. If V is in E_{p+1}^p, then $T(V \circ G - V \circ F) = 0$ on the t-axis, so $V \circ G - V \circ F$ is in $(m_T^\infty)^p$.

We will later prove:

$$(3.2) \qquad dfE_{n+1}^n + F^*E_{p+1}^p \supset (m_T^\infty)^p .$$

Assuming this, we have

$$(3.3) \qquad \begin{aligned} dgE_{n+1}^n + G^*E_{p+1}^p &\subseteq dfE_{n+1}^n + F^*E_{p+1}^p \\ &= dgE_{n+1}^n + G^*E_{p+1}^p + (m_T^\infty)^p . \end{aligned}$$

Our goal is to prove equality in (3.3). It then follows that $(m_T^\infty)^p \subset dgE_{n+1}^n + G^*E_{p+1}^p$. Multiplying both sides by G^*m_T, we have

$$\begin{aligned} G^*(m_T)(m_T^\infty)^p &\subset dgG^*m_T E_{n+1}^n + G^*(m_T^p) \\ &\subseteq dgm_T^n + G^*(m_T^p) . \end{aligned}$$

Since f is ∞-K-determined, it follows from the remark immediately following Lemma 2.1 of [11] that $(m_T^\infty)^p \subset dg(m_T^\infty)^n + G^*(m_T)(m_T^\infty)^p$. Thus,

$$(m_T^\infty)^p \subset dg m_T^n + G^*(m_T^p) .$$

Then the **argument of Mather in §6.3 of [4]** shows that g_t is equivalent to $g_0 = f$ for all t, and the diffeomorphisms depend smoothly on t.

Proof of 3.2. Recall that we assumed f is finitely K-determined. Choose $a_1, \cdots, a_s$ in E_n whose projections into $E_n/Jf + (f^*m_p)E_n$ form a real vector space basis. Since

$$(F^*m_{p+1})E_{n+1} \supset f^*(m_p)E_n + tE_{n+1} ,$$

$E_{n+1}/(Jf)E_{n+1} + (F^*m_{p+1})E_{n+1}$ is also generated as a real vector space by the projections of $a_1, \cdots, a_s$. By the Malgrange Preparation Theorem, $E_{n+1}/(Jf)E_{n+1}$ is generated by $a_1, \cdots, a_s$ as an F^*E_{p+1} module. In particular,

$$(3.4) \qquad (Jf)E_{n+1} + (F^*E_{p+1})E_n = E_{n+1} .$$

Since $df E_{n+1}^n \supset Jf E_{n+1}^p$ and

$$\begin{aligned}
df E_{n+1}^n + F^*E_{p+1}^p &\supset (df E_n^n + F^*E_{p+1}^p)F^*E_{p+1} \\
&\supset (df E_n^n + f^*E_p^p)F^*E_{p+1} \\
&\supset (m_n^\infty)^p F^*E_{p+1} \quad \text{by (3.1) ,}
\end{aligned}$$

it follows that

$$\begin{aligned}
df E_{n+1}^n + F^*E_{p+1}^p &\supset (m_n^\infty)^p(Jf E_{n+1} + F^*E_{p+1}E_n) \\
&= (m_n^\infty)^p E_{n+1} \quad \text{by (3.4) .} \qquad \square
\end{aligned}$$

Proof that equality holds in (3.3). Let tG denote $dg E_{n+1}^n$ and wG denote $G^*E_{p+1}^p$ (etc. for F). Then (3.3) says

$$(3.5) \qquad tG + wG \subsetneqq tF + wF = tG + wG + (m_T^\infty)^p .$$

Since G is ∞-K-determined,

$$(3.6) \qquad \begin{aligned}
(m_T^\infty)^p &= (tG + G^*m_{p+1}E_{n+1}^p)m_T^\infty \\
&\subset tG + G^*m_{p+1}(tF + wF) .
\end{aligned}$$

By (3.5) and (3.6),

$$(tG + wG) + G^*m_{p+1}(tF + wF) = tF + wF .$$

Let $A = (tF + wF)/tG$. We will show: (3.7) A is finitely generated as a G^*E_{p+1} module. Then, by Nakayama's lemma,

$$(tG + wG)/tG = (tF + wF)/tG , \quad \text{so}$$
$$tG + wG = tF + wF ,$$

which is what we want to prove.

To prove (3.7), it suffices by Gaffney's Preparation Theorem (see [2]) to show that (3.8) A is f.g. as an $(F, G)^*E_{2p+2}$ module, and (3.9) $A/(G^*m_{p+1})A$ has finite real dimension.

By (3.5), $A/(G^*m_{p+1})A$ is isomorphic to

$$(tG + wG + (m_T^\infty)^p)/(tG + G^*m_{p+1}(tG + wG + (m_T^\infty)^p)) \ .$$

Since the denominator contains $(G^*m_{p+1})^p$ and $tG + (G^*m_{p+1})(m_T^\infty)^p$, which contains $(m_T^\infty)^p$, the numerator is spanned by the denominator and the constant vector fields, proving (3.9).

Next we check that $(tF + wF)/tG$ is an $(F, G)^*E_{2p+2}$ module. Pick h in E_{2p+2}. Then $l = h(F, G) - h(F, F)$ is in m_T^∞. Since $tF + wF$ contains $(m_T^\infty)^p$,

$$h(F, G)(tF + wF) = h(F, F)(tF + wF) + l(tF + wF) \subset tF + wF \ .$$

tG is an E_{n+1} module, hence an $(F, G)^*E_{2p+2}$ module, and is a submodule of $tF + wF$ by (3.5).

Finally, we must prove (3.8). wF is f.g. over F^*E_{p+1}, hence over $(F, G)^*E_{2p+2}$. tF is generated over E_{n+1} by $\partial_i = \partial f/\partial x_i$, $i = 1, \cdots, n$. tG contains JGE_{n+1}^p. Thus

$$(tF + wF)/tG = \overline{wF} + \sum \bar\partial_i(E_{n+1}/JG) \ ,$$

where the bar denotes the projection into the quotient space. Since we assume f is finitely K-determined, so are F and G. Thus E_{n+1}/JG is finite as a G^*E_{p+1} module by the Malgrange Preparation Theorem. Thus A is finitely generated as an $(F, G)^*E_{2p+2}$ module.

This completes the proof that equality holds in (3.3) and hence completes the proof of Theorem 1.4.   □

4. Proof of Theorem 1.5. The proof that "(3) implies (2)" in Theorem 1.1 depends essentially on the coherence theorem for pushforward sheaves. Merrien in [9] (summarized in [7] and [8]) has developed a notion of semicoherence which plays a role for real analytic maps and semianalytic sets analogous to that of coherence for complex analytic maps and varieties. The two theorems below were communicated to me by Merrien. They generalize Theorems II. 7.1 and IV. 7.3 of [9], and are proved in the same way. Let $\mathcal{O}$ and $\mathcal{E}$ be the sheaves of real analytic and C^∞ functions on R^n (or R^p; which will be clear from context). For any sheaf $\mathcal{M}$ let $\mathcal{M}_U$ denote its restriction to U and $\mathcal{M}(U)$ the ring of sections into $\mathcal{M}$ over U. If $\mathcal{M} = \mathcal{E}^p/\mathcal{F}$, then $\mathcal{M}(U)$ is given the quotient topology of the compact-open C^∞ topologies, and $\widetilde{\mathcal{M}}(U)$ is, by definition, $\mathcal{E}^p(U)/\overline{\mathcal{F}(U)}$.

THEOREM 4.1. *Let* $f: U \to V$, $U \subset \mathbf{R}^n$, $V \subset \mathbf{R}^p$, *be analytic and let* $\mathscr{M}$ *be an analytic semicoherent sheaf on* U *such that*

(i) $f \mid \operatorname{Supp} \mathscr{M}: \operatorname{Supp} \mathscr{M} \to V$ *is proper*,

(ii) *for all* $x \in U$, $\mathscr{M}_x$ *is a finite* $\mathcal{O}_{f(x)}$ *module. Then* $f_*(\mathscr{M})$ *is semicoherent over* V.

THEOREM 4.2. *Let* f *and* $\mathscr{M}$ *be as above. Suppose* $\mathscr{M}'$ *is a locally finitely generated sheaf on* V *and* $\tau: \mathscr{M}' \to f_*(\mathscr{M})$ *is a morphism;* $\tilde{\tau}: \mathscr{M}' \widetilde{\otimes}_{\mathcal{O}_V} \mathscr{E}_V \to f_*(\mathscr{M}) \widetilde{\otimes}_{\mathcal{O}_V} \mathscr{E}_V$ *is the induced map. Then* $\operatorname{Im} \tilde{\tau}(V)$ *is closed in* $f_*(\mathscr{M}) \widetilde{\otimes}_{\mathcal{O}_V} \mathscr{E}_V(V) = f_*(\mathscr{M} \widetilde{\otimes}_{\mathcal{O}_U} \mathscr{E}_U)(V) = (\mathscr{M} \widetilde{\otimes}_{\mathcal{O}_U} \mathscr{E}_U)(U)$ *(all three of which have the same topology).*

Now let $f_0: \mathbf{R}^n, 0 \to \mathbf{R}^p, 0$ be analytic and finitely K-determined. Then there are neighborhoods U and V of 0 in $\mathbf{R}^n$ and $\mathbf{R}^p$ and a representative $f: U \to V$ of f_0 such that f and $\mathscr{M} = \mathcal{O}_U^p / df \mathcal{O}_U^n$ satisfy the hypotheses of Theorem 4.1. Let $\mathscr{M}' = \mathcal{O}_V^p$ and $\tau: \mathscr{M}' \to f_* \mathscr{M}$ be the map induced by $f^*: \mathcal{O}_V^p \to \mathcal{O}_U^p$. Then $\operatorname{Im} \tilde{\tau}(V)$ is the projection of $f^* \mathscr{E}^p(V)$ into $\mathscr{E}^p(U)/df\mathscr{E}^n(U)$ (note that $df\mathscr{E}^n(U)$ is closed in $\mathscr{E}^p(U)$ by VI. 1.5 of [10]). Theorem 4.2 implies that

(4.3) $df\mathscr{E}^n(U) + f^*\mathscr{E}^p(V)$ is closed in $\mathscr{E}^p(U)$.

Now assume that f_0 in addition satisfies condition (3a) (as a germ). Then f, U, and V of the above paragraph can be chosen so that, for any finite $S \subset U - \{0\}$, the germ of f at S is stable. Let $\mathscr{E}^p(U)_* = \{g \in \mathscr{E}^p(U): g \equiv 0 \text{ in a neighborhood of } 0\}$. Then $\mathscr{E}^p(U)_* \subset df\mathscr{E}^n(U) + f^*\mathscr{E}^p(V)$; this is proved by a partition of unity argument just like that on pp. 316–317 of [5]. Clearly $\mathscr{E}^p(U)_*$ is dense in the set of mappings infinitely flat at 0. Thus the latter are all in $df\mathscr{E}^n(U) + f^*\mathscr{E}^p(V)$. Passing to germs, $m_n^\infty E_n^p \subset df_0 E_n^n + f_0^* E_p^p$.   □

REFERENCES

1. G. R. Belickii, *Germs of mappings ω-determined with respect to a given group*, Mat. Sb., **94** (136) (1974), 452-467=Math. USSR-Sb., **23** (1974), 425-440.

2. T. Gaffney, *Properties of Finitely Determined Germs*, Thesis, Brandeis University, 1975.

3. M. Golubitsky and V. Guillemin, *Stable Mappings and Their Singularities*, Graduate Texts in Mathematics 14, Springer-Verlag, New York, 1973.

4. J. N. Mather, *Stability of* C^∞ *mappings, III: finitely-determined map germs*, Publ. Math. I. H. E. S., **35** (1968), 127-156.

5. ———, *Stability of* C^∞ *mappings, V: transversality*, Adv. in Math., **4** (1970), 301-336.

6. ———, *Infinite dimensional group actions*, Asterisque (Soc. Math. France), **32-33** (1976), 165-172.

7. J. Merrien, *Faisceaux analytiques semi-coherents*, C. R. Acad. Sci. Paris, **289** (1979), 791–793.

8. ———, *Applications des faisceaux analytiques semi-coherent aux fonctions differentiables*, C. R. Acad. Sci. Paris, **290** (1980), 37–39.

9. ———, *Faisceaux analytiques semi-coherents et fonctions differentiables*, These, Universite de Rennes, 1980.

10. J. C. Tougeron, *Ideaux de Fonctions Differentiables*, Ergebnisse Band 71, Springer-Verlag, New York, 1972.

11. L. C. Wilson, *Infinitely determined mapgerms*, Canad. J. Math., to appear.

12. ———, *A-codimension of mappings*, in preparation.

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