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EXPLICIT FORMULAE FOR A CLASS OF DIRICHLET SERIES

DON REDMOND

In this paper we shall prove explicit formulae for Dirichlet series satisfying functional equations involving multiple gamma factors. We shall illustrate the general theorem by giving a generalization of the von Mangoldt formula and by proving the nonvanishing on the line of absolute convergence for a subclass of the Dirichlet series considered in the main theorem.

1. Introduction. Explicit formulae have been around nearly as long as Dirichlet series. In 1895 C.J. de la Vallée Poussin [19] proved an explicit formula for the Riemann zeta function and from it deduced the prime number theorem with a good remainder term. Later A. Weil [20] proved general explicit formulae for the zeta function of Hecke with Grössencharakteren and used these formulae to study the distribution of the zeros of the Hecke zeta function. (See also S. Lang [13], Chapter 17.) More recently A. M. Odlyzko [15], [16] has used explicit formulae for the Dedekind zeta function to get lower bounds for the discriminants of the associated algebraic number fields. (See also G. Poitou [17].) C. J. Moreno, in [14], especially § 6, derives explicit formulae for automorphic forms which he uses to study the distribution of the zeros of these forms and the constant term of the associated Eisentein series. Finally, H. -J. Besenfelder [3] derived explicit formulae for tempered distributions in connection with the Riemann zeta function and later [4] used them to give a short proof of the nonvanishing of the Riemann zeta function on the line $\mathrm{Re}(s) = 1$ that uses no properties of the Riemann zeta function to the right of the line $\mathrm{Re}(s) = 1$, as is the case for most proofs (see, for example, [18], § 3.2).

In this paper we shall prove explicit formulae for Dirichlet series satisfying functional equations involving multiple gamma factors. Since we are not trying to get the most general theorems, some of the assumptions that we make are not essential to the argument, but are there to simplify the details. One could tackle more general cases with minor modifications in the proof. After proving the general explicit formula we shall give two examples that allow us to prove a generalization of the von Mangoldt formula and the nonvanishing on the line of absolute convergence for a subclass of the Dirichlet series here considered.

In the sequel we write the complex variable $s = \sigma + it$, where both σ and t are real. We denote by

$$\int_{(a,T)} \text{ the integral } \int_{a-iT}^{a+iT}$$

and by

$$\int_{(a)} \text{ the integral } \int_{a-i\infty}^{a+i\infty} .$$

Finally the letters c_j, $j = 1, 2, \cdots$, will denote positive absolute constants.

2. The statement of the main result.

Let

$$f(s) = \sum_{n=1}^{\infty} a(n)n^{-s} \text{ and } g(s) = \sum_{n=1}^{\infty} b(n)n^{-s}$$

be two Dirichlet series that converge absolutely for $\sigma > r$, where $r > 0$. We assume that

$$a(1)b(1) \neq 0 . \tag{2.1}$$

Let

$$\Delta(s) = \prod_{k=1}^{N} \Gamma(\alpha_k s + \beta_k) , \tag{2.2}$$

where, for $1 \leqq k \leqq N$, we have $\alpha_k > 0$ and $\beta_k = \mu_k + i\nu_k$ complex. Assume that $\Delta(s)$ has no poles in $\sigma \geqq r/2$. Suppose that there exist constants $C > 0$, Θ real and δ complex such that

$$f(s)\Delta(s) = C^{\,s+\delta}\Delta(r-s)g(r-s) . \tag{2.3}$$

We assume that $f(s)$ has at most a finite number of poles in the strip $0 < \sigma \leqq r$, say R of them, and that all nontrivial zeros of $f(s)$ (i.e., those zeros that do not arise from the cancelling poles of $\Delta(s)$) lie in the strip $0 \leqq \sigma \leqq r$. We will denote a pole by w_j and its multiplicity by δ_j, with $1 \leqq j \leqq R$. The nontrivial zeros of $f(s)$ we will denote by $\rho = \beta + i\gamma$.

If $a^{*-1}(n)$ and $b^{*-1}(n)$ denote the Dirichlet convolution inverses of $a(n)$ and $b(n)$, respectively, (which are well-defined due to assumption (2.1)) let

$$\Lambda_f(n) = \sum_{d|n} a(d)a^{*-1}(n/d) \log d \text{ and } \Lambda_g(n) = \sum_{d|n} b(d)b^{*-1}(n/d) \log d . \tag{2.4}$$

For $1 \leqq k \leqq N$ we define

$$m(k) = \max \left\{ m \text{ a nonnegative integer: } 0 \leqq -\frac{m + \operatorname{Re}(\beta_k)}{\alpha_k} \leqq r \right\} . \tag{2.5}$$

Let $l(x)$ be a complex-valued function of a real variable whose

Laplace transform,

$$L(s) = \int_{-\infty}^{+\infty} l(x) e^{(s-r/2)x} dx ,$$

exists and is analytic on $-a \leqq \sigma \leqq r + a$ for some $a > 0$. Suppose that $l(x)$ satisfies the following two conditions.

(1) l is continuous and continuously differentiable, except possibly at a finite number of points where the functions have jump discontinuities and are then defined to take on the mean value.

(2) There is a constant $b > \max(1, a)$ such that $l(x)$ and $l'(x)$ are both $O(e^{-(r/2+b)|x|})$ as $|x| \to +\infty$.

THEOREM 1. *Let*

$$A = \sum_{k=1}^{N} \alpha_k .$$

Then, if ρ runs over the nontrivial zeros of $f(s)$ we have the following explicit formula:

$$\begin{aligned} \lim_{T\to+\infty} \sum_{|I_m(\rho)|\leq T} L(\rho) &= \sum_{j=1}^{R} \delta_j L(w_j) + \sum_{k=1}^{N} \sum_{m=0}^{m(k)} L\left(-\frac{\beta_k + m}{\alpha_k}\right) \\ &-(\Theta \log C + 2A\gamma) l(0) \\ &- \sum_{n=1}^{\infty} \{\Lambda_f(n) l(\log n) + \Lambda_g(n) l(-\log n)\} n^{-r/2} \\ &+ \sum_{k=1}^{N} \alpha_k \int_{-\infty}^{+\infty} \frac{l(\alpha_k x) e^{-(\alpha_k r/2+\beta_k-1)|x|} - l(0)}{1 - e^{|x|}} dx , \end{aligned} \tag{2.6}$$

where γ denotes Euler's constant.

We could relax the conditions on the Dirichlet series. For example, one could allow r to be nonpositive. This could be done by considering a new function, say $f_1(s)$, given by

$$f_1(s) = f(s - u) ,$$

where $u > r$. Then $f_1(s)$ is easily seen to be absolutely convergent for $\sigma > u - r$ if $f(s)$ converges absolutely for $\sigma > r$. If we defined

$$\Delta_1(s) = \Delta(s - u) ,$$

then we have the functional equation

$$\Delta_1(s) f_1(s) = C^{\Theta s+\delta-\Theta u} \Delta_1(u - r - s) g(u - r - s) .$$

We would then work with f_1 and Δ_1 below. This brings in added complications and so we will not pursue the matter.

One could also relax the restriction that all the nontrivial zeros of $f(s)$ lie in the strip $0 \leqq \sigma \leqq r$. By a theorem of Berndt [2], Theorem 1, we know that there are numbers σ_1 and σ_2 such that all the nontrivial zeros of $f(s)$ lie on the strip $\sigma_1 \leqq \sigma \leqq \sigma_2$. Thus one could rearrange the proof to take into account nontrivial zeros outside the strip $0 \leqq \sigma \leqq r$.

Finally, one could relax the condition that the abcissa of absolute convergence of $f(s)$ and $g(s)$ is $\leqq r$. Again, this would only increase the details of the proof, but would add no additional complications.

The proof of the main theorem is similar to the proof of the explicit formulae for the Dedekind zeta function given in Lang [13], with additions from Besenfelder [3] and Poitou [17].

3. Preliminary results. If

$$P(s) = \prod_{k=1}^{N} \alpha_k \prod_{m=0}^{m(k)} \left(s + \frac{\beta_k + m}{\alpha_k}\right) \prod_{j=1}^{R} (s - w_j)^{\delta j} ,$$

then $P(s)\Delta(s)f(s)$ is holomorphic on $0 \leqq \sigma \leqq r$. Similarly, if

$$Q(s) = \prod_{k=1}^{N} \alpha_k \prod_{m=0}^{m(k)} \left(r - s + \frac{\beta_k + m}{\alpha_k}\right) \prod_{j=1}^{R} (r - s - w_j)^{\delta j} ,$$

then

$$Q(s) = P(r - s) \tag{3.1}$$

and so $Q(s)\Delta(s)g(s)$ is holomorphic on $0 \leqq \sigma \leqq r$. Let

$$T(s) = \begin{cases} P(s) & \text{if } P(s) = P(r - s) \\ Q(s)P(s) & \text{otherwise} . \end{cases}$$

Then, by (3.1),

$$T(s) = T(r - s) . \tag{3.2}$$

Let

$$F(s) = T(s)\Delta(s)f(s) \quad \text{and} \quad G(s) = T(s)C^{\Theta(r-s)+\delta}\Delta(s)g(s) . \tag{3.3}$$

Then, by (3.2) and (3.3),

$$G(r - s) = F(s) . \tag{3.4}$$

LEMMA 1. *Let $c > r$ and $h > 0$. If $-h \leqq \sigma \leqq c$, then as $|t| \rightarrow +\infty$*

$$f(s) \ll C^{\Theta\sigma} |t|^{A(r+2h)(c+h)^{-1}(c-\sigma)} .$$

Proof. If $\sigma \geqq c$, then $f(s)$ is an absolutely convergent Dirichlet series and so bounded. Similarly for $\sigma \leqq -h$ $g(r-s)$ is bounded. Thus for $\sigma = c$ we have

$$f(\sigma + it) \ll t^0$$

and for $\sigma = -h$ we have

$$f(\sigma + it) \ll t^{A(r+2h)} ,$$

by Stirling's formula and the functional equation. The result then follows by a standard Phragman-Lindelöf argument and so completes the proof.

COROLLARY. *For any fixed η we have, on the half-plane $\sigma \geqq \eta$, that $F(s)$ is an entire function of order* 1.

Proof. By Lemma 1 we see that on $\sigma \geqq \eta$

$$f(s) \ll |s|^{c_1} .$$

Since $T(s)$ is a polynomial in s we have that for any s

$$T(s) \ll |s|^{c_2} .$$

By Stirling's formula we have

$$\begin{aligned} \Delta(s) &\sim \prod_{k=1}^{N} e^{\alpha_k |s| \log |s|} \\ &= \exp (A|s| \log |s|) . \end{aligned}$$

Thus, for $\sigma \geqq \eta$,

$$F(s) \ll e^{C_3 |s| \log |s|}$$

and so $F(s)$ is of order 1 on $\sigma \geqq \eta$.

Since $\Delta(s)$ has no poles in $\sigma \geqq r$ and $f(s)$ converges absolutely for $\sigma > r$ we see that $F(s)$ is holomorphic for $\sigma > r$. By the definition of $T(s)$ we see that $F(s)$ is holomorphic for $0 \leqq \sigma \leqq r$. Similarly $G(s)$ is holomorphic for $\sigma \geqq 0$. Then by the functional equation (3.4) we see that $F(s)$ is holomorphic for $\sigma < 0$ and so is an entire function. This completes the proof.

Since $F(s)$ is an entire function of order 1 we may use the Hadamard factorization theorem (see [5], Chapter 11) to write

$$F(s) = e^{as+b} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho} ,$$

where a and b are constants. Now the only zeros of $F(s)$ are the nontrivial zeros of $f(s)$ which lie in the strip $0 \leqq \sigma \leqq r$, since the other zeros of $f(s)$ are cancelled by the poles of $\Delta(s)$. If we let

$$F_1(s) = \frac{F(s)}{T(s)} \quad \text{and} \quad G_1(s) = \frac{G(s)}{T(s)}\,, \tag{3.5}$$

then we have

$$F_1(s) = G_1(r - s)$$

and also that

$$F_1(s) = e^{as+b}T^{-1}(s)\prod_\rho\Big(1 - \frac{s}{\rho}\Big)e^{s/\rho}\,. \tag{3.6}$$

Since $T^{-1}(s)$ is a nonzero rational function we see that in the strip $0 \leqq \sigma \leqq r$ the only zeros of $F_1(s)$ are the nontrivial zeros of $f(s)$.

From (3.5), we have

$$\frac{F_1'}{F_1}(s) = b + \sum_\rho\Big(\frac{1}{s-\rho} + \frac{1}{\rho}\Big) - \frac{T'}{T}(s)\,, \tag{3.7}$$

where

$$\frac{T'}{T}(s) = \sum_{j=1}^{R}\frac{\delta_j}{s - w_j} + \sum_{k=1}^{N}\sum_{m=0}^{m(k)}\Big(s + \frac{m+\beta_k}{\alpha_k}\Big)^{-1}\,.$$

Now the infinite sum on ρ in (3.7) converges absolutely since $F(s)$ is of order 1 and so

$$\sum_\rho \frac{1}{|\rho|^2}$$

is convergent. We also have, by (3.3) and (3.5), that

$$\frac{F_1'}{F_1}(s) = \frac{\Delta'}{\Delta}(s) + \frac{f'}{f}(s)\,. \tag{3.8}$$

LEMMA 2. *There exists an $\alpha > 0$ such that for every integer m, $|m| \geqq 2$, there exists a T_m in the open interval $(m, m+1)$ such that $f(s)$ has no zeros in the region $\{s = \sigma + it\colon |t - T_m| \leqq \alpha\log^{-1}|m|,\ 0 \leqq \sigma \leqq r\}$.*

Proof. Let $N_m = N(m+1) - N(m)$, where $N(T)$ is the number of zeros of $f(s)$ is $|t| \leqq T$. Now divide the interval $(m, m+1)$ on the t-axis into $N_m + 1$ subintervals and extend these across the vertical strip $0 \leqq \sigma \leqq r$. Then at least one of these rectangles contains no zeros in its interior. Let T_m be the midpoint of the corresponding subinterval on the t-axis. By Berndt [2], Theorems 3 and 9, there exist constants c_4 and c_5 such that

$$c_4 \log|m| \leqq N_m \leqq c_5 \log|m| \,.$$

Thus if

$$|t - T_m| \leqq (3N_m)^{-1} \leqq (3c_4 \log|m|)^{-1} \,,$$

then there are no zeros of $f(s)$ in the rectangle

$$\{s = \sigma + it \colon |t - T_m| \leqq (3c_4 \log|m|)^{-1},\ 0 \leqq \sigma \leqq r\} \,.$$

This completes the proof.

LEMMA 3. *Let* $0 < a \leqq 1$ *and let* m *be an integer,* $|m| \geqq 2$. *Let* $s = \sigma_0 + iT_m$, *where* T_m *is as in Lemma* 2 *and* $-a \leqq \sigma_0 \leqq r + a$. *Then as* $m \to \infty$,

$$\left|\frac{f'}{f}(s)\right| \ll \log^2|m| \,.$$

Proof. Let $\rho = \beta + i\gamma$ run through the nontrivial zeros of $f(s)$ in the strip $0 \leqq \sigma \leqq r$. Then by Theorem 5 of [2] we have

$$\frac{f'}{f}(s) = \sum_{|t-\gamma|\leqq 1} \frac{1}{s - \rho} + O(\log|t|) \,, \tag{3.9}$$

uniformly for $-a \leqq \sigma \leqq r + a$, since $0 < a \leqq 1$.

By definition of T_m we have, as $|m| \to +\infty$,

$$\log|T_m| \sim \log|m| \,. \tag{3.10}$$

Thus, by (3.9) and (3.10), we have

$$\begin{aligned}\frac{f'}{f}(\sigma_0 + iT_m) &= \sum_{|\gamma - T_m|\leqq 1} \frac{1}{s-\rho} + O(\log|m|) \\ &= \sum_{\alpha \log^{-1}|m| \leqq |\gamma - T_m| \leqq 1} \frac{1}{s-\rho} + O(\log|m|) \,,\end{aligned}$$

since $f(s)$ has no zeros with $|\gamma - T_m| \leqq \alpha \log^{-1}|m|$, by Lemma 2.

Let δ_0 be the distance from T_m to the nearest zero of $f(s)$ in the strip $-a \leqq \sigma \leqq r + a$. Then, by Lemma 2, $\delta_0 \gg \log^{-1}|m|$. Let $\delta = \min(1, \delta_0)$, Then $\delta^{-1} \ll \log|m|$. Also, since $\delta \leqq 1$ and $0 < a \leqq 1$, we have

$$\begin{aligned}|s - \rho|^2 &= (T_m - \gamma)^2 + (\sigma_0 - \beta)^2 \\ &\geqq \frac{1}{2}\delta^2(a^2 + (T_m - \gamma)^2) \,.\end{aligned}$$

Thus, for the ρ being summed over in (3.11),

$$(3.12)\qquad \frac{1}{|s-\rho|} \leqq \sqrt{\frac{2}{\delta^2(a^2+(T_m-\gamma)^2)}} \leqq \frac{\sqrt{2}}{\delta|T_m-\gamma|} \ll \log^2|m| \ .$$

Since σ_0 is fixed, the sum in (3.11) contains only a finite number of terms. Thus, by (3.11) and (3.12),

$$\left|\frac{f'}{f}(\sigma_0+iT_m)\right| \ll \log^2|m| \ ,$$

which completes the proof.

LEMMA 4. *Let* $0<a\leqq 1$ *and* m *be an integer,* $|m|\geqq 2$. *Let* $s=\sigma_0+iT_m$, *where* T_m *is given in Lemma 2, and* $-a\leqq\sigma_0\leqq r+a$. *Then, as* $|m|\to+\infty$, *we have*

$$(3.13)\qquad \left|\frac{F_1'}{F_1}(s)\right| \ll \log^2|m| \ .$$

Proof. By Stirling's formula (see [1], p. 259, (6.3.18)), we have

$$\frac{\Gamma'}{\Gamma}(z) = \log z + O\left(\frac{1}{|z|}\right)$$

as $|z|\to+\infty$ off the negative x-axis. Thus, for the s in question, we have

$$\left|\frac{\varDelta'}{\varDelta}(s)\right| = \left|\sum_{k=1}^{N}\alpha_k\frac{\Gamma'}{\Gamma}(\alpha_k s+\beta_k)\right| \ll \log|m| \ .$$

The result, (3.13), follows from Lemma 3 and (3.8) and completes the proof.

LEMMA 5. *Let* $\psi(z)$ *denote the digamma function. Let* $M(x)$ *be a continuous function, except for at most a finite number of jump discontinuities, where it takes on the mean value, that satisfies the estimates*

$$M(0)-M(x) \ll |x|/(1+x^2),\ \ as\ \ |x|\to+\infty$$

and

$$\hat{M}(t) = \int_{-\infty}^{+\infty} M(x)e^{itx}dx \ll (1+|t|)^{-1}$$

as $|t|\to+\infty$. *If* $\alpha, r>0$ *and* β *is a complex number such that* $\alpha r+2\operatorname{Re}(\beta)>0$, *then we have*

$$\int_{-\infty}^{+\infty}\psi(\alpha(r/2+it)+\beta)\hat{M}(t)dt$$
$$= 2\pi M(0)\psi(\alpha r/2+\beta) + \frac{2\pi}{\alpha}\int_0^{\infty} e^{-(\alpha r+2\beta)x/2\alpha}\frac{M(0)-M(x)}{1-e^{-x/\alpha}}dx \ .$$

Proof. We prove this only for the case when $M(x)$ is continuous. The more general case can be obtained by approximating by continuous functions, since for any $\varepsilon > 0$ we can find a continuous function $f(x)$ such that $|M(x) - f(x)| < \varepsilon$, uniformly for all x, and $M(x) = f(x)$, except on a finite number of intervals of arbitrarily small total length.

Suppose first that $M(0) = 0$. For $c > 0$ let

$$p(t) = t^{-2}\sin^2 t \text{ and } q(t) = p(tc^{-1}) .$$

Then, [9, p. 19, (8)],

$$\hat{q}(x) = \begin{cases} \pi\left(c - \frac{1}{2}xc^2\right) & |x| < 2c^{-1} \\ 0 & |x| \geqq 2c^{-1} . \end{cases} \tag{3.14}$$

Let

$$H(t) = q(t)\psi(\alpha(r/2 + it) + \beta) .$$

Then, as $c \to +\infty$,

$$\int_{-\infty}^{+\infty} H(t)\hat{M}(t)dt \longrightarrow \int_{-\infty}^{+\infty}\psi(\alpha(r/2 + it) + \beta)\hat{M}(t)dt . \tag{3.15}$$

By Parseval's formula we have

$$\int_{-\infty}^{+\infty} H(t)\hat{M}(t)dt = \int_{-\infty}^{+\infty}\hat{H}(x)M(x)dx . \tag{3.16}$$

If $x \neq 0$, then

$$\begin{aligned}\hat{H}(x) &= \int_{-\infty}^{+\infty}\psi(\alpha(r/2 + it) + \beta)q(t)e^{itx}dt \\ &= c\int_{-\infty}^{+\infty}\psi(\alpha(r/2 + ict) + \beta)p(t)e^{ictx}dt .\end{aligned}$$

Since $\psi(\alpha(r/2 + it) + \beta)p(t) \ll t^{-2}\log|t|$, as $|t| \to +\infty$, we have

$$\hat{H}(x) \ll \frac{\log c}{|x|} , \tag{3.17}$$

since $p^1(t) \ll t^{-3}$ and $\psi^1(z) \ll |z|^{-1}$ as $|z| \to +\infty$ (see [1, p. 260, 6.4.12]).

Now, for $\mathrm{Re}(z) > 0$, we have (see [1, p. 259, 6.3.21])

$$\psi(z) = \int_0^{+\infty}\left[\frac{e^{-u}}{u} - \frac{e^{-zu}}{1 - e^{-u}}\right]du .$$

Since $\mathrm{Re}(\alpha(r/2 + itc) + \beta) = \alpha r/2 + \mathrm{Re}(\beta) > 0$, we have

$$\psi(\alpha(r/2 + ict) + \beta) = \int_0^{+\infty}\left[\frac{e^{-u}}{u} - \frac{e^{-(\alpha(r/2+ict)+\beta)u}}{1 - e^{-u}}\right]du .$$

Thus, for $|x| > 10c^{-1}$, we have, since $\hat{q}(x) = 0$ for $|x| > 10c^{-1}$,

$$\hat{H}(x) = \int_{-\infty}^{+\infty}\left\{\int_0^{+\infty}\left[\frac{e^{-u}}{u} - \frac{e^{-(\alpha(r/2+it)+\beta)u}}{1-e^{-u}}\right]du\right\}q(t)e^{ixt}dt$$

$$= \begin{cases} -\Pi\int_{(x-2c^{-1})/\alpha}^{(x+2c^{-1})/\alpha}\frac{e^{-(\alpha r/2+\beta)u}}{1-e^{-u}}\left(c - \frac{x-\alpha u}{2}c^2\right)du & \text{if } x > \frac{10}{c} \\ 0 & \text{if } x < \frac{-10}{c}, \end{cases}$$

by (3.14). Thus, as $c \to +\infty$,

$$\hat{H}(x) = -\frac{2\pi}{\alpha}\cdot\frac{e^{-(\alpha r/2+\beta)x/\alpha}}{1-e^{-x/\alpha}} + 0\left\{\frac{e^{-(\alpha r/2+\mathrm{Re}(\beta))x/\alpha}}{xc}\right\}. \tag{3.18}$$

If we combine (3.17) and (3.18), we have

$$\int_{-\infty}^{+\infty}\hat{H}(x)M(x)dx = -\frac{\pi}{\alpha}\int_0^{+\infty}\frac{e^{-(\alpha r/2+\beta)x/\alpha}}{1-e^{-x/\alpha}}M(x)dx + O\left(\frac{\log c}{c}\right),$$

as $c \to +\infty$. Letting $c \to +\infty$ the result follows by (3.15) and (3.16).

Now suppose that $M(0) \neq 0$. Then in the above argument we replace $M(x)$ by the function

$$M(0)e^{-x^2/B} - M(x),$$

where $B > 0$. Since the Fourier transform of $e^{-x^2/B}$ is $\sqrt{\pi B}e^{-Bt^2/4}$, which approximates the delta function, we have, as $B \to +\infty$

$$\int_{-\infty}^{+\infty}\psi(\alpha(r/2+it)+\beta)\sqrt{\pi B}e^{-Bt^2/4}dt \longrightarrow 2\pi\psi(\alpha r/2+\beta).$$

This implies the result of the lemma and completes the proof.

The proof of this lemma is adapted from the proof of Odlyzko for the case $\alpha = r = 1$, $\beta = 0$. The result is also stated for $\alpha = 1|2$, $r = 1$, $\beta = 0$.

In order to get the result in the form that we shall need later we use the following identity [1, p. 259, 6.3.22]:

$$\psi(z) + \gamma = \int_0^{+\infty}\frac{e^{-t} - e^{-zt}}{1-e^{-t}}dt. \tag{3.19}$$

In Lemma 5 let $z = (1/2)\alpha r + \beta$. We have

$$\frac{2\pi}{\alpha}\int_0^{+\infty}\frac{e^{-zx/\alpha}(M(0)-M(x))}{1-e^{-x/\alpha}}dx = 2\pi\int_0^{+\infty}\frac{e^{-zx}(M(0)-M(\alpha x))}{1-e^{-x}}dx.$$

Also

$$-\int_0^{+\infty}\frac{e^{-(z-1)x}M(\alpha x) - M(0)}{1-e^{x}}dx = \int_0^{+\infty}\frac{e^{-zx}M(\alpha x) - M(0)e^{-x}}{1-e^{-x}}dx. \tag{3.20}$$

Thus, by (3.19) and (3.20),

$$2\pi\int_0^{+\infty}\frac{e^{-zx}(M(0)-M(\alpha x))}{1-e^{-x}}dx - 2\pi\int_0^{+\infty}\frac{e^{-(z-1)x}M(\alpha x)-M(0)}{1-e^{x}}dx$$
$$= 2\pi M(0)\int_0^{+\infty}\frac{e^{-zx}-e^{-x}}{1-e^{-x}}dx$$
$$= -2\pi M(0)(\psi(z)+\gamma)\ .$$

We rewrite this result as

$$2\pi M(0)\psi(\alpha r/2+\beta) + \frac{2\pi}{\alpha}\int_0^{+\infty}\frac{e^{-(\alpha r/2+\beta)x/\alpha}(M(0)-M(x))}{1-e^{-x/\alpha}}dx$$
$$= -2\pi M(0)\gamma + 2\pi\int_0^{+\infty}\frac{e^{-(\alpha r/2+\beta-1)x}M(\alpha x)-M(0)}{1-e^{x}}dx\ .$$

This gives the following corollary to Lemma 5.

COROLLARY 2. *Under the hypotheses of Lemma* 5 *we have*

$$\int_{-\infty}^{+\infty}\psi(\alpha(r/2+it)+\beta)\hat{M}(t)dt = -2\pi M(0)\gamma$$
$$+ 2\pi\int_0^{+\infty}\frac{e^{-(\alpha r/2+\beta-1)x}M(\alpha x)-M(0)}{1-e^{x}}dx\ .$$

4. Proof of Theorem 1. By the assumption that

(4.1) $$l(x) \ll e^{-(r/2+b)|x|},\ \text{as } |x| \longrightarrow +\infty\ ,$$

we see that uniformly for $-a \leqq \sigma \leqq r+a$, $a<b$, we have

(4.2) $$L(\sigma+it) \ll |t|^{-1}\ ,$$

as $|t| \to +\infty$.

Let $T>2$. Then the number of zeros whose imaginary part is between T and the nearest T_m of Lemma 2 is $\ll \log T$. Also the number of zeros whose imaginary part is between $-T$ and the nearest T_l of Lemma 2 is $\ll \log T$. Thus the sum $\sum L(\rho)$ over those zeros tends to zero as $T\to+\infty$, since the sum is $\ll T^{-1}\log T$.

Let C be the rectangle with vertices $-a+iT_m$, $-a+iT_l$, $r+a+iT_m$ and $r+a+iT_l$. Then, by the residue theorem, for T sufficiently large,

$$\sum_{|\gamma|<T} L(\rho) - \sum_{j=1}^{B}\delta_j L(w_j) - \sum_{k=1}^{N}\sum_{m=0}^{m(k)} L\left(-\frac{m+\beta_k}{\alpha_k}\right)$$
$$= \frac{1}{2\pi i}\int_C \frac{F_1'}{F_1}(s)L(s)ds\ .$$

By Lemma 4 and (4.2), we have, as $T\to+\infty$,

$$\int_{-a+iT_l}^{r+a+iT_l} \frac{F_1'}{F_1}(s)L(s)ds \ll T^{-1}\log^2 T \tag{4.3}$$

and

$$\int_{r+a+iT_m}^{-a+iT_m} \frac{F_1'}{F_1}(s)L(s)ds \ll T^{-1}\log^2 T\,. \tag{4.4}$$

Thus, by (4.3) and (4.4),

$$\begin{aligned}\sum_{|\gamma|<T} L(\rho) - \sum_{j=1}^{R}\delta_j L(w_j) - \sum_{k=1}^{N}\sum_{m=0}^{m(k)} L\left(-\frac{\beta_k+m}{\alpha_k}\right) \\ = \frac{1}{2\pi i}\left\{\int_{r+a+iT_l}^{r+a+iT_m} - \int_{-a+iT_l}^{-a+iT_m}\right\}\frac{F_1'}{F_1}(s)L(s)ds + O(T^{-1}\log^2 T)\,,\end{aligned} \tag{4.5}$$

as $T \to +\infty$.

Since $F_1(s) = G_1(r-s)$ we have

$$\frac{F_1'}{F_1}(s) = -\frac{G_1'}{G_1}(r-s)\,.$$

Thus we may rewrite the right hand side of (4.5) as

$$\frac{1}{2\pi i}\left\{\int_{r+a+iT_l}^{r+a+iT_m} \frac{F_1'}{F_1}(s)L(s)ds + \int_{-a+iT_l}^{-a+iT_m}\frac{G_1'}{G_1}(r-s)L(s)ds\right\} + O(T^{-1}\log^2 T)\,.$$

As we observed above the sums over $L(\rho)$ over γ between T_m and T and T_l and $-T$ are $O(T^{-1}\log T)$. Thus we may shift the lines of integration to T and $-T$, since, as in (4.3) and (4.4), the integrals along the horizontals are $O(T^{-1}\log^2 T)$. Thus (4.5) takes the form

$$\begin{aligned}\sum_{|\gamma|<T} L(\rho) - \sum_{j=1}^{R}\delta_j L(w_j) - \sum_{k=1}^{N}\sum_{m=0}^{m(k)} L\left(-\frac{\beta_k+m}{\alpha_k}\right) \\ = \frac{1}{2\pi i}\int_{(r+a,T)} \frac{F_1'}{F_1}(s)L(s)ds + \frac{1}{2\pi i}\int_{(-a,T)}\frac{G_1'}{G_1}(r-s)L(s)ds \\ + O(T^{-1}\log^2 T)\,.\end{aligned} \tag{4.6}$$

By (3.3) and (3.5), we have

$$\frac{F_1'}{F_1}(s) = \frac{\Delta'}{\Delta}(s) + \frac{f'}{f}(s)$$

and

$$\frac{G_1'}{G_1}(r-s) = -\Theta\log C + \frac{\Delta'}{\Delta}(r-s) + \frac{g'}{g}(r-s)\,.$$

Thus

$$\begin{aligned}&\frac{1}{2\pi i}\int_{(r+a,T)}\frac{F_1'}{F_1}(s)L(s)ds\\ (4.7)\qquad &=\frac{1}{2\pi i}\int_{(r+a,T)}\left\{\frac{\Delta'}{\Delta}(s)+\frac{f'}{f}(s)\right\}L(s)ds\\ &=I_1+I_2\,,\end{aligned}$$

say, and

$$\begin{aligned}&\frac{1}{2\pi i}\int_{(-a,T)}\frac{G_1'}{G_1}(r-s)L(s)ds\\ (4.8)\qquad &=\frac{1}{2\pi i}\int_{(-a,T)}\left\{-\Theta\log C+\frac{\Delta'}{\Delta}(r-s)+\frac{g'}{g}(r-s)\right\}L(s)ds\\ &=-I_3+I_4+I_5\,,\end{aligned}$$

say.

We have

$$I_3=(\Theta\log C)\frac{1}{2\pi i}\int_{(-a,T)}L(s)ds\,.$$

Since $L(s)$ is holomorphic on $\sigma=-a$ we can move the line of integration to $\sigma=r/2$. Thus

$$I_3=\frac{\Theta\log C}{2\pi i}\int_{-T}^{T}L(r/2+it)dt\,.$$

Since $L(r/2+it)$ is the Fourier transform of $l(x)$, we have, by the Fourier inversion formula, that, as $T\to+\infty$,

$$(4.9)\qquad I_3\longrightarrow(\Theta\log C)l(0)\,.$$

We have

$$\frac{f'}{f}(s)=-\sum_{n=1}^{+\infty}\Lambda_f(n)n^{-s}\,.$$

Thus, since $f(s)$ converges absolutely for $\sigma>r$,

$$\begin{aligned}\frac{1}{2\pi i}\int_{(r+a,T)}\frac{f'}{f}(s)L(s)ds&=-\sum_{n=1}^{+\infty}\Lambda_f(n)\frac{1}{2\pi i}\int_{(r+a,T)}n^{-s}L(s)ds\\ &=-\frac{1}{2\pi}\int_{-T}^{T}\int_{-\infty}^{+\infty}\sum_{n=1}^{+\infty}\Lambda_f(n)n^{-r/2}l(u+\log n)e^{(r/2+a)u}e^{itu}dudt\\ &=-\frac{1}{2\pi}\int_{-T}^{T}\int_{-\infty}^{+\infty}\sum_{n=1}^{+\infty}H_{n,f}(u)e^{itu}dudt\,,\end{aligned}$$

say, where

$$H_{n,f}(u)=\Lambda_f(n)n^{-r/2}l(u+\log n)e^{(r/2+a)u}\,.$$

By (4.1), we have

$$|H_{n,f}(u)| \ll |\Lambda_f(n)| n^{-(r+b)} .$$

Thus $\sum_{n=1}^{+\infty} H_{n,f}(u)$ is absolutely and uniformly convergent and defines an L^2 function $H_f(u)$. In a similar way we have that

$$\frac{1}{2\pi i}\int_{(-a,T)} \frac{g'}{g}(r-s)L(s)ds = \frac{-1}{2\pi}\int_{-T}^{T}\int_{-\infty}^{+\infty} H_{n,g}(u)e^{itu}dudt ,$$

where $H_{n,g}(u) = \Lambda_g(n)n^{-r/2}l(u - \log n)e^{-(r/2+a)u}$ and satisfies the estimate $|H_{n,g}(u)| \ll |\Lambda_g(n)| n^{-(r+b)}$. Thus $\sum_{n=1}^{+\infty} H_{n,g}(u)$ is absolutely and uniformly convergent and defines an L^2 function $H_g(u)$. If we let

$$H(u) = H_f(u) + H_g(u) .$$

Then we have

$$I_2 + I_5 = \frac{-1}{2\pi}\int_{-T}^{T}\int_{-\infty}^{+\infty} H(u)e^{itu}dudt .$$

Also we see that $H(u)$ is continuous and differentiable since l is, except at the points $\pm \log n$, where $\Lambda_f(n) \neq 0$ or $\Lambda_g(n) \neq 0$. At such points it only has jump discontinuities. Thus, by the Fourier inversion formula, we have, as $T \to +\infty$,

$$I_2 + I_5 \longrightarrow -H(0) , \tag{4.10}$$

where

$$H(0) = \sum_{n=1}^{+\infty} \{\Lambda_f(n)l(\log n) + \Lambda_g(n)l(-\log n)\}n^{-r/2} . \tag{4.11}$$

By assumption we see that $\Delta'/\Delta(s)$ has no poles for $\sigma > r/2$ and is $O(\log |t|)$ in any vertical strip outside of neighborhoods of its poles. In I_1 and I_4 we move the line of integration to the segment $\sigma = r/2$, $|t| \leqq T$. Then, by (4.2),

$$\int_{(r+a,T)} \frac{\Delta'}{\Delta}(s)L(s)ds = \int_{(r/2,T)} \frac{\Delta'}{\Delta}(s)L(s)ds + O(T^{-1}\log T)$$

and

$$\int_{(-a,T)} \frac{\Delta'}{\Delta}(r-s)L(s)ds = \int_{(r/2,T)} \frac{\Delta'}{\Delta}(r-s)L(s)ds + O(T^{-1}\log T) .$$

Thus

$$\begin{aligned} I_1 + I_4 &= \frac{1}{2\pi i}\int_{(r/2,T)} \left\{\frac{\Delta'}{\Delta}(s) + \frac{\Delta'}{\Delta}(r-s)\right\}L(s)ds + O(T^{-1}\log T) \\ &= \frac{1}{2\pi i}\int_{(r/2,T)} \sum_{k=1}^{N} \alpha_k \left\{\frac{\Gamma'}{\Gamma}(\alpha_k s + \beta_k) + \frac{\Gamma'}{\Gamma}(\alpha_k(r-s) + \beta_k)\right\}L(s)ds \\ &\quad + O(T^{-1}\log T) \end{aligned}$$

$$= \sum_{k=1}^{N} \alpha_k J_k + O(T^{-1} \log T) ,$$

where

$$J_k = \frac{1}{2\pi i}\int_{(r/2,T)} \left\{\frac{\Gamma'}{\Gamma}(\alpha_k s + \beta_k) + \frac{\Gamma'}{\Gamma}(\alpha_k(r-s) + \beta_k)\right\} L(s)ds .$$

As $T \to +\infty$, we have, by Proposition 4 of Lang [13, p. 347],

$$\begin{aligned} J_k \longrightarrow \frac{1}{2\pi}\int_{-\infty}^{+\infty}\Big\{&\frac{\Gamma'}{\Gamma}(\alpha_k r/2 + \beta_k + \alpha_k it) \\ &+ \frac{\Gamma'}{\Gamma}(\alpha_k r/2 + \beta_k - \alpha_k it)\Big\} L(r/2 + it)dt . \end{aligned} \tag{4.12}$$

Since $L(r/2 + it)$ is the Fourier transform of $l(x)$ and $\Delta(s)$ is assumed to have no poles in the half plane $\sigma > r/2$, which is equivalent to saying that $\alpha_k r/2 + \mu_k > 0$ for $1 \leqq k \leqq N$, we see that the hypotheses of Corollary 2 are satisfied. Thus the right hand side of (4.12) is

$$\begin{aligned} &-l(0)\gamma + \int_0^{+\infty} \frac{e^{-(\alpha_k r/2+\beta_k-1)x} l(\alpha_k x) - l(0)}{1 - e^x} dx \\ &-l(0)\gamma + \int_0^{+\infty} \frac{e^{-(\alpha_k r/2+\beta_k-1)x} l(-\alpha_k x) - l(0)}{1 - e^x} dx \\ &= -2l(0)\gamma + \int_{-\infty}^{+\infty} \frac{e^{-(\alpha_k r/2+\beta_k-1)|x|} l(\alpha_k x) - l(0)}{1 - e^{|x|}} dx . \end{aligned} \tag{4.13}$$

By (4.1) we see that this infinite integral is convergent.

Thus combining (4.6), (4.9), (4.10) and (4.13), we have the explicit formula (2.6) and complete the proof of Theorem 1.

5. A generalization of the von Mangoldt formula. In this section we give as an example of Theorem 1 a result that relates the analytic behavior of f to the summatory function of the associated von Mangoldt function Λ_f.

Let $y > 1$ and

$$l(x) = \begin{cases} e^{rx/2}, & \text{if } 0 < x < \log y \\ \frac{1}{2}, & \text{if } x = 0 \\ \frac{1}{2}\sqrt{y}, & \text{if } x = \log y \\ 0 & \text{else} . \end{cases}$$

Then it is easy to see that

$$L(s) = \begin{cases} \dfrac{y^s - 1}{s}, & \text{if } s \neq 0 \\ 0, & \text{if } s = 0 . \end{cases}$$

We have

$$\sum_{n=1}^{+\infty} \{\Lambda_f(n) l(\log n) + \Lambda_g(n) l(-\log n)\} n^{-r/2} = \sum_{n \leqq y}' \Lambda_f(n) , \tag{5.1}$$

where the' indicates that if $n = y$ we add only $(1/2)\Lambda_f(n)$, if we note that by (2.4) $\Lambda_g(1) = 0$.

We have

$$\begin{aligned} &\sum_{j=1}^{R} \delta_j L(w_j) + \sum_{k=1}^{N} \sum_{m=0}^{m(k)} L\left(-\frac{m + \beta_k}{\alpha_k}\right) \\ &= \sum_{j=1}^{R}{}^{*} \frac{\delta_j}{w_j}(y^{w_j} - 1) - \sum_{k=1}^{N}{}^{*} \alpha_k \sum_{m=0}^{m(k)} \frac{y^{-(\beta_k + m)/\alpha_k} - 1}{m + \beta_k} , \end{aligned} \tag{5.2}$$

where the * indicates that if $w_j = 0$, for some j, then the term is $\delta_j \log y$ and if $\beta_k = -m$, for some k, then the term is $\log y$. We have

$$\begin{aligned} &\int_{-\infty}^{+\infty} \frac{e^{-(\alpha_k r/2 + \beta_k - 1)|x|} l(\alpha_k x) - l(0)}{1 - e^{|x|}} dx \\ &= \int_0^{(\log y)/\alpha_k} \frac{e^{(1-\beta_k)x} - 1}{1 - e^x} dx + \log(1 - y^{-1/\alpha_k}) . \end{aligned} \tag{5.3}$$

If we combine (5.1)-(5.3) we have the result of Theorem 2, which is the generalization of the von Mangoldt formula.

THEOREM 2. *With the notation as above, we have*

$$\begin{aligned} \sum_{n \leqq y}' \Lambda_f(n) &= \sum_{j=1}^{R}{}^{*} \frac{\delta_j}{w_j}(y^{w_j} - 1) - \sum_{k=1}^{N}{}^{*} \alpha_k \sum_{m=0}^{m(k)} \frac{y^{-(\beta_k + m)/\alpha_k} - 1}{m + \beta_k} \\ &- \lim_{T \to +\infty} \sum_{|\iota| < T}{}^{*} \frac{y^\rho - 1}{\rho} - \frac{1}{2}(\Theta \log C + 2A\gamma) \\ &+ \sum_{k=1}^{N} \alpha_k \left\{ \log(1 - y^{-1/\alpha_k}) + \int_0^{(\log y)/\alpha_k} \frac{e^{(1-\beta_k)x} - 1}{1 - e^x} dx \right\} . \end{aligned}$$

Knowing more about the location of the zeros of $f(s)$ and the w_j, $1 \leqq j \leqq R$, would maybe allow one to prove a prime number theorem for the coefficients of $f(s)$. In the next section we will prove a theorem relating to the location of the zeros of $f(s)$.

6. The nonvanishing on the line of absolute convergence. In this section we adapt the method of Besenfelder [4] to show the nonvanishing on the line of absolute convergence for a certain subclass of the Dirichlet series being studied in this paper.

Let $y > 0$, t be a real number and

$$l(x) = \frac{1}{2\sqrt{\pi y}} e^{-x^2/4y - itx} .$$

Then the corresponding Laplace transform is easily seen to be

$$L(s) = e^{y(s-r/2-it)^2} .$$

Thus, by Theorem 1, we have

$$\begin{aligned}\lim_{T\to+\infty} \sum_{|Im(\rho)|\leq T} & e^{y(\rho-r/2-it)^2} \\ &= \sum_{j=1}^{R} \delta_j e^{y(wj-r/2-it)^2} + \sum_{k=1}^{N} \sum_{m=0}^{m(k)} e^{y((\beta_k+m)/\alpha_k+r/2+it)^2} \\ &- \frac{1}{2\sqrt{\pi y}} \sum_{n=1}^{\infty} e^{-(\log n)^2/4y} n^{-r/2} \{\Lambda_f(n)n^{-it} + \Lambda_g(n)n^{it}\} \\ &- \frac{\Theta \log C + 2A\gamma}{2\sqrt{\pi y}} + \frac{1}{2\sqrt{\pi y}} \sum_{k=1}^{N} \alpha_k W_k(y) ,\end{aligned}$$

where

$$\begin{aligned} W_k(y) &= \int_{-\infty}^{+\infty} \frac{e^{-(\alpha_k x)^2/4y - it(\alpha_k x) - (\alpha_k r/2+\beta_k-1)|x|} - 1}{1 - e^{|x|}} dx \\ &= 2\int_0^{+\infty} \frac{e^{-\alpha_k^2 x^2/4y - (\alpha_k r/2+\beta_k-1)x} \cos(\alpha_k t x) - 1}{1 - e^x} dx . \end{aligned} \tag{6.2}$$

In [2, Theorem 10] it is shown that there exists a positive constant C such that

$$N(T) \sim CT \log T , \tag{6.3}$$

as $T \to +\infty$. Using (6.3) and writing the sum on the left hand side of (6.1) as a Stieltjes integral one can show that the sum over all ρ is convergent, i.e.,

$$\lim_{T\to\infty} \sum_{|Im\rho|\leq T} e^{y(\rho-r/2-it)^2} = \sum_{\rho} e^{y(\rho-r/2-it)^2} .$$

We are concerned with the behavior of the right hand side of (6.1) as $y \to +\infty$. We have

$$\frac{\Theta \log C + 2A\gamma}{2\sqrt{\pi y}} \ll y^{-1/2} . \tag{6.4}$$

Let $\varepsilon_k = (1/2)\alpha_k r + \beta_k - 1$ and $\eta_k = (1/2)\alpha_k r + \mu_k - 1$, where $\mu_k = \text{Re}(\beta_k)$. Then

$$\frac{1}{2}W_k(y) = \int_0^{+\infty} \frac{1-\cos(\alpha_k tx)}{e^x - 1}dx + \int_0^{+\infty} \frac{\cos(\alpha_k tx)(1 - e^{-\alpha_k^2 x^2/4y - \varepsilon_k x})}{e^x - 1}dx . \tag{6.5}$$

The first integral in (6.5) is convergent and has the value

$$2\{\gamma + \mathrm{Re}\,[\psi(1 + \alpha_k i)]\} .$$

(See [1], p. 78, 4.3.138 and p. 259, 6.3.17.) If $0 < x < 4\eta_k y/\alpha_k^2$, then $-\alpha_k^2 x^2/4y - \eta_k x > 0$ and if $x > 4\eta_k y/\alpha_k^2$, $-\alpha_k^2 x^2/4y - \eta_k x < 0$. Also if $\eta_k > 0$, then $-\alpha_k^2 x^2/4y - \eta_k x < 0$. Let

$$\Theta_k = \max(0,\ 4\eta_k/\alpha_k^2) .$$

Then

$$\left| \int_0^{\Theta_k y} \frac{\cos(\alpha_k tx)(1 - e^{-\alpha_k^2 x^2/4y - \varepsilon_k x})}{e^x - 1}dx \right| \leqq \int_0^{\Theta_k y} e^{-(1+\eta_k)x}dx \ll 1 ,$$

as $y \to +\infty$, if $\eta_k > -1$. Also

$$\left| \int_{\Theta_k y}^{+\infty} \frac{\cos(\alpha_k tx)(1 - e^{-\alpha_k^2 x^2/4y - \varepsilon_k x})}{e^x - 1}dx \right| \leqq \int_{\Theta_k y}^{+\infty} \frac{dx}{e^x - 1} = -\log(1 - e^{-\Theta_k y}) = o(1) ,$$

as $y \to +\infty$. Thus, if $\eta_k > -1$, i.e., $\alpha_k r + 2\mu_k > 0$, then $W_k(y) \ll 1$, as $y \to +\infty$. Thus, as $y \to +\infty$,

$$\frac{1}{2\sqrt{\pi y}} \sum_{k=1}^{N} \alpha_k W_k(y) = O(1) .$$

Let

$$L_n(y, t) = \frac{1}{2\sqrt{\pi y}} n^{-r/2} e^{-(\log^2 n)/4n}(n^{-it}\Lambda_f(n) + n^{it}\Lambda_g(n)) = L_n(y)M_n(t) ,$$

say. Then, if $\alpha_k r + 2\mu_k > 0$, we have, as $y \to +\infty$,

$$\sum_\rho e^{y(\rho - r/2 - it)^2} = \sum_{j=1}^{R} \delta_j e^{y(w_j - r/2 - it)^2} + \sum_{k=1}^{N} \sum_{m=0}^{m(k)} e^{y(r/2 + it + (m+\beta_k)/\alpha_k)^2} - \sum_{n=1}^{\infty} L_n(y)M_n(t) + o(1) . \tag{6.6}$$

Suppose $\rho_0 = r + i\gamma_0$ is a zero of $f(s)$. Suppose $f(s)$ has no zeros in the region

$$U = \{z = u + iv: (u - r/2)^2 \geqq v^2\} .$$

Then $|\gamma_0| > r/2$.

From (6.6), with $t = 0$, we have

$$\sum_{\rho} e^{y(\rho - r/2)^2} = \sum_{\rho} e^{y[(\mathrm{Re}(\rho) - r/2)^2 - (\mathrm{Im}(\rho))^2 + 2(\mathrm{Re}(\rho) - r/2)(\mathrm{Im}(\rho))i]}$$
$$= o(1) ,$$

as $y \to +\infty$, since f has no zeros in the region U. Thus

$$(6.7) \qquad \sum_{m=1}^{\infty} L_m(y, 0) = \sum_{j=1}^{R} \delta_j e^{y(w_j - r/2)^2} + \sum_{k=1}^{N} \sum_{m=0}^{m(k)} e^{y(r/2 + (\beta_k + m)/\alpha_k)^2} + o(1) ,$$

as $y \to +\infty$.

In (6.6) let $t = \gamma_0$ and let ν be the multiplicity of ρ_0. Then ρ_0 contributes a term $\nu e^{yr^2/4}$. Since $r \geqq \mathrm{Re}(\rho)$ for all nontrivial zeros of $f(s)$ and since $f(s)$ has no zeros in U imples $\mathrm{Im}(\rho) \geqq r/2$ the rest of the sum on the left hand side of (6.6) is $o(e^{yr^2/4})$, as $y \to +\infty$. Since the first two sums on the right hand side of (6.6) arise from poles of $f(s)$ and $\Delta(s)$ in the strip $0 \leqq \sigma \leqq r$ and $t = \gamma_0 > r/2$, we see that, as $y \to +\infty$, these terms are $o(e^{yr^2/4})$. Thus (6.6) becomes, in this case,

$$(6.8) \qquad \sum_{m=1}^{\infty} L_m(y, \gamma_0) = -\nu e^{yr^2/4} + o(e^{yr^2/4}) ,$$

as $y \to +\infty$.

We now suppose the coefficients of $f(s)$ are real and that $f(s) = g(s)$. Then $\Lambda_f(n) = \Lambda_g(n)$, by (2.4), and so

$$\begin{aligned} L_m(y, t) &= L_m(y) M_m(t) \\ &= 2L_m(y)\Lambda_f(m) \cos(t \log m) \\ &= N_m(y) \cos(t \log m) , \end{aligned}$$

say. Note that

$$L_m(y, 0) = N_m(y)$$

and

$$L_m(y, \gamma_0) = N_m(y) \cos(\gamma_0 \log m) .$$

By Schwarz's inequality and the double angle formula, we have

$$\left(\sum_{m=1}^{+\infty} L_m(y, \gamma_0)\right)^2 = \left\{\sum_{m=1}^{+\infty} \sqrt{N_m(y)} \sqrt{N_m(y)} \cos(\gamma_0 \log m)\right\}^2$$
$$\leqq \frac{1}{2}\left(\sum_{m=1}^{+\infty} N_m(y)\right)^2 + \frac{1}{2} \sum_{m=1}^{+\infty} N_m(y) \sum_{m=1}^{+\infty} N_m(y) \cos(2\gamma_0 \log m)\right) .$$

Thus

$$(6.9)\quad \left\{\frac{\sum_{m=1}^{+\infty} N_m(y)\cos(\gamma_0 \log m)}{\sum_{m=1}^{+\infty} N_m(y)}\right\}^2 \leqq \frac{1}{2} + \frac{1}{2}\left\{\frac{\sum_{m=1}^{+\infty} N_m(y)\cos(2\gamma_0 \log m)}{\sum_{m=1}^{+\infty} N_m(y)}\right\}.$$

We now apply (6.7) and (6.8). Then, as $y \to +\infty$, the left hand side of (6.9) approaches

$$(6.10)\quad \left\{\frac{-\nu e^{yr^2/4} + o(e^{yr^2/4})}{\sum_{j=1}^{R} \delta_j e^{y(w_j - r/2)^2} + \sum_{k=1}^{N}\sum_{m=0}^{m(k)} e^{y(r/2+(\beta_k+m)/\alpha_k)^2} + o(1)}\right\}^2.$$

If we let $p = \max_{j,k,m}(|w_j - r/2|^2, |r/2 + (m + \beta_k)/\alpha_k|^2)$, then, as $y \to +\infty$; (6.10) approaches

$$(6.11)\quad \frac{-\nu e^{yr^2/4} + o(e^{yr^2/4})^2}{e^{yp} + o(e^{yp})} \longrightarrow \{-\nu e^{y(r^2/4-p)} + o(e^{y(r^2/4-p)})\}^2.$$

Now $p \leqq r^2/4$, since $\mathrm{Re}(w_j) \leqq r$. Thus the right hand side of (6.11) is $\geqq \nu^2$.

On the other hand, the right hand side of (6.9) is $\leqq 1/2$. For if $2\gamma_0$ is an ordinate of a zero of $f(s)$, then, by (6.8), we see that, since $\delta_j > 0, 1 \leqq j \leqq R$, the quotient is negative as $y \to +\infty$. If $2\gamma_0$ is not the ordinate of a zero, then, as $y \to +\infty$, the quotient vanishes, by (6.6) and (6.7), since $2\gamma_0 > r$ and $\delta_j > 0$, $1 \leqq j \leqq R$.

This proves the following theorem.

THEOREM 3. *Suppose $f(s)$ is a Dirichlet series with real coefficients that satisfies the functional equation (2.3) with $f(s) = g(s)$. Suppose that $\alpha_k r + 2\mu_k > 0$, $1 \leqq k \leqq N$, and $f(s)$ has no zeros in the region $U = \{s = \sigma + it = (\sigma - r/2)^2 \geqq t^2\}$. Then $f(s)$ has no zeros on the line $\sigma = r$.*

We illustrate Theorem 3 with some examples.

EXAMPLE 1. The Riemann zeta function. In [18, p. 22] it is shown that $\zeta(s)$ satisfies the functional equation (2.3) with $N = 1$, $\alpha_1 = 1/2$, $\beta_1 = 0$, $C = \pi$, $\Theta = 1$, $\delta = -1/2$ and $r = 1$. Then

$$\alpha_1 r + 2\mu_1 = 1/2 > 0.$$

On p. 330 of [18] it is shown that the zeros closest to the real axis are $1/2 \pm i\gamma$, where γ is approximately 14.13. Thus, by Theorem 3, $\zeta(s)$ has no zeros on the line $\sigma = 1$.

EXAMPLE 2. Hecke zeta functions over the Gaussian field. In

[13, Chapter 14] it is shown that if k is a positive integer and

$$\zeta(s, \lambda^k) = \frac{1}{4} \sum_{\substack{m=-\infty \\ (m,n)\neq(0,0)}}^{+\infty} \sum_{n=-\infty}^{+\infty} \frac{e^{4ik \arg(m+in)}}{(m^2+n^2)^s},$$

then $\zeta(s, \lambda^k)$ satisfies the functional equation (2.3) with $N=1$, $\alpha_1 = 1$, $\beta_1 = 2k$, $C = \pi$, $\Theta = 2$, $\delta = -1$ and $r = 1$. Then

$$\alpha_1 r + 2\mu_1 = 2 + 4k > 0 .$$

From Table 2 on p. 501 of [6] we see that for $1 \leqq k \leqq 10$ all the zeros of $\zeta(s, \lambda^k)$ are outside the region U. Thus, by Theorem 3, $\zeta(s, \lambda^k)$ is nonzero on the line $\sigma = 1$ for $1 \leqq k \leqq 10$.

EXAMPLE 3. L series with real characters. In [5, p. 71] it is shown that χ is a real character modulo k, a positive intger, and

$$L(s, \chi) = \sum_{n=1}^{+\infty} \chi(n) n^{-s},$$

then $L(s, \chi)$ satisfies the functional equation (2.3) with $N=1$, $\alpha_1 = 1/2$, $\beta_1 = (1/2)a$, $C = \pi/k$, $\Theta = 1$, $\delta = -a/2 + (\log \varepsilon)/\log C$ and $r = 1$, where $a = \{\chi(1) - \chi(-1)\}/2$ and $\varepsilon = k^{-1/2} G(\chi)\{(1-a) - ai\}$, with $G(\chi)$ the usual Gaussian sum associated to χ. Then

$$\alpha_1 r + 2\mu_1 = 1/2 > 0 .$$

From [7, Table 2] it is known that $L(s, \chi)$, for $k = 3, 4, 5, 7, 8, 11, 12, 13, 15, 19, 24, 43, 67$, and 163, has no zeros in the region U since the zeros are all of the form $1/2 \pm i\gamma$ with $\gamma > 0$. Thus, by Theorem 3, $L(s, \chi)$ has no zeros on the line $\sigma = 1$ for these k.

EXAMPLE 4. Artin L-functions attached to certain cubic fields. Let $\zeta_{Q(a^{1/3})}(s)$ be the Dedekind zeta function for the field $Q(a^{1/3})$ and let

$$L_a(s) = \zeta_{Q(a_{1/3})}(s)/\zeta(s) .$$

Then it is known [13, p. 254] that $L_a(s)$ satisfies the functional equation (2.3) with $N=1$, $\alpha_1 = 1$, $\beta_1 = 0$, $C = 3\sqrt{3}\,a/2\pi$, $\Theta = -2$, $\delta = 1$ and $r = 1$, then

$$\alpha_1 r + 2\mu_1 = 1 > 0 .$$

From Table 1 of [12] we see that $L_a(s)$ has no zeros in U for $a = 2, 3, 6$ and 12. Thus, by Theorem 3, we know that $L_a(s)$ has no zeros on the line $\sigma = 1$ for $a = 2, 3, 6$ and 12.

REMARKS. (1) This result, in this generality, seems to be new.

(2) One may also prove the same result if the abscissa of absolute convergence is a, where $r/2 \leqq a < r$.

(3) It seems reasonable that this same proof would work, with not too many modifications, for a wider class of Dirichlet series. It should not be necessary to assume that the $a(n)$ are real or that $f(s) = g(s)$.

(4) Finally, it seems quite likely that this proof can be modified to give a zero-free region for $f(s)$, since many of the proofs that yield the nonvanishing of $\zeta(s)$ at $s = 1$ can be amended to produce zero-free regions. See, for example, [18, pp. 40 and 48].

(5) In [10] Jaquet and Shalika show that the L-functions attached to certain representations of $GL(n)$ do not vanish on their line of absolute convergence. Their method of proof is unconditional in that they do not need to check for nonvanishing in the region U. In [11] they use this result to show that certain related L-functions are absolutely convergent in the half plane to the right of this line of absolute convergence and hence nonzero there.

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Southern Illinois University
Carbondale, IL 62901