

LOCALLY INVARIANT TOPOLOGIES ON FREE GROUPS

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In 1948, M. I. Graev proved that the free topological group on a completely regular Hausdorff space is Hausdorff, by showing that the free group admits a certain locally invariant Hausdorff group topology. In 1964, S. Świerczkowski gave a different proof, which also depends on the construction of a locally invariant topology. Yet another such construction follows from work of K. Bicknell and S. A. Morris. Graev's topology has proved to be essential in the investigation of free products of topological groups; Świerczkowski's topology is the key to the work of W. Taylor on varieties and homotopy laws; and Bicknell and Morris extend results of Abels on norms on free topological groups. In this paper, the three topologies are investigated in detail. It is seen that the Graev topology contains the Świerczkowski topology, which in turn contains that of Bicknell and Morris. These containments are shown to be proper in general. It is known that the topology of the free topological group is in general finer than each of these three topologies.

Introduction. If X is a completely regular Hausdorff space, let $F(X)$ denote the free group on the set X . Clearly the finest group topology on $F(X)$ which gives X its original topology must make $F(X)$ the free topological group on X . Because of this, a number of authors have constructed Hausdorff group topologies on $F(X)$ as a means of proving that the free topological group is Hausdorff. Moreover, most other proofs of this fact are easily seen to contain implicitly the construction of some group topology on $F(X)$.

In this paper we shall examine and compare the topologizations of $F(X)$ arising from three such constructions.

The topologies we study will all have the additional property of *local invariance*; that is, they have bases at the identity of sets invariant under inner automorphisms, or, equivalently, they are defined by families of invariant pseudometrics. (A pseudometric ρ on a group is *(two-sided) invariant* if $\rho(axb, ayb) = \rho(x, y)$ for all group elements a, b, x, y .) Such topologies arise naturally in the present context since they necessarily make the group operations continuous.

1. **The topologies and their properties.** We remind the reader that the Graev free topological group on a completely regular Hausdorff pointed space X with basepoint e is a topological group $F(X)$ which, algebraically, is a free group with $X \setminus \{e\}$ as a free basis, and is such that any continuous map from X to a topological group, sending e to the identity, extends uniquely to a continuous homomorphism on $F(X)$. The basepoint e becomes the identity of $F(X)$, and the generating copy of X has the topology of the original space. See [10] for a survey of the elementary theory of free topological groups.

In this paper we shall in fact use $F(X)$ to denote the underlying group of the free topological group — that is, the free group generated by the set $X \setminus \{e\}$, with e the identity of $F(X)$.

We now define the first topology of interest.

Graev's construction: By a *continuous pseudometric* on X , we mean a pseudometric inducing a topology contained in the given topology on X .

For any continuous pseudometric ρ on X , we define a pseudometric ρ' on $F(X)$ as follows. Let $w = \alpha_1^{\varepsilon_1} \cdots \alpha_n^{\varepsilon_n}$ be a (not necessarily reduced) word in symbols $\{\alpha_i\}$, with $\varepsilon_i = \pm 1$, $i = 1, \dots, n$. For $u, v \in F(X)$ we define

$$\rho'(u, v) = \inf \left\{ \sum_{i=1}^n \rho(a_i, b_i) \right\},$$

where $a_i, b_i \in X$, $i = 1, \dots, n$, are such that $\alpha_1^{\varepsilon_1} \cdots \alpha_n^{\varepsilon_n} = u$ and $b_1^{\varepsilon_1} \cdots b_n^{\varepsilon_n} = v$, and the infimum is taken over all words w and all possible choices of $\{a_i\}$ and $\{b_i\}$.

A definition of ρ' equivalent to ours was given by Graev [4], who showed that ρ' is an invariant pseudometric (and a metric if ρ is), which coincides with ρ on X . Denote by τ_1 the topology induced on $F(X)$ by the extensions ρ' of all the continuous pseudometrics ρ on X . The properties of ρ' and τ_1 will be discussed fully later, but one straightforward fact will be noted now.

It is easily checked that two-sided invariance is equivalent to the requirement that

$$\rho(ab, cd) \leq \rho(a, c) + \rho(b, d)$$

for all group elements a, b, c, d . From the definition of ρ' we have the following:

PROPOSITION 1. *The pseudometric ρ' is the largest two-sided invariant extension of ρ , for any pseudometric ρ on X , and τ_1 is the finest locally invariant topology on $F(X)$ inducing the original topology on X .*

Świerczkowski's construction: In [8], Świerczkowski defined a topology on general free algebras, giving an explicit construction of neighborhoods of points. His construction may be rephrased easily in terms of pseudometrics, giving rise, in the case of free groups, to the following definition.

Let $w = w(\alpha_1, \dots, \alpha_n)$ be any word in the symbols $\alpha_1, \dots, \alpha_n$. (Thus, each α_i may occur many times in the expression for w .) Given ρ and X as before, and $u, v \in F(X)$, we define

$$\rho^2(u, v) = \inf \left\{ \sum_{i=1}^n \rho(a_i, b_i) \right\},$$

where again the infimum is taken over all words w and all choices of $\{a_i\}, \{b_i\}$ such that $w(a_1, \dots, a_n) = u$ and $w(b_1, \dots, b_n) = v$. Let τ_2 be the topology defined by all the pseudometrics ρ^2 .

We shall now discuss the properties of τ_1 and τ_2 in detail. Several of the arguments we shall use here have certain features in common, so we shall spend a little time isolating some of these first.

Let $w = a_1^{\epsilon_1} \dots a_n^{\epsilon_n}$ be a word in symbols $\{a_i\}$, with $\epsilon_i = \pm 1, i = 1, \dots, n$, and for $a_i, b_i \in X$ write u and v for the group elements represented by $a_1^{\epsilon_1} \dots a_n^{\epsilon_n}$ and $b_1^{\epsilon_1} \dots b_n^{\epsilon_n}$ respectively. We may select in the word $a_1^{\epsilon_1} \dots a_n^{\epsilon_n}$ a *cancellation order*; that is, a sequence of steps of the form "delete e " or "replace an adjacent x and x^{-1} by e ", at the end of which the word representing u is reduced. Each symbol $a_i^{\epsilon_i}$ remaining after reduction will be called *essential*, and the others *inessential*. Note that the cancellation order may not be unique, and that the essential symbols $a_i^{\epsilon_i}$ arising from different cancellation orders may occur at different values of the subscript i ; though clearly the final string of essential symbols in the reduced word is uniquely determined. Of course, a given symbol may occur at one point as an essential symbol, and at another as an inessential one.

Let a cancellation order be fixed for both u and v . Then for any essential symbol $s = a_i^{\epsilon_i}$ of u we may form a sequence of columns

$$\begin{pmatrix} s \\ t_1 \end{pmatrix} \begin{pmatrix} s_1^{-1} \\ t_1^{-1} \end{pmatrix} \begin{pmatrix} s_1 \\ t_2 \end{pmatrix} \begin{pmatrix} s_2^{-1} \\ t_2^{-1} \end{pmatrix} \dots,$$

where each pair $(s, t_1), (s_1^{-1}, t_1^{-1}), \dots$ is one of $(a_1^{\epsilon_1}, b_1^{\epsilon_1}), (a_2^{\epsilon_2}, b_2^{\epsilon_2}), \dots$,

where $t_1 t_1^{-1}, t_2 t_2^{-1}, \dots$ and $s_1^{-1} s_1, s_2^{-1} s_2, \dots$ are replaced by e in the cancellation orders for v and u respectively, and where the sequence ends with the next occurrence of an essential symbol of u or v , or of e . Such a sequence may be formed from each essential symbol of u , and then, similarly, from each essential symbol of v : we observe that any two of these sequences are either disjoint or are identical except for the order in which their columns are displayed. (We may form similar sequences from the remaining inessential symbols, but these will be of no interest to us.)

Given any column sequence, its *normalization* will be the sequence of columns $C_1 C_2 \dots C_m$ obtained by replacing each $x^{-1}, x \in X$, by x in the original sequence. For each $i, 1 \leq i \leq m$, let $C_i(1)$ and $C_i(2)$ denote respectively the upper and lower entries of C_i . Then it is clear from its construction that the normalized sequence has a *pairing*, in that either

$$\begin{cases} C_1(1) = C_2(1), C_3(1) = C_4(1), \dots \\ C_2(2) = C_3(2), C_4(2) = C_5(2), \dots \end{cases}$$

or

$$\begin{cases} C_2(1) = C_3(1), C_4(1) = C_5(1), \dots \\ C_1(2) = C_2(2), C_3(2) = C_4(2) \dots \end{cases}$$

There are exactly two entries left *unpaired*, namely, either $C_1(1)$ or $C_1(2)$, and either $C_m(1)$ or $C_m(2)$.

We may perform a process of *column-deletion* in a normalized sequence $C_1 C_2 \dots C_m$, as follows. If $C_i = C_j$ for $i < j$, we delete columns $C_{i+1}, C_{i+2}, \dots, C_j$ if $j - i$ is even, and we delete columns C_i, C_{i+1}, \dots, C_j if $j - i$ is odd. This procedure is carried out repeatedly (the steps chosen, of course, may not be unique) until we obtain a sequence, say $C_{i_1}, C_{i_2}, \dots, C_{i_k}$, which has distinct columns. It is easily checked that k and m have the same parity, and that (if $k > 0$) $C_{i_1} C_{i_2} \dots C_{i_k}$ has a pairing like that of $C_1 C_2 \dots C_m$, with its two unpaired elements being equal to those of $C_1 C_2 \dots C_m$, and occurring in the same positions.

(For example, if $C_1(2)$ and $C_m(1)$ are unpaired, then so are $C_{i_1}(2)$ and $C_{i_k}(1)$, and $C_1(2) = C_{i_1}(2)$ and $C_m(1) = C_{i_k}(1)$.)

In [4], Graev used an argument based on the idea of partitioning into column sequences to prove that for any pair of representations $a_1^{e_1} \dots a_n^{e_n}$ and $b_1^{e_1} \dots b_n^{e_n}$ of $u, v \in F(X)$, there is another pair, $c_1^{e_1} \dots c_n^{e_n}$ and $d_1^{e_1} \dots d_n^{e_n}$ respectively, with the properties:

(1) $\{c_1, \dots, c_n, d_1, \dots, d_n\}$ are taken only from among e and the letters occurring in the reduced forms of u and v , and

(2) $\sum_{i=1}^n \rho(a_i, b_i) \geq \sum_{i=1}^n \rho(c_i, d_i)$

(3) at most $p + q$ of the terms $\rho(c_i, d_i)$ are nonzero, where u and v have reduced lengths p and q respectively. It follows that the infimum $\rho'(u, v)$ is actually attained for some pair of representations, and, using this, Graev proved that the restriction of ρ' to X is ρ (so that τ_1 induces the original topology on X) and that τ_1 is Hausdorff.

Our next result is that the infimum $\rho^2(u, v)$ need *not* be attained, but we shall go on to show that results parallel to those just mentioned for ρ' and τ_1 still hold for ρ^2 and τ_2 . Indeed it will be observed that our proofs carry over routinely to the case of ρ' and τ_1 also, so that Graev's result on the infimum may be dispensed with entirely.

PROPOSITION 2. *The infimum defining $\rho^2(u, v)$ need not be attained by any pair of representations of u and v .*

Proof. Take X to be the plane with the point a deleted, and choose $x, y, e \in X$ so that x, y, e are the vertices of an equilateral triangle of side $l = \sqrt{3}\rho(e, a)$ and center a , where ρ is the Euclidean metric on the plane. If $w(\alpha, \beta, \gamma) = \alpha\beta^{-1}\gamma\beta^{-1}$, the choices $(\alpha, \beta, \gamma) = (x, e, y)$ and $(\alpha, \beta, \gamma) = (b, b, b)$ for any $b \in X$, yield representations of xy and e , respectively, giving an associated sum of distances $\rho(x, b) + \rho(y, b) + \rho(e, b)$. It is easy to see that this sum approaches $\sqrt{3}l$ as b approaches a , so that $\rho^2(xy, e) \leq \sqrt{3}l$. We shall show that no pair of representations can achieve this value.

Suppose that $w(a_1, \dots, a_n) = xy$ and $w(b_1, \dots, b_n) = e$, for some w , and some $a_i, b_i \in X, i = 1, \dots, n$. As described above, select a cancellation order, essential symbols and a partitioning into column sequences of these representations. One may easily check that the sequences containing the essential occurrences of x and y must be disjoint, so that both end with an occurrence of e . Normalize both sequences.

Now perform the column deletion process on the sequence for x , writing the transformed sequence as $C_1C_2 \dots C_p$ and, noting that C_1 is of the form $\begin{pmatrix} x \\ u \end{pmatrix}$ for some u , with x unpaired. Suppose that the normalized sequence for y is $D_1D_2 \dots D_q$, and let i_0 be the least $i, 1 \leq i \leq q$, for which D_i equals some C_j , if such an i exists. Say $D_{i_0} = C_{j_0}$, noting that the choice of j_0 is unique. Assume for concreteness that i_0 is even.

Now transform $D_1 \cdots D_{i_0}$ into a sequence $D_{i_1} \cdots D_{i_k}$ (for some k) by column deletions. (By our choice of i_0 , $k \neq 0$.) Then k is even, $D_1(1) = D_{i_1}(1) = y$, and if $D_{i_0} = \begin{pmatrix} s \\ t \end{pmatrix}$ then $D_{i_k} = \begin{pmatrix} s \\ t' \end{pmatrix}$ for some t' , with y and s the unpaired elements of both sequences.

Write $c_j = \rho(C_j(1), C_j(2))$ and $d_m = \rho(D_{i_m}(1), D_{i_m}(2))$, for $1 \leq j \leq \rho$ and $1 \leq m \leq k$. If $t \neq t'$ (so that in fact none of D_{i_1}, \dots, D_{i_k} equals any of C_1, \dots, C_p), then

$$\sum_{i=1}^n \rho(a_i, b_i) \geq \begin{cases} \sum_{j=1}^{j_0} c_j + \sum_{j=j_0+1}^p c_j + \sum_{m=1}^k d_m, & j_0 \text{ even} \\ \sum_{j=1}^{j_0-1} c_j + \sum_{j=j_0}^p c_j + \sum_{m=1}^k d_m, & j_0 \text{ odd} \end{cases} \\ \geq \rho(x, s) + \rho(s, e) + \rho(y, s).$$

If $t = t'$, so that $D_{i_k} = D_{i_0} = C_{j_0} = \begin{pmatrix} s \\ t \end{pmatrix}$, then

$$\sum_{i=1}^n \rho(a_i, b_i) \geq \begin{cases} \sum_1^{j_0-1} c_i + \sum_{j_0}^p c_j + \sum_1^{i_{k-1}} d_m, & j_0 \text{ even} \\ \sum_1^{j_0} c_j + \sum_{j_0+1}^p c_j + \sum_1^{i_{k-1}} d_m, & j_0 \text{ odd} \end{cases} \\ \geq \rho(x, t) + \rho(t, e) + \rho(y, t).$$

A similar argument applies when i_0 is odd, so that in all cases, $\sum \rho(a_i, b_i)$ is greater than or equal to an expression $\rho(x, b) + \rho(b, e) + \rho(y, b)$, for some $b \in X$. Finally, if no integer i_0 exists, $\sum \rho(a_i, b_i)$ is easily seen to be at least $\rho(x, e) + \rho(y, e)$. Since the latter equals $2l$, and any sum $\rho(x, b) + \rho(b, e) + \rho(y, b)$ is strictly larger than $\sqrt{3}l$ (for $b \neq a$), the result follows.

We remark further that for representations $w(a_1, \dots, a_n) = xy$ and $w(b_1, \dots, b_n) = e$ with $a_i, b_i \in \{x, y, e\}$, $i = 1, \dots, n$, $\sum_{i=1}^n \rho(a_i, b_i)$ must be at least $2l$.

PROPOSITION 3. *Let G be the subgroup of $F(X)$ generated by $\{x_1, \dots, x_k\} \subseteq X$, and set m equal to the minimum distance among $\{\rho(x_i, x_j) : x_i \neq x_j\}$ and $\{\rho(x_i, e) : x_i \neq e\}$. Then $\rho^2(w, e) \geq m$ for any $w \in G$, $w \neq e$.*

Proof. Let $w = y_1^{e_1} \cdots y_p^{e_p} \neq e$, in reduced form, with $y_i \in \{x_1, \dots, x_k\}$, $i = 1, \dots, p$, and let $w(a_1, \dots, a_n)$ and $w(b_1, \dots, b_n)$ be representations of w and e respectively. Select a cancellation order, essential symbols, and a column partitioning as usual.

If the column sequence for any $y_i^{e_i}$ ends in e , a column deletion

argument shows the distance sum from our representations to be at least $\rho(y_i, e)$, giving the desired conclusion. Similarly, $\rho^2(w, e) \geq m$ if the normalized sequence for any y_i^i ends with any y_j different from y_i . But this must happen for some y_i^i , since if each normalized sequence begins and ends with the same essential symbol, we may replace all inessential symbols by e (which cannot affect the reduced forms of $w(a_1, \dots, a_n)$ and $w(b_1, \dots, b_n)$), and we then see that $w(a_1, \dots, a_n)$ reduces to e , contradicting $w \neq e$. Thus $\rho^2(w, e) \geq m$ as required.

COROLLARY. *If X is (completely regular) Hausdorff, so is $(F(X), \tau_2)$.*

Proof. If X is Hausdorff and x_1, \dots, x_k are distinct, there is a continuous pseudometric ρ on X for which $m \geq 1$, and then $\rho^2(w, e) \geq 1$ for any nontrivial word w in the symbols $\{x_1, \dots, x_k\}$. Hence τ_2 is Hausdorff.

We note the next two propositions without proof; the arguments needed are very much in the spirit of those already presented.

PROPOSITION 4. *For $x, y \in X$, $\rho^2(x, y) = \rho(x, y)$, and τ_2 induces the original topology on X .*

PROPOSITION 5. *If x_1, x_2, \dots, x_n are (not necessarily distinct) elements of X different from e , then*

$$\rho'(x_1 x_2 \cdots x_n, e) = \sum_{i=1}^n \rho(x_i, e).$$

Of interest in light of Proposition 2, and also of use later, is the following result and its corollary.

PROPOSITION 6. *Let ρ be the usual metric on $X = [0, 1]$, and let $u, v \in F(X)$. Given representations $w(a_1, \dots, a_n)$ and $w(b_1, \dots, b_n)$ of u and $v(a_i, b_i \in X, i = 1, \dots, n)$, there is another pair of representations $w'(a'_1, \dots, a'_m), w'(b'_1, \dots, b'_m)$, with $a'_1, \dots, a'_m, b'_1, \dots, b'_m$ chosen from e and the letters occurring in the reduced forms of u and v , such that*

$$\sum_{i=1}^m \rho(a'_i, b'_i) \leq \sum_{j=1}^n \rho(a_j, b_j).$$

COROLLARY. *The infimum $\rho^2(u, v)$ is achieved for some pair of representations $w(a_1, \dots, a_n)$ and $w(b_1, \dots, b_n)$ with each a_i and b_i*

chosen from e and the letters occurring in the reduced forms of u and v .

Proof of Proposition 6. Let $x_1 < x_2 < \dots < x_p$ run through e and the distinct letters occurring in the reduced forms of u and v . If any a_i or b_i , $i = 1, \dots, n$ is less than x_i , it does not occur in the reduced form of u or v , and so we may replace it at each occurrence by x_1 without affecting the reduced forms of our representations. It is easy to see, moreover, that the distance sum from the altered representations is no larger than that from the original ones. Similarly replacing all a_i and b_i larger than x_p by x_p , we may assume that $x_1 \leq a_i, b_i \leq x_p$ for each i .

For any $a, b \in [x_1, x_p]$, let $d(a, b)$ be the number of integers j for which $a < x_j < b$ or $b < x_j < a$. We shall show how to exchange our representations for ones in which $d(a_i, b_i) = 0$ for each i .

Suppose that $d(a_i, b_i) > 0$. If $a_i < b_i$, let j be the least integer such that $a_i < x_j < b_i$, and if $b_i < a_i$ let j be the greatest such that $b_i < x_j < a_i$.

Define

$$\begin{aligned} w_1(\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_n, \alpha, \beta, \gamma) \\ = w(\alpha_1, \dots, \alpha_{i-1}, (\alpha\beta^{-1}\gamma), \alpha_{i+1}, \dots, \alpha_n). \end{aligned}$$

Then

$$w_1(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n, a_i, b_i, b_i)$$

and

$$w_1(b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_n, x_j, x_j, b_i)$$

are clearly representations of u and v respectively, and their associated distance sum differs from the previous one only in that $\rho(a_i, b_i)$ has been replaced by $\rho(a_i, x_j) + \rho(x_j, b_i) + \rho(b_i, b_i)$. The latter, though, is equal to $\rho(a_i, b_i)$, so the distance sum is in fact unaltered. We observe that $d(a_i, x_j) = d(b_i, b_i) = 0$, and that $d(x_j, b_i) = d(a_i, b_i) - 1$, so it follows that a repeated application of the above procedure will yield representations of u and v , which we again write as $w(a_1, \dots, a_n), w(b_1, \dots, b_n)$, in which $d(a_i, b_i) = 0$ for each i . That is, for each i there is a $j, 1 \leq j \leq p - 1$, such that $a_i, b_i \in [x_j, x_{j+1}]$.

Fix a $j, 1 \leq j \leq p - 1$. For each i , write $C_i = [a_i, b_i]$ if $a_i \leq b_i$ and $C_i = [b_i, a_i]$ if $b_i \leq a_i$, and suppose without loss of generality that $\{i: C_i \subseteq [x_j, x_{j+1}]\} = \{1, 2, \dots, k\}$ for some k . For $t \in (x_j, x_{j+1})$, with $t \neq a_i, b_i, i = 1, \dots, k$, define $B(t)$ to be the number of intervals $C_i, i = 1, \dots, k$, for which $t \in C_i$. Equivalently, $B(t) = \sum_{i=1}^k \chi_{C_i}(t)$,

for all t in the domain of B (wher χ_C is the characteristic function of C). Then we have

$$\begin{aligned} \sum_{i=1}^k \rho(a_i, b_i) &= \sum |b_i - a_i| \\ &= \sum \int_{x_j}^{x_{j+1}} \chi_{C_i}(t) dt \\ &= \int_{x_j}^{x_{j+1}} B(t) dt . \end{aligned}$$

Now choose a t_0 for which $B(t_0)$ is minimal, and, in our representations, replace each a_i in $[x_j, t_0)$ by $a'_i = x_j$, and each a_i in $(t_0, x_{j+1}]$ by $a'_i = x_{j+1}$. Similarly, replace $b_i \in [x_j, t_0)$ by $b'_i = x_j$, and $b_i \in (t_0, x_{j+1}]$ by $b'_i = x_{j+1}$.

The transformed representations must still represent u and v , since no symbol which has been (nontrivially) replaced occurs in the reduced forms of u or v .

Denote the closed interval with endpoints a'_i, b'_i by $C'_i, i = 1, \dots, k$, and set $B'(t) = \sum_{i=1}^k \chi_{C'_i}(t), t \in (x_j, x_{j+1})$. As above, $\sum_{i=1}^k \rho(a'_i, b'_i) = \int_{x_j}^{x_{j+1}} B'(t) dt$. But exactly $B(t_0)$ of the intervals C'_i are equal to $[x_j, x_{j+1}]$, while the rest are either $\{x_j\}$ or $\{x_{j+1}\}$, so that $B'(t) = B(t_0)$ for all t .

Therefore

$$\begin{aligned} \sum_{i=1}^k \rho(a'_i, b'_i) &= (x_{j+1} - x_j) B(t_0) \\ &\leq \int_{x_j}^{x_{j+1}} B(t) dt \\ &= \sum_{i=1}^k \rho(a_i, b_i) . \end{aligned}$$

Hence the distance sum from the transformed representations is no larger than that from the earlier ones, and, carrying out this process for each $j, 1 \leq j \leq p-1$, we obtain representations of the desired kind.

Proof of the corollary. We observe that any representations $w(a_1, \dots, a_n)$ and $w(b_1, \dots, b_n)$ of u and v may be replaced by others in which the pairs $\{(a_i, b_i)\}$ are distinct, and which give a sum of distances no greater than $\sum_{i=1}^n \rho(a_i, b_i)$. For, if $(a_i, b_i) = (a_j, b_j)$ for $i \neq j$, define

$$w_1(\alpha_1, \dots, \alpha_{j-1}, \alpha_{j+1}, \dots, \alpha_n) = w(\alpha_1, \dots, \alpha_{j-1}, a_i, \alpha_{j+1}, \dots, \alpha_n) ;$$

then clearly $w_1(a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_n)$ and $w_1(b_1, \dots, b_{j-1}, b_{j+1}, \dots, b_n)$ are again representations of u and v , and the distance sum has been decreased by $\rho(a_j, b_j)$. This procedure may be repeated until all pairs (a_i, b_i) remaining are distinct.

From this observation, and Proposition 6, it follows easily that the infimum defining $\rho^2(u, v)$ may be taken over a finite set only, and the corollary follows.

Let τ_0 denote the topology of the free topological group $F(X)$. The next proposition extends to τ_1 and τ_2 a result proved in [4] for τ_0 . The proofs proceed by a straightforward use of the definitions of ρ' and ρ^2 , and will be omitted.

PROPOSITION 7. *The topologies τ_0, τ_1 and τ_2 are independent of the choice of basepoint. More precisely, if $e_1, e_2 \in X$, let $F(X, e_1), F(X, e_2)$ be the groups obtained by selecting e_1, e_2 (respectively) as basepoint. If $\phi: (X, e_1) \rightarrow F(X, e_2)$ is the (basepoint-preserving) map defined by $\phi(x) = xe_1^{-1}, x \in X$, then the extension of ϕ to a homomorphism $\Phi: F(X, e_1) \rightarrow F(X, e_2)$ is a topological isomorphism with respect to τ_0, τ_1 or τ_2 .*

We now define and discuss the third topology of interest.

The construction using the path-connected embedding: If G is a group, let G^* be the set of continuous-from-the-right step functions from the half-open interval $[0, 1)$ into G ; that is, functions f for which there is a partition $0 = a_0 < a_1 < \dots < a_n = 1$ of $[0, 1]$ such that f is constant on each $[a_i, a_{i+1})$. Given an invariant pseudometric ρ on G we may define another such pseudometric ρ^* on G^* by

$$\rho^*(f, g) = \int_0^1 \rho(f(t), g(t)) dt, \quad f, g \in G^* .$$

With the topology defined by ρ^* , G^* becomes a path-connected topological group (see the more general Theorem 1 of [6]), and the map i defined by $i(g)(t) = g, t \in [0, 1), g \in G$, is an isometric isomorphism of G into G^* . We define the path $P \equiv P(f, g)$ between f and g in G^* by

$$P(s)(t) = \begin{cases} g(t), & 0 \leq t < s \\ f(t), & s \leq t < 1, \end{cases}$$

for $t \in [0, 1), s \in [0, 1]$.

Now suppose that $S = \{s_1, s_2, \dots, s_n\}$ is a finite ordered subset of X containing e . Let $G(S)$ be the subgroup of $F(X)$ generated by S , let d_s be the discrete metric on $G(S)$ (that is, all nonzero distances are equal to 1), and let d_s^* be the extension of d_s to $G(S)^*$. Clearly $\bigcup_{j=1}^{n-1} P(i(s_j), i(s_{j+1}))$ is homeomorphic to $[0, 1]$, so by the complete regularity of X there is a continuous map $\phi: X \rightarrow \bigcup_{j=1}^{n-1} P(i(s_j), i(s_{j+1}))$ such that $\phi(s_j) = i(s_j), j = 1, \dots, n$. Extending ϕ to a continuous

homomorphism $\Phi: F(X) \rightarrow G(S)^*$, where $F(X)$ is now the free topological group, we define a continuous invariant pseudometric $\rho_{S,\phi}$ on $F(X)$ by

$$\rho_{S,\phi}(u, v) = d_S^*(\Phi(u), \Phi(v)), \quad u, v \in F(X).$$

(Note that $\rho_{S,\phi}$ is invariant since d_S and d_S^* are.)

A proof that the free topological group is Hausdorff was given in [2] using path-connected embedding, although no explicit construction was given there of a topology on the free group. The topology τ_s generated by all the pseudometrics $\rho_{S,\phi}$, for all choices of S , and all choices of ϕ of the kind indicated, may be seen as an expression in terms of invariant pseudometrics of the proof of [2].

PROPOSITION 8. *If X is completely regular and Hausdorff, then τ_s is Hausdorff, and X has its original topology and is a closed subspace of $(F(X), \tau_s)$.*

Proof. If w is any nontrivial word in letters from $S = \{x_1, \dots, x_n, e\} \subseteq X$, we form the discrete metric d_S on $G(S)$ and its extension d_S^* to $G(S)^*$. Let $\phi: X \rightarrow \bigcup_{i=1}^{n-1} P(i(x_j), i(x_{j+1})) \cup P(i(x_n), i(e)) \subseteq G(S)^*$ be any continuous map such that $\phi(x_j) = i(x_j)$, $j = 1, \dots, n$, and extend ϕ to a homomorphism $\Phi: F(X) \rightarrow G(S)^*$. Then clearly $\Phi(w)(t) = w$ for all $t \in [0, 1)$, so $\rho_{S,\phi}(w, e) = 1$, and τ_s is Hausdorff. Indeed we see further that if $u \notin X$, $\rho_{S,\phi}(x, u) = 1$ for each $x \in X$, and this shows that X is closed in τ_s .

By construction, each $\rho_{S,\phi}$ is continuous on X , in its original topology, so the latter contains the restriction of τ_s to X . On the other hand, let $U \neq X$ be open in the original topology, and let x be any point in U . If $e \in U$, select any $y \in X \setminus U$, set $T = \{x, y, e\}$, and form d_T, d_T^* and $\rho_{T,\psi}$, where $\psi: X \rightarrow G(T)^*$ satisfies $\psi(x) = i(x)$, $\psi(e) = i(e)$ and $\psi(X \setminus U) = \{i(y)\}$, and has extension Ψ on $F(X)$. If $B = \{w \in F(X): \rho_{T,\psi}(w, x) < 1\}$, then $B \cap X$ is open in the restriction of τ_s to X . But for $z \in B \cap X$ we have

$$\begin{aligned} 1 > \rho_{T,\psi}(z, x) &= d_T^*(\Psi(z), \Psi(x)) \\ &= \int_0^1 d_T(\Psi(z)(t), x) dt, \end{aligned}$$

so that for some $t \in [0, 1)$, $\Psi(z)(t) = x$, implying that $z \in U$. That is, $x \in B \cap X \subseteq U$, and U is open in the restriction of τ_s to X . (Note that this argument works even when $x = e$.) A very similar argument gives the same conclusion in the case when $e \notin U$, and the equality of the two topologies follows.

REMARK. We have not included proofs that X is closed in τ_1 and τ_2 . This may be proved directly, but will in fact follow from Proposition 8 and Theorem 1.

2. The relations between the topologies.

THEOREM 1. *If X is completely regular and Hausdorff, then*

$$\tau_0 \supseteq \tau_1 \supseteq \tau_2 \supseteq \tau_3 .$$

THEOREM 2. *The above inclusions are in general strict. Specifically, if $\tau_0 = \tau_1$, or if $\tau_1 = \tau_2$, then X has the property that any countable union of its closed subsets must be closed. When $X = [0, 1]$, $\tau_2 \neq \tau_3$.*

Proof of Theorem 1. Of course τ_0 is the strongest group topology on $F(X)$ inducing the original topology on X , and so $\tau_1 \subseteq \tau_0$. The inclusion $\tau_2 \subseteq \tau_1$ follows from Proposition 1.

To show that $\tau_3 \subseteq \tau_2$, fix a finite set $S \subseteq X$ containing e , and define d_s^* and $\rho_{S,\phi}$ as earlier, for any choice of ϕ . If we let ρ be the restriction of $\rho_{S,\phi}$ to X , ρ is a continuous pseudometric on X , and we shall show that $\rho^2 \geq \rho_{S,\phi}$.

Suppose that for $\delta > 0$, $\rho^2(u, v) < \delta$, for some $u, v \in F(X)$. Then there is a word $w(\alpha_1, \dots, \alpha_n)$ and $a_1, \dots, a_n, b_1, \dots, b_n \in X$ such that $w(a_1, \dots, a_n) = u$, $w(b_1, \dots, b_n) = v$, and such that $\sum_{i=1}^n \delta_i < \delta$, where

$$\delta_i = \rho(a_i, b_i) = \int_0^1 d_s(\phi(a_i)(t), \phi(b_i)(t)) dt .$$

Since d_s is the discrete metric we have $\phi(a_i)(t) = \phi(b_i)(t)$ for t in a set A_i of measure $1 - \delta_i$. Suppose that $w(\alpha_1, \dots, \alpha_n) = \alpha_{i_1}^{\epsilon_1} \dots \alpha_{i_k}^{\epsilon_k}$, a (not necessarily reduced) word in the symbols $\{\alpha_1, \dots, \alpha_n\}$, with each ϵ_i equal to ± 1 . For each l , $1 \leq l \leq k$, the pair (a_{i_l}, b_{i_l}) is one of the pairs $(a_1, b_1), \dots, (a_n, b_n)$, and so $\phi(a_{i_l})(t) = \phi(b_{i_l})(t)$ for t in one of the sets A_1, \dots, A_n .

Hence $\Phi(u)(t) = \phi(a_{i_1})(t)^{\epsilon_1} \dots \phi(a_{i_k})(t)^{\epsilon_k}$ is equal to $\Phi(v)(t) = \phi(b_{i_1})(t)^{\epsilon_1} \dots \phi(b_{i_k})(t)^{\epsilon_k}$ for $t \in A = \bigcap_{i=1}^n A_i$, and the measure of A is at least $1 - \sum \delta_i$, which is larger than $1 - \delta$. Therefore $\rho_{S,\phi}(u, v) = \int_0^1 d_s(\Phi(u)(t), \Phi(v)(t)) dt < \delta$, and $\rho_{S,\phi} \leq \rho^2$ as required, proving the theorem.

Proof of Theorem 2.

Case 1: τ_0 and τ_1 . The result stated here will appear with proof in [5]. See [3] and [7] for related results.

Case 2: τ_1 and τ_2 . The following result will be needed.

LEMMA. Suppose that $\{A_n\}$ is a sequence of closed sets in X and $e \notin \cup A_n$. If $\{k_n\}$ is an increasing sequence of positive integers for which $\sum (1/k_n)$ converges, and if $B = \cup B_n = \cup (A_n)^{k_n}$, then $e \notin \bar{B}$, the closure of B in τ_1 .

Proof. Since, for each n , A_n is closed and $e \notin A_n$, there are continuous pseudometrics ρ_n on X such that $\rho_n(e, A_n) \geq 1, n \in N$. If we define

$$\rho(x, y) = \sum_{n=1}^{\infty} \min\{\rho_n(x, y), (k_n)^{-1}\}, \quad x, y \in X,$$

we see that ρ is again a continuous pseudometric on X and that $\rho(e, A_n) \geq (k_n)^{-1}, n \in N$. Then if $w = x_1 x_2 \cdots x_k \in (A_n)^k$ (writing $k = k_n$), Proposition 5 shows that $\rho'(w, e) = \sum \rho(x_i, e) \geq 1$, and so $\rho'(e, B) \geq 1$ and $e \notin \bar{B}$, which proves the lemma.

With $\{A_n\}$, A and B as in the lemma, suppose that A is not in X , and that $e \in \bar{A} \setminus A$. We shall show that B is not closed in τ_2 . For if ρ is any continuous pseudometric on X , and $\varepsilon > 0$, there is an x in some A_n such that $\rho(x, e) < \varepsilon$. Setting $w(\alpha) = \alpha^k (k = k_n)$, we see that $w(x) = x^k$ and $w(e) = e$, so that

$$\rho^2(x^k, e) \leq \rho(x, e) < \varepsilon.$$

Then since $x^k \in B, e$ is in the closure of B in τ_2 . Thus, in the case when $e \in \bar{A} \setminus A, \tau_1 \neq \tau_2$. An appeal to Proposition 7 now proves Case 2 in general.

Case 3: τ_2 and τ_3 . We shall show that $\tau_2 \neq \tau_3$ in the case $X = [0, 1]$. Let $I = [a, b] \subset [0, 1]$ be a closed interval of length $1/4$, such that $\rho(I, e) \geq 1/4$, where ρ is the usual metric on $[0, 1]$. For $n \in N$ and $1 \leq l \leq 3n$, set $x_l^n = a + l/12n \in I$, define

$$w_{nk} = x_{3k-2}^n (x_{2k-1}^n)^{-1} x_{3k}^n (x_{2k-2}^n)^{-1} x_{2k-1}^n (x_{3k}^n)^{-1} \quad \text{for } 1 \leq k \leq n,$$

and set $w_n = w_{n1} w_{n2} \cdots w_{nn}$. We shall show that $\{w_n\}$ converges to e in τ_3 , but does not in τ_2 . In general the superscripts n will be omitted.

Let $S = \{s_1, \dots, s_m\}$ be a finite ordered subset of X containing e , define $G(S), d_S, d_S^*$ as earlier, and let $\phi: X \rightarrow \prod_{j=1}^{m-1} P(i(s_j), i(s_{j+1}))$ be any continuous map such that $\phi(s_j) = i(j, s), j = 1, \dots, m$. Extend ϕ to a homomorphism $\Phi: F(X) \rightarrow G(S)^*$. Since ϕ is uniformly continuous on X , there is an n_0 such that when $n > n_0$,

$$(1) \quad \left\{ \begin{array}{l} d_S^*(\phi(x_{3k-2}), \phi(x_{3k-1})) < 1 \\ d_S^*(\phi(x_{3k-1}), \phi(x_{3k})) < 1 \\ \text{and } d_S^*(\phi(x_{3k-2}), \phi(x_{3k})) < 1 \end{array} \right.$$

for $1 \leq k \leq n$. We shall show that for each $t \in [0, 1)$, at least two of $\phi(x_{3k-2})(t)$, $\phi(x_{3k-1})(t)$ and $\phi(x_{3k})(t)$ are equal.

Consider a fixed value of k . Then for $l = 0, 1, 2$ there is a p_l , $1 \leq p_l \leq m$, and an α_l , $0 \leq \alpha_l < 1$, such that

$$\phi(x_{3k-l})(t) = \begin{cases} s_{p_{l+1}}, & 0 \leq t < \alpha_l \\ s_{p_l}, & \alpha_l \leq t < 1. \end{cases}$$

For distinct l_1 and l_2 , we must have $|p_{l_1} - p_{l_2}| \leq 1$, since otherwise $\phi(x_{3k-l_1})(t)$ and $\phi(x_{3k-l_2})(t)$ are different for all $t \in [0, 1)$, and hence $d_S^*(\phi(x_{3k-l_1}), \phi(x_{3k-l_2})) = 1$, contradicting (1). Therefore, two of $\{p_0, p_1, p_2\}$ are equal, and the third differs from them by at most one. Suppose without loss of generality that $p_0 = p_1$, so that either (a) $p_2 = p_0 - 1$, or (b) $p_2 = p_0$, or (c) $p_2 = p_0 + 1$.

Consider case (a). By (1), we must have $\alpha_2 > \alpha_0$ and $\alpha_2 > \alpha_1$. Then, assuming without loss of generality that $\alpha_0 \leq \alpha_1$, we have

$$\begin{aligned} \phi(x_{3k})(t) &= \phi(x_{3k-1})(t) = s_{p_0+1}, & 0 \leq t < \alpha_0, \\ \phi(x_{3k})(t) &= \phi(x_{3k-2})(t) = s_{p_0}, & \alpha_0 \leq t < \alpha_1, \\ \text{and } \phi(x_{3k})(t) &= \phi(x_{3k-1})(t) = s_{p_0}, & \alpha_1 \leq t < 1. \end{aligned}$$

A similar analysis in the other cases shows that two of $\phi(x_{3k-2})(t)$, $\phi(x_{3k-1})(t)$ and $\phi(x_{3k})(t)$ are always equal, so that $\Phi(w_n)(t) = e$ for all $t \in [0, 1)$. Therefore $\Phi(w_n)(t) = e$ for all t and for $n > n_0$, and hence $\rho_{S,\phi}(w_n, e) = 0$ for $n > n_0$, and $w_n \rightarrow e$ with respect to $\rho_{S,\phi}$. Thus $w_n \rightarrow e$ in τ_3 .

To prove that $w_n \not\rightarrow e$ in τ_2 , we use the corollary to Proposition 6 to give us, for a fixed n , representations $w(a_1, \dots, a_p) = w_n$ and $w(b_1, \dots, b_p) = e$, where $a_1, \dots, a_p, b_1, \dots, b_p$ are taken from among e and $\{x_l^i: 1 \leq l \leq 3n\}$, and where $\rho^2(w_n, e) = \sum_{i=1}^p \rho(a_i, b_i)$, with ρ the usual metric on $[0, 1]$. For a given k , $1 \leq k \leq n$, consider all pairs (a_i, b_i) for which $\{a_i, b_i\} \cap \{x_{3k-2}, x_{3k-1}, x_{3k}\} \neq \emptyset$.

A straightforward argument shows that for (at least) one such pair we must have $a_i \neq b_i$. Write the pair as $(x(k), y(k))$, and carry out this procedure for each k . If any $x(k)$ or $y(k) = e$, we have $\rho^2(w_n, e) \geq 1/4$ immediately, by our choice of the interval I . If no $x(k)$ or $y(k) = e$, then $\rho(x(k), y(k)) \geq 1/12n$, $1 \leq k \leq n$. But the number of distinct pairs in $\{(x(k), y(k)): 1 \leq k \leq n\}$ is at least $n/2$, as any such pair may equal at most one other. Therefore $\rho^2(w_n, e) \geq (n/2) \cdot (1/12n) = (1/24)$, and $w_n \not\rightarrow e$ in τ_2 .

This completes the proof of Theorem 2.

EXAMPLE. Let X be any uncountable set. Fixing $e \in X$, define a topology on X by specifying that a set is closed if it is countable, or it contains e . Then X is a (Hausdorff) normal nondiscrete space with the property that countable unions of closed sets are always closed. Thus Theorem 2 leaves the question of equality between τ_0 and τ_1 , and τ_1 and τ_2 , unsettled for a nontrivial class of spaces.

REMARK. We conclude by noting that if X is a "large" metric space then $\tau_3 \neq \tau_1$.

More precisely let X be any metric space, or even any completely regular Hausdorff space which admits a continuous metric. Then Thompson [11] shows that $F(X)$ with the topology τ_1 has no small subgroups. On the other hand, it is clear that if the cardinal number of the set X is strictly greater than 2^c , then from the definition, the group $F(X)$ with the topology τ_3 does have small subgroups (because the cardinality of $G(S)^*$ is only 2^c). Hence $\tau_3 \neq \tau_1$ in this case.

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