PACIFIC JOURNAL OF MATHEMATICS
Vol. 104, No. 2, 1983

CRITERIA FOR OSCILLATORY SUBLINEAR SCHRÖDINGER EQUATIONS

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The semilinear Schrödinger equation

$$Lu \equiv \Delta u + f(x, u) = 0, \qquad x \in \Omega \tag{1}$$

will be considered in an exterior domain $\Omega \subset R^n$, $n \geq 2$, where f is nonnegative and locally Hölder continuous in $\Omega \times (0, \infty)$. One objective is to find sharp necessary conditions for (1) to be oscillatory in Ω under the *sublinear* hypothesis that $\max_{|x|=r} t^{-1} f(x, t)$ is a nonincreasing function of t in $(0, \infty)$ for each fixed $r > 0$. The necessary conditions below are proved in §2:

$$\int^{\infty} r \max_{|x|=r} f(x, c \log r)\, dr = +\infty \qquad \text{if } n = 2;$$

$$\int^{\infty} r \max_{|x|=r} f(x, c)\, dr = +\infty \qquad \text{if } n \geq 3$$

for some positive constant c. Sufficient conditions for (1) to be oscillatory in Ω are proved in §3 under a modified sublinear hypothesis. These results are then combined to yield characterizations of oscillatory sublinear equations of the Emden-Fowler type in exterior domains.

The sublinear Emden-Fowler (or Lane-Emden) equation is the prototype

$$\Delta u + p(x) | u |^{\gamma} \operatorname{sgn} u = 0, \qquad 0 < \gamma < 1, x \in \Omega, \tag{2}$$

where $p(x)$ is nonnegative and locally Hölder continuous in Ω. A theorem of Kitamura and Kusano [7] states in particular that (2) is oscillatory in a exterior domain Ω in R^n, $n \geq 2$, if

$$\int^{\infty} r P_1(r)\, dr = +\infty, \tag{3}$$

where $P_1(r) = \min_{|x|=r} p(x)$. The same is true if $P_1(r)$ is replaced by the spherical mean of $p(x)$ over the sphere of radius r (see §3). Under additional regularity hypotheses on $p(x)$ it was proved by E. S. Noussair and the writer [12] that (3) is in fact *necessary and sufficient* for (2) to be oscillatory in $\Omega \subset R^n$ if $n \geq 3$. However, this is not so if $n = 2$; an easy counterexample is provided in the case that (2) is a radial equation:

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{du}{dr} \right) + p(r) |u|^{\gamma} \operatorname{sgn} u = 0, \tag{4}$$

where $p(r) = r^{-2}(\log r)^{-\delta}$, $1 < \delta < 1 + \gamma$, $r > 0$. Liouville's transformation $r = e^s$, $h(s) = u(e^s)$ sends (4) into the canonical form

$$h''(s) + e^{2s}p(e^s)|h(s)|^{\gamma} \operatorname{sgn} h(s) = 0,$$

which is oscillatory at $s = \infty$ if and only if

$$\int^{\infty} s^{\gamma} e^{2s} p(e^s)\, ds = +\infty$$

by Belohorec's well-known theorem **[2]**, or equivalently, if and only if

$$\int^{\infty} r(\log r)^{\gamma} p(r)\, dr = +\infty. \tag{5}$$

In the present example, (5) is satisfied while (3) fails. Therefore (3) is not necessary for oscillation of (2) in 2 dimensions. It might be expected that a necessary condition for oscillation of (2) (or (1)) is similar to (5), and in fact Theorem 2.4 below shows, under the hypotheses that $p(x)$ is nonnegative and Hölder continuous, that a necessary condition for (2) to be oscillatory in an exterior domain $\Omega \subset R^2$ is

$$\int^{\infty} r(\log r)^{\gamma} \Big[\max_{|x|=r} p(x)\Big]\, dr = +\infty. \tag{6}$$

One of our main objectives is to improve (3) in R^2; this is accomplished in Theorems 3.3 and 3.4. In §2 the necessary conditions for oscillation mentioned above (i.e. the nonoscillation results) are extended to the general sublinear case (1). Theorem 3.6 and Corollary 3.7 contain characterizations of oscillatory sublinear equations (1) or (2) in R^n, $n \geq 3$, under suitable regularity hypotheses on $f(x, u)$ or $p(x)$.

2. Necessary conditions for oscillation. Points in Euclidean n-space R^n are denoted by $x = (x_1, \ldots, x_n)$, and the Euclidean length of x is written $|x|$. The following notation will be used throughout the sequel:

$$S_a = \{x \in R^n : |x| = a\}, \qquad a > 0;$$
$$G_a = \{x \in R^n : |x| > a\}.$$

An *exterior* domain Ω in R^n is defined by the property that $G_a \subset \Omega$ for some $a > 0$.

For a bounded domain $M \subset R^n$, the Hölder norms of a function $u: \overline{M} \to R^1$ are defined by

$$\|u\|_{\lambda, \overline{M}} = \sup_{\substack{x, y \in \overline{M} \\ x \neq y}} \frac{|u(x) - u(y)|}{|x - y|^{\lambda}}$$

$$\|u\|_{m+\lambda, \overline{M}} = \sum_{|i|=m} \|D^i u\|_{\lambda, \overline{M}} + \sum_{|i| \leq m} \sup_{x \in \overline{M}} |D^i u(x)|,$$

$$0 < \lambda < 1, \quad m = 1, 2, \ldots,$$

where i denotes a multi-index of length $|i|$. The Hölder space $C^{m+\lambda}(\overline{M})$ is defined as the set of all continuous real-valued functions on $\overline{M}$ with partial derivatives on $\overline{M}$ such that $\|u\|_{m+\lambda, \overline{M}}$ is finite, $0 < \lambda < 1$, $m = 0, 1, \ldots$. The notation $C_{\text{loc}}^{m+\lambda}(\Omega)$ denotes the set of all $u: \Omega \to R$ such that $u \in C^{m+\lambda}(\overline{M})$ for every bounded domain $M \subset \Omega$. The notation $C_{\text{loc}}^{m+\lambda}(\Omega \times R^+)$ is defined similarly, where $R^+ = (0, \infty)$.

Equation (1) is to be considered in an exterior domain $\Omega \subset R^n$ subject to the assumptions below.

ASSUMPTIONS

(A) $f \in C_{\text{loc}}^{\lambda}(\Omega \times R^+)$ for some λ in $0 < \lambda < 1$, fixed in the sequel.

(B) $0 \leq f(x, t) \leq t g(|x|, t)$ for all $x \in \Omega$ and for all $t > 0$, where $g \in C_{\text{loc}}^{\lambda}(R^+ \times R^+)$ and $g(r, t)$ is a nonincreasing function of t in R^+ for each fixed $r > 0$.

The above nonincreasing property of $g(r, t)$ is a *sublinear* condition for equation (1). For example, assumptions (A) and (B) hold in the Emden-Fowler prototype (2), i.e. $f(x, t) = p(x)t^{\gamma}$, $0 < \gamma < 1$, where $p(x)$ is nonnegative in Ω and $p \in C_{\text{loc}}^{\lambda}(\Omega)$. In this case an example of a function g in (B) is

$$g(r, t) = \left[\max_{|x|=r} p(x)\right] t^{\gamma - 1}.$$

A *solution* of $Lu = 0$ [$Lu \leq 0$, $Lu \geq 0$] in Ω is a function $u \in C_{\text{loc}}^{2+\lambda}(\Omega)$, with λ as in (A), such that $(Lu)(x) = 0$ [$(Lu)(x) \leq 0$, $(Lu)(x) \geq 0$, respectively] for all $x \in \Omega$. The operator L given by (1) is called *oscillatory* in Ω whenever every solution of (1) defined in $G_a \subset \Omega$ for some $a > 0$ changes sign in G_r for all $r \geq a$. Then L is nonoscillatory in Ω whenever (1) has a positive solution $u(x)$ in G_b for some $b \geq a$.

2.1. THEOREM. *Let L be the operator defined by* (1) *where f is nonnegative and satisfies assumption* (A) *in an exterior domain Ω, and suppose that $G_a \subset \Omega$ for some $a > 0$. If there exists a positive solution v and a nonnegative solution w of $Lv \leq 0$ and $Lw \geq 0$, respectively, in G_a such that $w(x) \leq v(x)$ throughout $G_a \cup S_a$, then equation* (1) *has at least one solution $u(x)$ satisfying $u(x) = v(x)$ on S_a and $w(x) \leq u(x) \leq v(x)$ throughout G_a.*

A proof was given by Noussair and the writer **[12]**. A variation with an additional monotony hypothesis appears in **[11]**. Versions of this theorem for bounded domains appear in the works of Nagumo **[9]**, Cohen **[5]**, Keller **[6]**, Simpson and Cohen **[13]**, and Amann **[1]**.

The corollary below applies to the case that the ordinary differential equation

$$\frac{d}{dr}\left(r^{n-1}\frac{d\zeta}{dr}\right) + r^{n-1}\zeta g(r, \zeta) = 0 \tag{7}$$

has a positive solution $\zeta \in C^{2+\lambda}[a, b]$ for some $a > 0$ and for all $b > a$.

2.2. COROLLARY. *If* (A) *and* (B) *hold, equation* (1) *is nonoscillatory in an exterior domain* $\Omega \subset R^n$, $n \geq 2$, *if* (7) *has a positive solution* $\zeta(r)$ *in* $a < r < \infty$ *for some* $a > 0$ *such that* $\zeta \in C^{2+\lambda}[a, b]$ *for all* $b > a$.

Proof. Let v be the function defined in G_a by $v(x) = \zeta(r)$, $r = |x| \geq a$. Then

$$\begin{aligned} r^{n-1}(Lv)(x) &= \frac{d}{dr}\left(r^{n-1}\frac{d\zeta}{dr}\right) + r^{n-1}f(x, v(x)) \\ &\leq \frac{d}{dr}\left(r^{n-1}\frac{d\zeta}{dr}\right) + r^{n-1}\zeta(r)g(r, \zeta(r)), \end{aligned}$$

and hence $(Lv)(x) \leq 0$ for all $x \in G_a$ by (7). Since $w(x) \equiv 0$ satisfies $(Lw)(x) \geq 0$, Theorem 2.1 shows that (1) has a solution $u(x)$ satisfying $0 \leq u(x) \leq v(x) = \zeta(r)$ for $|x| \geq a$. However, $(\Delta u)(x) \leq 0$ in the annulus $G_{a,b} = \{x \in R^n\colon a < |x| < b\}$, $u(x) = v(x) > 0$ for $|x| = a$, and $u(x) \geq 0$ for $|x| = b$, and therefore $u(x) > 0$ throughout $G_{a,b}$ by the maximum principle. Since b is arbitrary, $u(x)$ is a positive solution of (1) in G_a.

To apply Corollary 2.2, we shall appeal to the following theorem of Belohorec [**3**, Theorem 3], Coffman and Wong [**4**, Theorem 2], concerning the ordinary differential equation

$$\frac{d^2u}{dt^2} + ug(t, u) = 0, \qquad 0 < t < \infty. \tag{8}$$

2.3. THEOREM (*Belohorec, Coffman and Wong*). *Let* $f(t, u) = ug(t, u)$ *be continuous and nonnegative for* $0 < t < \infty$, $0 < u < \infty$, *and suppose that* $g(t, u)$ *is nonincreasing in* u *for each* t. *Then equation* (8) *has an unbounded positive solution* $u(t)$ *in* (a, ∞) *for some* $a > 0$ *if and only if*

$$\int^{\infty} tg(t, ct)\, dt < \infty \tag{9}$$

for some $c > 0$.

Although not relevant here, it is proved in **[3]** and **[4]** that (9) characterizes equations (8) possessing an asymptotically linear unbounded solution.

2.4. THEOREM. *Under assumptions* (A) *and* (B), (1) *has a positive solution* $u(x)$ *in an exterior domain* $G_a \subset R^n$ *for some* $a > 0$ *if*

$$\int^{\infty} r \log r \, g(r, c \log r) \, dr < \infty \qquad (n = 2) \tag{10}$$

$$\int^{\infty} r g(r, c) \, dr < \infty \qquad (n \geq 3) \tag{11}$$

for some $c > 0$.

Proof. If $n = 2$, Liouville's transformation $r = e^s$, $h(s) = \zeta(e^s)$ changes (7) into the standard form

$$h''(s) + e^{2s} h(s) g(e^s, h(s)) = 0. \tag{12}$$

By Theorem 2.3, (12) has a positive solution $h(s)$ in some interval (A, ∞) if and only if

$$\int^{\infty} s e^{2s} g(e^s, cs) \, ds < \infty$$

for some $c > 0$, which is equivalent to (10). Since $g \in C^{\lambda}$ by assumption (B), standard regularity theorems (see e.g. **[8]**) show that $h \in C^{2+\lambda}[A, B]$ for all $B > A$, or equivalently $\zeta \in C^{2+\lambda}[a, b]$ for all $b > a = e^A$. Then Corollary 2.2 shows that (10) is sufficient for (1) to have a positive solution $u(x)$ in $G_a \subset R^2$ for some $a > 0$.

Similarly if $n \geq 3$ the change of variables

$$r = \beta(s) = (\nu s)^{\nu}, \qquad h(s) = s\zeta(\beta(s)),$$

where $\nu = 1/(n-2)$, transforms (7) into

$$h''(s) + s^{-4}[\beta(s)]^{2n-2} h(s) g\left(\beta(s), \frac{h(s)}{s}\right) = 0. \tag{13}$$

By Theorem 2.3, (13) has a positive solution in some interval (A, ∞) if and only if

$$\int^{\infty} s^{-3} [\beta(s)]^{2n-2} g(\beta(s), c) \, ds < \infty$$

for some $c > 0$, establishing the sufficiency of (11) for (1) to have a positive solution in $G_a \subset R^n$, $n \geq 3$, for some $a > 0$.

In the special case (2) of (1), i.e. $f(x, u) = p(x)u^\gamma$, $0 < \gamma < 1$, where $p(x)$ is nonnegative in Ω and $p \in C^\lambda_{\mathrm{loc}}(\Omega)$, we choose $g(r, u)$ in (B) to be $P_2(r)u^{\gamma-1}$, where

$$P_2(r) = \max_{|x|=r} p(x).$$

In this case, conditions (10) and (11) reduce to, respectively

$$\int^\infty r(\log r)^\gamma P_2(r)\, dr < \infty, \qquad n = 2 \tag{14}$$

$$\int^\infty rP_2(r)\, dr < \infty, \qquad n \geq 3. \tag{15}$$

3. Sufficient conditions for oscillation. Oscillation criteria for (1) or (2) will be generated by developing necessary conditions for (1) or (2) to have a positive solution $u(x)$ in some exterior domain G_a, $a > 0$, under the following alternative to assumption (B):

ASSUMPTION

(C) $f(x, u) \geq p(x)\phi(u)$ for all $x \in \Omega$ and for all $u \geq 0$, where p is continuous and nonnegative in Ω; $\phi \in C^1[0, \infty)$; $\phi(u) > 0$, $\phi'(u) > 0$, and $\Phi(u) < \infty$ for all $u > 0$, where

$$\Phi(u) = \int_0^u \frac{dt}{\phi(t)}, \qquad u > 0. \tag{16}$$

The *sublinear* condition $\Phi(u) < \infty$ is satisfied, for example, if $\phi(u) = |u|^\gamma \operatorname{sgn} u$, $0 < \gamma < 1$. A *solution* of (1) is now a classical solution, i.e. $u \in C^2(\Omega)$ and $Lu = 0$ at every point in Ω.

The extension of (1) to negative u can be made by adjoining the conditions $f(x, -u) = -f(x, u)$ and $\phi(-u) = -\phi(u)$ for all $u > 0$, or weaker requirements. Then the existence of a negative solution $u(x)$ of (1) is equivalent to the existence of a positive solution, and our theorems imply criteria for the nonexistence of any one-signed solutions.

3.1. LEMMA. *If u is a positive solution of* (1) *in G_a for some $a > 0$, and $z \in C^2(G_a)$ is an arbitrarypositive function in G_a, then $\Phi(u)$ satisfies the differential inequality*

$$\begin{aligned} &-\Delta[z(x)\Phi(u(x))] \\ &\quad \geq z(x)p(x) - \Phi(u(x))(\Delta z)(x) - \frac{|(\nabla z)(x)|^2}{z(x)\phi'(u(x))} \end{aligned} \tag{17}$$

for all $x \in G_a$.

Proof. An easy calculation on the basis of (1), (16), and assumption (C) shows that

$$\Delta\Phi(u) \le -p - \phi'(u)|\nabla\Phi(u)|^2$$

in G_a, from which

$$-\Delta(z\Phi(u)) \ge zp + z\phi'(u)\left|\nabla\Phi(u) - [z\phi'(u)]^{-1}\nabla z\right|^2$$
$$-\Phi(u)\Delta z - [z\phi'(u)]^{-1}|\nabla z|^2,$$

implying (17) since $\phi'(u) > 0$ for $u > 0$.

The next lemma is a specialization to the case $n = 2$, $z(x) = \phi(h)$, $h = \log|x|$.

3.2. LEMMA. *If u is a positive solution of* (1) *in $G_a \subset R^2$ for some $a > 0$, then*

$$(18) \qquad -\Delta[\phi(h)\Phi(u)] \ge \phi(h)p(x) - \frac{[\phi'(h)]^2 + \phi(h)\phi''(h)\Phi(u)\phi'(u)}{r^2\phi(h)\phi'(u)}$$

where $h = \log r$, $r = |x| \ge a$.

Proof. If $z(x) = \phi(h)$, then

$$\nabla z = \frac{\phi'(h)x}{r^2}, \qquad \Delta z = \frac{\phi''(h)}{r^2}$$

and (17) reduces to (18).

In the classical sublinear case, $\phi(u) = u^\gamma$, $0 < \gamma < 1$, $\Phi(u)\phi'(u) = \gamma/(1-\gamma)$, and hence

$$[\phi'(h)]^2 + \phi(h)\phi''(h)\Phi(u)\phi'(u) \equiv 0.$$

Then (18) simplifies to

$$(19) \qquad -\Delta w(x) \ge (\log r)^\gamma p(x), \qquad r = |x| \ge a$$

where

$$w(x) = (1-\gamma)^{-1}(\log r)^\gamma[u(x)]^{1-\gamma}.$$

The *spherical mean* $U(r) = m(r; u)$ of a function u: $R^n \to R^1$ over the sphere S_r of radius r is defined by

$$(20) \qquad m(r; u) = \frac{1}{s(S_r)}\int_{S_r} u(x)\, ds = \frac{1}{\omega(S_1)}\int_{S_1} u(x)\, d\omega,$$

where s and ω denote the measure on S_r and S_1, respectively; $ds = r^{n-1}\, d\omega$.

3.3. THEOREM. *If assumption* (C) *holds where* $\phi(u) = u^\gamma$, $0 < \gamma < 1$, *a necessary condition for* (1) *to have a positive solution* $u(x)$ *in some exterior domain* $G_a \subset R^2$ *is*

$$\int^\infty r(\log r)^\gamma m(r; p)\, dr < \infty. \tag{21}$$

Proof. The spherical mean of any function $w \in C^2(G_a)$ satisfies [**10**; Lemma 2, pp. 69–70]

$$\frac{d}{dr}\left[r^{n-1}\frac{dm(r; w)}{dr}\right] = \frac{r^{n-1}}{\omega(S_1)}\int_{S_1} \Delta w(x)\, d\omega. \tag{22}$$

For $n = 2$, (19), (20) and (22) imply the differential inequality

$$-\frac{d}{dr}\left[r\frac{dm(r; w)}{dr}\right] \geq r(\log r)^\gamma m(r; p) \tag{23}$$

for $a < r < \infty$. Define $W(r) = m(r;\, w)$. Then $Z(r) = rW'(r)$ is nonincreasing for $r > a$ by (23). Since $w(x) > 0$ by hypothesis, so also $W(r) > 0$ by (20), it follows that $W'(r) > 0$ for all $r > a$; for if

$$W(r) - W(R) = \int_R^r \frac{Z(t)}{t}\, dt \leq Z(R) \log \frac{r}{R},$$

contradicting the positivity of $W(r)$ for all $r > a$. Integration of (23) over (a, r) gives

$$-rW'(r) + aW'(a) \geq \int_a^r t(\log t)^\gamma m(t;\, p)\, dt. \tag{24}$$

Since $W'(r) > 0$ for all $r > a$, (24) implies the conclusion (21) of Theorem 3.3.

Comparison of (21) with (10) or (14) indicates the sharpness of these criteria. In fact, as the theorem below states, condition (21) characterizes equations (2) possessing a positive solution in some exterior domain in R^2 provided the condition

$$\limsup_{r\to\infty} \frac{P_2(r)}{m(r;\, p)} < \infty, \qquad P_2(r) = \max_{|x|=r} p(x) \tag{25}$$

is added to the other hypotheses.

3.4. THEOREM. *Suppose that* $p(x)$ *in* (2) *is nonnegative in an exterior domain* Ω *in* R^2, $p \in C^\lambda_{\mathrm{loc}}(\Omega)$, *and* (25) *is satisfied. Then* (21) *is necessary and sufficient for equation* (2) *to have a positive solution in some exterior domain in* R^2.

Proof. The necessity of (21) is the content of Theorem 3.3 and the sufficiency follows from (14) and (25).

The theorem below is a slight modification of a theorem of Kitamura and Kusano [7], with an alternative proof which seems more direct and elementary.

3.5. THEOREM (*Kitamura and Kusano*). *If assumption* (C) *holds, a necessary condition for* (1) *to have a positive solution in some exterior domain in* R^n, $n \geq 2$, *is*

$$\int^{\infty} rm(r;\, p)\, dr < \infty. \tag{26}$$

Proof. If $u(x)$ is a positive solution of (1) in G_a for some $a > 0$, and $z(x) \equiv 1$ in Lemma 3.1, then (17) reduces to $-\Delta\Phi(u(x)) \geq p(x)$, and (22) implies the differential inequality

$$-\frac{d}{dr}\left[r^{n-1}\frac{dW}{dr}\right] \geq r^{n-1}m(r;\, p), \tag{27}$$

where $W(r) = m(r;\, \Phi(u))$. For $n \geq 3$ the change of variables

$$r = \beta(s) = (\nu s)^{\nu}, \quad h(s) = sW(\beta(s)), \quad \nu = \frac{1}{n-2}$$

transforms (27) into

$$-h''(s) \geq s^{-3}[\beta(s)]^{2n-2}m(\beta(s);\, p). \tag{28}$$

Integration over (A, s_0) gives

$$-h'(s_0) + h'(A) \geq \nu\int_a^{r_0} rm(r;\, p)\, dr, \tag{29}$$

where $a = \beta(A)$, $r_0 = \beta(s_0)$. Since $h'(s)$ is nonincreasing by (28) and $h(s) > 0$ for $s > A$, $h'(s_0) > 0$ for all $s_0 > A$ by a standard argument, and (26) follows from (29).

If $n = 2$, (26) follows directly from integration of (27), but this is unnecessary because of the stronger conclusion (21) of Theorem 3.3.

Condition (26) characterizes equations (1) with a positive solution in some exterior domain in R^n, $n \geq 3$, provided the extra hypothesis

$$\limsup_{r\to\infty} \frac{g(r, c)}{m(r;\, p)} < \infty, \tag{30}$$

for some constant $c > 0$, is adjoined to hypotheses (B) and (C).

3.6 THEOREM. *If* (A), (B), (C), *and* (30) *hold, then* (26) *is necessary and sufficient for* (1) *to have a positive solution in an exterior domain in* R^n, $n \geq 3$.

Proof. Theorem 3.5 establishes the necessity of (26), and Theorem 2.4 establishes the sufficiency in view of (30).

If (1) is specialized to (2), condition (30) reduces to (25) and the following corollary results.

3.7 COROLLARY. *Suppose that $p(x)$ is nonnegative in an exterior domain Ω in R^n, $n \geq 3$, $p \in C^{\lambda}_{\mathrm{loc}}(\Omega)$, $0 < \lambda < 1$, and* (25) *is satisfied. Then* (26) *is necessary and sufficient for* (2) *to have a positive solution in some exterior domain in R^n, $n \geq 3$.*

An equivalent statement is that the condition

$$\int^{\infty} rm(r; p)\, dr = +\infty \tag{31}$$

is necessary and sufficient for (2) to be oscillatory in Ω, i.e. for every solution of (2) in $G_a \subset \Omega$ to change sign in G_r for all $r \geq a$. The same applies to (1) under the hypotheses of Theorem 3.6 if the function f in (1) is odd in u, i.e. $f(x, -u) = -f(x, u)$ for all $u > 0$.

The original version of Theorem 3.3 contained a slightly weaker version of condition (21). I am grateful to Professor Takeshi Kura for supplying me with his typescript "Oscillation criteria for a class of semilinear elliptic equations of the second order", indicating the present sharp version.

Added in proof: A variant of Theorem 3.3 has recently been given by Hiroshi Onose, "Oscillation criteria for the sublinear Schrödinger equation", Proc. Amer. Math. Soc., **85** (1982), 69–72.

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Received September 18, 1980. Support from the Natural Sciences and Engineering Research Council of Canada is acknowledged with gratitude.

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