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SUFFICIENCY AND RELATIVE ENTROPY IN $*$-ALGEBRAS WITH APPLICATIONS IN QUANTUM SYSTEMS

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The sufficiency and weak sufficiency in $*$-algebras are discussed. Some properties are studied concerning the relative entropy and the sufficiency for invariant states and KMS states in W^*- and C^*-dynamical systems.

Introduction. The concept of sufficiency is very important in mathematical statistics. The abstract measure theoretic investigation of sufficient statistics was initiated by Halmos and Savage [**13**]. Kullback and Leibler [**19**] gave the characterization of sufficiency in terms of the information (i.e., the classical relative entropy). Umegaki [**33, 34**] studied the sufficiency and the relative entropy in the noncommutative case of semi-finite von Neumann algebras.

Araki [**4, 5**] extended the relative entropy to the case for normal positive linear functionals of general von Neumann algebras and showed its several properties. Furthermore Uhlmann [**32**] showed the general WYDL concavity using a quadratic interpolation theory and defined the relative entropy of positive linear functionals of arbitrary $*$-algebras.

In the previous paper [**14**], we discussed the sufficiency and the relative entropy in von Neumann algebras and gave the characterizations of invariant states and KMS states with respect to the modular automorphism group of a faithful normal state.

In this paper, we further develop the sufficiency and the relative entropy in $*$-algebras. In §1, we introduce besides the sufficiency another notion of weak sufficiency and establish the relation between them. In §2, we deal with the weak sufficiency of positive linear maps between $*$-algebras. In §3, we mention the Araki's and Uhlmann's relative entropies which are equal in the von Neumann algebra case. We further give a formula of relative entropy for states of C^*-algebras. In §4, we establish some properties of invariant states and KMS states in W^*-dynamical systems and C^*-dynamical systems through the relative entropy and the sufficiency. The theorems there improve or extend the results obtained in [**14**]. Finally we give an application to the Gibbs states of quantum lattice systems.

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1. Sufficiency and weak sufficiency of $*$-subalgebras. In this paper, we shall assume that all $*$-algebras, C^*-algebras and von Neumann algebras have the unity I and their $*$-subalgebras always contain I. Let $\mathcal{A}$ be a $*$-algebra and $\mathbb{S}$ be the set of all states of $\mathcal{A}$.

DEFINITION 1.1. A $*$-subalgebra $\mathcal{B}$ of $\mathcal{A}$ is said to be *sufficient* for $S \subset \mathbb{S}$ if there exists a projection ε of $\mathcal{A}$ onto $\mathcal{B}$ such that

(i) $\varepsilon(A^*) = \varepsilon(A)^*$ for all $A \in \mathcal{A}$,
(ii) $\varepsilon(A)^*\varepsilon(A) \leq \varepsilon(A^*A)$ for all $A \in \mathcal{A}$,
(iii) $\varepsilon(B_1AB_2) = B_1\varepsilon(A)B_2$ for all $A \in \mathcal{A}$ and $B_1, B_2 \in \mathcal{B}$,
(iv) $\varphi = \varphi \circ \varepsilon$ for all $\varphi \in S$.

We here call a projection ε of $\mathcal{A}$ onto $\mathcal{B}$ satisfying (i)–(iii) a *conditional expectation* of $\mathcal{A}$ onto $\mathcal{B}$. If $\mathcal{A}$ is a C^*-algebra and $\mathcal{B}$ is a C^*-subalgebra, then a conditional expectation of $\mathcal{A}$ onto $\mathcal{B}$ is nothing but a norm one projection of $\mathcal{A}$ onto $\mathcal{B}$ (cf. [**31**]).

We first give some examples of sufficiency in von Neumann algebras. Let $\mathfrak{N}$ be a von Neumann algebra and $\mathfrak{S}$ be the set of all normal states of $\mathfrak{N}$. The definition in [**14**] of sufficiency of a von Neumann subalgebra for $S \subset \mathfrak{S}$ is somewhat different from Definition 1.1. However these are equivalent if S contains a faithful normal state (this is the case dealt in [**14**]).

EXAMPLE 1.2. Let $\varphi \in \mathfrak{S}$ be faithful and σ_t^φ be its modular automorphism group (cf. [**28**]). We showed in [**14**] that the centralizer of Z_φ of φ is sufficient for the set of all σ_t^φ-invariant states in $\mathfrak{S}$ and the center $\mathfrak{Z} = \mathfrak{N} \cap \mathfrak{N}'$ is sufficient for the set of all states in $\mathfrak{S}$ satisfying the KMS condition with respect to σ_t^φ (at $\beta = 1$).

EXAMPLE 1.3. Assume that $\mathfrak{N}$ is semi-finite with a faithful normal semi-finite trace τ of $\mathfrak{N}$. For each $\varphi \in \mathfrak{S}$, there exists a unique positive self-adjoint operator $\rho_\varphi = d\varphi/d\tau$ such that $\varphi(A) = \tau(\rho_\varphi A)$ for all $A \in \mathfrak{N}$. For any set $S \subset \mathfrak{S}$, the von Neumann subalgebra $\mathfrak{M}$ generated by $\{d\varphi/d\tau\colon \varphi \in S\}$ is proved to be sufficient for S (see [**16**, p. 72]).

EXAMPLE 1.4. Let $\{\mathfrak{N}, G, \alpha\}$ be a W^*-dynamical system where $g \mapsto \alpha_g$ is a representation of a group G in Aut$(\mathfrak{N})$. Let $\mathfrak{N}^\alpha$ be the fixed point subalgebra of α and $\mathfrak{S}_\alpha$ be the set of all α-invariant states in $\mathfrak{S}$. Then the result of Kovács and Szűcs [18] asserts that if $\mathfrak{N}$ is G-finite, i.e., $\varphi(A^*A) = 0$ for all $\varphi \in \mathfrak{S}_\alpha$ implies $A = 0$, then $\mathfrak{N}^\alpha$ is sufficient for $\mathfrak{S}_\alpha$.

For $*$-subalgebras $\mathcal{B}$ of $\mathcal{A}$, the existence of a conditional expectation of $\mathcal{A}$ onto $\mathcal{B}$ is usually a rather strict condition. In the sequel, we introduce another weak notion of sufficiency by using cyclic representations of $\mathcal{A}$. Unbounded $*$-representations of $*$-algebras were studied in [23]. A *$*$-representation* π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is a map of $\mathcal{A}$ into linear operators all defined on a common dense domain $D(\pi) \subset \mathcal{H}$ which satisfies $\pi(I) = I$ and

(i) $\pi(\alpha A + \beta B)\Phi = \alpha\pi(A)\Phi + \beta\pi(B)\Phi$ for all $A, B \in \mathcal{A}$, $\alpha, \beta \in \mathbf{C}$ and $\Phi \in D(\pi)$,

(ii) $\pi(A)D(\pi) \subset D(\pi)$ for all $A \in \mathcal{A}$ and $\pi(A)\pi(B)\Phi = \pi(AB)\Phi$ for all $A, B \in \mathcal{A}$ and $\Phi \in D(\pi)$,

(iii) $\langle \Phi, \pi(A)\Psi \rangle = \langle \pi(A^*)\Phi, \Psi \rangle$ for all $\Phi, \Psi \in D(\pi)$, i.e., $\pi(A^*) \subset \pi(A)^*$ for all $A \in \mathcal{A}$.

The *unbounded commutant* $\pi(\mathcal{A})^c$ of $\pi(\mathcal{A})$ consists of all linear operators T: $D(\pi) \to \mathcal{H}$ such that

$$\langle \Phi, T\pi(A)\Psi \rangle = \langle \pi(A^*)\Phi, T\Psi \rangle, \qquad A \in \mathcal{A}, \Phi, \Psi \in D(\pi).$$

The *commutant* $\pi(\mathcal{A})'$ of $\pi(\mathcal{A})$ is the set of all bounded operators T on $\mathcal{H}$ such that $T \upharpoonright D(\pi) \in \pi(\mathcal{A})^c$. For each positive linear functional φ of $\mathcal{A}$, the GNS construction gives rise to a cyclic representation $\{\mathcal{H}_\varphi, \pi_\varphi, \Omega_\varphi\}$ of $\mathcal{A}$ induced by φ which is unique up to unitary equivalence, that is, π_φ is a $*$-representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}_\varphi$ with $\Omega_\varphi \in D(\pi_\varphi)$ such that

$$D(\pi_\varphi) = \pi_\varphi(\mathcal{A})\Omega_\varphi, \qquad \mathcal{H}_\varphi = \overline{\pi_\varphi(\mathcal{A})\Omega_\varphi},$$
$$\varphi(A) = \langle \Omega_\varphi, \pi_\varphi(A)\Omega_\varphi \rangle, \qquad A \in \mathcal{A}.$$

If for every $A \in \mathcal{A}$ there exists a $c > 0$ with $A^*A \le cI$ (particularly if $\mathcal{A}$ is a C^*-algebra), then π_φ becomes a bounded $*$-representation of $\mathcal{A}$ on $\mathcal{H}_\varphi$. We shall use in this paper the following three conditions of absolute continuity.

(1) A positive linear functional ψ is *absolutely continuous* with respect to φ (we write $\psi \ll \varphi$) if $\varphi(A^*A) = 0$ implies $\psi(A^*A) = 0$.

(2) A linear functional ψ is *strongly absolutely continuous* with respect to φ (we write $\psi \prec \varphi$) if for each sequence $\{A_n\}$ in $\mathcal{A}$, $\varphi(A_n^*A_n) \to 0$ implies $\psi(BA_n) \to 0$ for all $B \in \mathcal{A}$.

(3) A positive linear functional ψ is *dominated* by φ if $\psi \le c\varphi$ for some $c > 0$.

Note that for any positive ψ, (3) implies (2) and (2) implies (1). If ψ is a linear functional of $\mathcal{A}$ with $\psi \prec \varphi$, then by [11, Theorem 1] there exists a unique $T \in \pi_\varphi(\mathfrak{A})^c$ (we denote by $T = d\psi/d\varphi$) such that

$$\psi(A) = \langle T\Omega_\varphi, \pi_\varphi(A)\Omega_\varphi \rangle, \qquad A \in \mathfrak{A}.$$

Then ψ is positive if and only if T is positive, and moreover ψ is dominated by φ if and only if T is bounded so that $T \in \pi_\varphi(\mathfrak{A})'$. For each $*$-subalgebra $\mathfrak{B}$ of $\mathfrak{A}$, let $\mathfrak{B}_\varphi = \pi_\varphi(\mathfrak{B})\Omega_\varphi$ and $\overline{\mathfrak{B}_\varphi} = \overline{\pi_\varphi(\mathfrak{B})\Omega_\varphi}$. For every $A \in \mathfrak{A}$, we define a vector $P_\varphi(A \mid \mathfrak{B})$ in $\overline{\mathfrak{B}_\varphi}$ by

$$P_\varphi(A \mid \mathfrak{B}) = P_{\overline{\mathfrak{B}_\varphi}}(\pi_\varphi(A)\Omega_\varphi)$$

where $P_{\overline{\mathfrak{B}_\varphi}}$ is the orthogonal projection onto $\overline{\mathfrak{B}_\varphi}$.

DEFINITION 1.5. A $*$-subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ is said to be *weakly sufficient* for $S \subset \mathfrak{S}$ if for each $A \in \mathfrak{A}$ there exists a sequence $\{B_n\}$ in $\mathfrak{B}$ such that

$$P_\varphi(A \mid \mathfrak{B}) = s\text{-}\lim \pi_\varphi(B_n)\Omega_\varphi, \qquad \varphi \in S.$$

THEOREM 1.6. *Assume that there is a finite subset $\{\varphi_1, \ldots, \varphi_k\}$ of S such that every $\varphi \in S$ is dominated by $\rho = \sum_{i=1}^k \varphi_i$. Then a $*$-subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ is weakly sufficient for S if and only if $(d\varphi/d\rho)\mathfrak{B}_\rho \subset \overline{\mathfrak{B}_\rho}$ for every $\varphi \in S$.*

Proof. Suppose that $\mathfrak{B}$ is weakly sufficient for S. For each $A \in \mathfrak{A}$, there exists a sequence $\{B_n\}$ in $\mathfrak{B}$ such that

$$P_\varphi(A \mid \mathfrak{B}) = s\text{-}\lim \pi_\varphi(B_n)\Omega_\varphi, \qquad \varphi \in S.$$

Since $\{\pi_\rho(B_n)\Omega_\rho\}$ is Cauchy, it follows that $\Psi = s\text{-}\lim \pi_\rho(B_n)\Omega_\rho$ exists in $\overline{\mathfrak{B}_\rho}$. If $B \in \mathfrak{B}$, then we have

$$\begin{aligned}
\|\pi_\rho(A)\Omega_\rho - \pi_\rho(B)\Omega_\rho\|^2 &= \sum_{i=1}^k \|\pi_{\varphi_i}(A)\Omega_{\varphi_i} - \pi_{\varphi_i}(B)\Omega_{\varphi_i}\|^2 \\
&\geq \sum_{i=1}^k \|\pi_{\varphi_i}(A)\Omega_{\varphi_i} - P_{\varphi_i}(A \mid \mathfrak{B})\|^2 \\
&= \lim \sum_{i=1}^k \|\pi_{\varphi_i}(A)\Omega_{\varphi_i} - \pi_{\varphi_i}(B_n)\Omega_{\varphi_i}\|^2 \\
&= \lim \|\pi_\rho(A)\Omega_\rho - \pi_\rho(B_n)\Omega_\rho\|^2 \\
&= \|\pi_\rho(A)\Omega_\rho - \Psi\|^2,
\end{aligned}$$

so that $P_\rho(A \mid \mathfrak{B}) = \Psi = s\text{-}\lim \pi_\rho(B_n)\Omega_\rho$. For each $\varphi \in S$, let $T = d\varphi/d\rho$ and $\hat{T} = d(\varphi \restriction \mathfrak{B})/d(\rho \restriction \mathfrak{B})$ where the cyclic representation of $\mathfrak{B}$ induced by $\rho \restriction \mathfrak{B}$ is given by $\{\overline{\mathfrak{B}_\rho}, \pi_\rho \restriction \mathfrak{B}, \Omega_\rho\}$. Then for every $B \in \mathfrak{B}$ we have

$$\begin{aligned}\langle T\pi_\rho(B)\Omega_\rho, \pi_\rho(A)\Omega_\rho\rangle = \varphi(B^*A) &= \langle \pi_\varphi(B)\Omega_\varphi, P_\varphi(A \mid \mathfrak{B})\rangle \\ &= \lim \varphi(B^*B_n) = \langle \hat{T}\pi_\rho(B)\Omega_\rho, P_\rho(A \mid \mathfrak{B})\rangle \\ &= \langle \hat{T}\pi_\rho(B)\Omega_\rho, \pi_\rho(A)\Omega_\rho\rangle.\end{aligned}$$

Since this holds for each $A \in \mathfrak{A}$, we obtain

$$T\pi_\rho(B)\Omega_\rho = \hat{T}\pi_\rho(B)\Omega_\rho \in \overline{\mathfrak{B}_\rho}, \qquad B \in \mathfrak{B},$$

and hence $T\mathfrak{B}_\rho \subset \overline{\mathfrak{B}_\rho}$.

Conversely suppose that $(d\varphi/d\rho)\mathfrak{B}_\rho \subset \overline{\mathfrak{B}_\rho}$ for all $\varphi \in S$. Let $A \in \mathfrak{A}$ and take a sequence $\{B_n\}$ in $\mathfrak{B}$ such that $P_\rho(A \mid \mathfrak{B}) = s\text{-}\lim \pi_\rho(B_n)\Omega_\rho$. For each $\varphi \in S$, since φ is dominated by ρ, it follows that $\{\pi_\varphi(B_n)\Omega_\varphi\}$ is Cauchy, so that $\Phi = s\text{-}\lim \pi_\varphi(B_n)\Omega_\varphi$ exists in $\overline{\mathfrak{B}_\varphi}$. If $B \in \mathfrak{B}$, then we have

$$\begin{aligned}\langle \pi_\varphi(B)\Omega_\varphi, P_\varphi(A \mid \mathfrak{B})\rangle = \varphi(B^*A) &= \langle (d\varphi/d\rho)\pi_\rho(B)\Omega_\rho, P_\rho(A \mid \mathfrak{B})\rangle \\ &= \lim \varphi(B^*B_n) = \langle \pi_\varphi(B)\Omega_\varphi, \Phi\rangle,\end{aligned}$$

and hence $P_\varphi(A \mid \mathfrak{B}) = \Phi = s\text{-}\lim \pi_\varphi(B_n)\Omega_\varphi$. Thus $\mathfrak{B}$ is weakly sufficient for S. □

REMARK. Theorem 1.6 is considered as the noncommutative extension of Halmos-Savage's theorem [**13**]. For the proof of "only if" part of Theorem 1.6, we need only $\varphi \prec \rho$ for every $\varphi \in S$. If π_ρ is a bounded $*$-representation (particularly if $\mathfrak{A}$ is a C^*-algebra), we see that $(d\varphi/d\rho)\mathfrak{B}_\rho \subset \overline{\mathfrak{B}_\rho}$ is equivalent to $(d\varphi/d\rho)\Omega_\rho \in \overline{\mathfrak{B}_\rho}$ since $T\pi_\rho(A)\Omega_\rho = \pi_\rho(A)T\Omega_\rho$ for all $A \in \mathfrak{A}$ and $T \in \pi_\rho(\mathfrak{A})^c$.

In the following theorem, we state the elementary facts of weak sufficiency which are immediately seen from the definition and Theorem 1.6.

THEOREM 1.7. (1) *If a $*$-subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ is weakly sufficient for $\{\varphi, \psi\}$ and $\varphi = \psi$ on $\mathfrak{B}$, then $\varphi = \psi$ on $\mathfrak{A}$.*

When the assumption in Theorem 1.6 *is satisfied, then*:

(2) *If a $*$-subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ is weakly sufficient for S, then $\mathfrak{B}$ is weakly sufficient for the convex hull of S.*

(3) *If a $*$-subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ is weakly sufficient for S and a $*$-subalgebra $\mathfrak{C}$ of $\mathfrak{B}$ is weakly sufficient for $\{\varphi \restriction \mathfrak{B}: \varphi \in S\}$, then $\mathfrak{C}$ is weakly sufficient for S.*

(4) *If a $*$-subalgebra $\mathfrak{B}$ of a C^*-algebra $\mathfrak{A}$ is weakly sufficient for S, then any $*$-subalgebra $\mathfrak{C}$ with $\mathfrak{B} \subset \mathfrak{C} \subset \mathfrak{A}$ is weakly sufficient for S.*

THEOREM 1.8. (1) *If a $*$-subalgebra $\mathcal{B}$ of $\mathcal{A}$ is sufficient for S, then $\mathcal{B}$ is weakly sufficient for S.*

(2) *Assume that there is a $\varphi \in S$ such that $\psi \prec \varphi$ for all $\psi \in S$. Then a $*$-subalgebra $\mathcal{B}$ of $\mathcal{A}$ is sufficient for S if and only if $\mathcal{B}$ is weakly sufficient for S and there exists a conditional expectation ε_φ of $\mathcal{A}$ onto $\mathcal{B}$ with $\varphi = \varphi \circ \varepsilon_\varphi$.*

Proof. (1) Let ε be a conditional expectation of $\mathcal{A}$ onto $\mathcal{B}$ with $\varphi = \varphi \circ \varepsilon$ for all $\varphi \in S$. If $A \in \mathcal{A}$, $B \in \mathcal{B}$ and $\varphi \in S$, then we have

$$\begin{aligned}\left\langle P_\varphi(A \mid \mathcal{B}), \pi_\varphi(B)\Omega_\varphi \right\rangle &= \varphi(A^*B) = \varphi(\varepsilon(A)^*B) \\ &= \left\langle \pi_\varphi(\varepsilon(A))\Omega_\varphi, \pi_\varphi(B)\Omega_\varphi \right\rangle,\end{aligned}$$

and hence $P_\varphi(A \mid \mathcal{B}) = \pi_\varphi(\varepsilon(A))\Omega_\varphi$. Thus $\mathcal{B}$ is weakly sufficient for S.

(2) Suppose that $\mathcal{B}$ is weakly sufficient for S and there exists a conditional expectation ε_φ of $\mathcal{A}$ onto $\mathcal{B}$ with $\varphi = \varphi \circ \varepsilon_\varphi$. We show that $\psi = \psi \circ \varepsilon_\varphi$ for all $\psi \in S$. For each $\psi \in S$, since $(d\psi/d\varphi)\Omega_\varphi \in \overline{\mathcal{B}_\varphi}$ by Theorem 1.6 (Remark), we can choose $\{B_n\}$ in $\mathcal{B}$ such that

$$(d\psi/d\varphi)\Omega_\varphi = s\text{-}\lim \pi_\varphi(B_n)\Omega_\varphi.$$

Then $\psi = \psi \circ \varepsilon_\varphi$ follows from

$$\begin{aligned}\psi(\varepsilon_\varphi(A)) &= \left\langle (d\psi/d\varphi)\Omega_\varphi, \pi_\varphi(\varepsilon_\varphi(A))\Omega_\varphi \right\rangle \\ &= \lim \left\langle \pi_\varphi(B_n)\Omega_\varphi, \pi_\varphi(\varepsilon_\varphi(A))\Omega_\varphi \right\rangle = \lim \varphi(B_n^* \varepsilon_\varphi(A)) \\ &= \lim \varphi(B_n^* A) = \left\langle (d\psi/d\varphi)\Omega_\varphi, \pi_\varphi(A)\Omega_\varphi \right\rangle = \psi(A), \qquad A \in \mathcal{A}. \quad \square\end{aligned}$$

EXAMPLE 1.9. We recall the usual concept of sufficiency in the classical probability theory (cf. [7, **13**]). Let $(X, \mathcal{F})$ be a measurable space and S be a set of probability measures on $\mathcal{F}$. A σ-subalgebra $\mathcal{G}$ of $\mathcal{F}$ is sufficient for S if and only if for each $A \in \mathcal{F}$ there exists a $\mathcal{G}$-measurable function g such that $g = E_\mu(1_A \mid \mathcal{G})$ a.e. $[\mu]$ for every $\mu \in S$, where $E_\mu(1_A \mid \mathcal{G})$ denotes the conditional expectation of the characteristic function 1_A of A with respect to μ and $\mathcal{G}$. Let $\mathcal{A}$ (resp. $\mathcal{B}$) be the set of all complex-valued $\mathcal{F}$(resp. $\mathcal{G}$)-measurable simple functions. Under the pointwise operations, $\mathcal{A}$ becomes a $*$-algebra and $\mathcal{B}$ is a $*$-subalgebra of $\mathcal{A}$. Each $\mu \in S$ is naturally regarded as a state of $\mathcal{A}$. The cyclic representation $\{\mathcal{H}_\mu, \pi_\mu, \Omega_\mu\}$ is given as follows: $\mathcal{H}_\mu = L^2(X, \mathcal{F}, \mu)$, $\pi_\mu(f)$ is the multiplication operator by $f \in \mathcal{A}$, and $\Omega_\mu = 1$. Moreover $\overline{\mathcal{B}_\mu} = L^2(X, \mathcal{G}, \mu)$ and $P_\mu(f \mid \mathcal{B}) = E_\mu(f \mid \mathcal{G})$. Then it is easy to see that if S is dominated, i.e., there is a measure m on $\mathcal{F}$ with $\mu \ll m$ for all $\mu \in S$, then $\mathcal{G}$ is sufficient for S if and only if $\mathcal{B}$ is weakly sufficient for S.

EXAMPLE 1.10. Let $\mathfrak{N}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ with a cyclic and separating vector Ω with $\|\Omega\| = 1$, and φ be a faithful normal state given by $\varphi(A) = \langle \Omega, A\Omega \rangle$. For each von Neumann subalgebra $\mathfrak{M}$ of $\mathfrak{N}$, let S be the set of all states ψ defined by $\psi(A) = \langle T\Omega, A\Omega \rangle$ with $T \in \mathfrak{N}'_+$, $T\Omega \in \overline{\mathfrak{M}\Omega}$ and $\|T^{1/2}\Omega\| = 1$. Then it follows from Theorem 1.6 that $\mathfrak{M}$ is weakly sufficient for S. Furthermore Theorem 1.8 shows that $\mathfrak{M}$ is sufficient for S if and only if there exists a conditional expectation ε_φ of $\mathfrak{N}$ onto $\mathfrak{M}$ with $\varphi = \varphi \circ \varepsilon_\varphi$, which is if and only if $\mathfrak{M}$ is invariant under the modular automorphism group σ_t^φ (cf. [**29**]).

2. Weak sufficiency of positive linear maps. In this section, let $\mathcal{A}$ and $\mathcal{B}$ be two $*$-algebras and γ: $\mathcal{B} \to \mathcal{A}$ be a linear map such that $\gamma(I) = I$, $\gamma(B^*) = \gamma(B)^*$ and $\gamma(B)^*\gamma(B) \le \gamma(B^*B)$ for all $B \in \mathcal{B}$. We also assume that for every $B \in \mathcal{B}$ there is a $c > 0$ with $B^*B \le cI$, which is satisfied if $\mathcal{B}$ is a C^*-algebra. Let $\mathbb{S}_\mathcal{A}$ and $\mathbb{S}_\mathcal{B}$ be the sets of all states of $\mathcal{A}$ and $\mathcal{B}$. Then it is immediate that $\varphi \in \mathbb{S}_\mathcal{A}$ implies $\varphi \circ \gamma \in \mathbb{S}_\mathcal{B}$. For each $\varphi \in \mathbb{S}_\mathcal{A}$ and $A \in \mathcal{A}$, define a linear functional φ_A of $\mathcal{A}$ by $\varphi_A(A_1) = \varphi(A^*A_1)$. Then we have $\varphi_A \circ \gamma \prec \varphi \circ \gamma$ since

$$\begin{aligned} |(\varphi_A \circ \gamma)(BB_1)| &= |\varphi(A^*\gamma(BB_1))| \\ &\le \varphi(A^*A)^{1/2}\varphi(\gamma(BB_1)^*\gamma(BB_1))^{1/2} \\ &\le \varphi(A^*A)^{1/2}\varphi(\gamma(B_1^*B^*BB_1))^{1/2} \\ &\le \varphi(A^*A)^{1/2}c^{1/2}(\varphi \circ \gamma)(B_1^*B_1)^{1/2} \end{aligned}$$

for every B, $B_1 \in \mathcal{B}$ where $B^*B \le cI$. Therefore $d(\varphi_A \circ \gamma)/d(\varphi \circ \gamma) \in \pi_{\varphi \circ \gamma}(\mathcal{B})^c$ is defined.

DEFINITION 2.1. We call γ to be *weakly sufficient* for S if for each $A \in \mathcal{A}$ there exists a sequence $\{B_n\}$ in $\mathcal{B}$ such that

$$[d(\varphi_A \circ \gamma)/d(\varphi \circ \gamma)]\Omega_{\varphi \circ \gamma} = s\text{-}\lim \pi_{\varphi \circ \gamma}(B_n)\Omega_{\varphi \circ \gamma}, \qquad \varphi \in S.$$

Definition 2.1 is compatible with Definition 1.5. Indeed we have

THEOREM 2.2. *Let* γ: $\mathcal{B} \to \mathcal{A}$ *be a $*$-homomorphism. Then* γ *is weakly sufficient for* $S \subset \mathbb{S}_\mathcal{A}$ *if and only if the $*$-subalgebra* $\gamma\mathcal{B}$ *of* $\mathcal{A}$ *is weakly sufficient for* S.

Proof. If $\{\mathcal{H}_\varphi, \pi_\varphi, \Omega_\varphi\}$ is the cyclic representation of $\mathcal{A}$ induced by $\varphi \in \mathbb{S}_\mathcal{A}$, then the cyclic representation of $\mathcal{B}$ induced by $\varphi \circ \gamma$ is obtained by $\{\overline{(\gamma\mathfrak{B})_\varphi}, \pi_\varphi \circ \gamma, \Omega_\varphi\}$. Now it suffices to show that

$$[d(\varphi_A \circ \gamma)/d(\varphi \circ \gamma)]\Omega_\varphi = P_\varphi(A \mid \gamma\mathfrak{B}), \qquad \varphi \in \mathfrak{S}_{\mathfrak{A}}, A \in \mathfrak{A}.$$

This follows from

$$\begin{aligned}\left\langle [d(\varphi_A \circ \gamma)/d(\varphi \circ \gamma)]\Omega_\varphi, \pi_\varphi(\gamma B)\Omega_\varphi \right\rangle &= (\varphi_A \circ \gamma)(B) \\ &= \varphi(A^*(\gamma B)) = \left\langle \pi_\varphi(A)\Omega_\varphi, \pi_\varphi(\gamma B)\Omega_\varphi \right\rangle \\ &= \left\langle P_\varphi(A \mid \gamma\mathfrak{B}), \pi_\varphi(\gamma B)\Omega_\varphi \right\rangle, \qquad B \in \mathfrak{B}. \qquad \square\end{aligned}$$

We assume further that $\mathfrak{A}$ is abelian and $\gamma\colon \mathfrak{B} \to \mathfrak{A}$ is completely positive, i.e.,

$$\sum_{i,j=1}^{n} A_i^*\gamma(B_i^*B_j)A_j \geq 0$$

for every $A_1,\ldots,A_n \in \mathfrak{A}$ and $B_1,\ldots,B_n \in \mathfrak{B}$. Note (see [**30**, IV. 3]) that when $\mathfrak{A}$ and $\mathfrak{B}$ are C^*-algebras, any completely positive map $\gamma\colon \mathfrak{B} \to \mathfrak{A}$ with $\gamma(I) = I$ satisfies automatically $\gamma(B)^*\gamma(B) \leq \gamma(B^*B)$ for all $B \in \mathfrak{B}$, and any positive linear map $\gamma\colon \mathfrak{B} \to \mathfrak{A}$ is completely positive if either $\mathfrak{A}$ or $\mathfrak{B}$ is abelian. Let $\mathfrak{A} \otimes \mathfrak{B}$ be the $*$-algebraic tensor product of $\mathfrak{A}$ and $\mathfrak{B}$. For each $\varphi \in \mathfrak{S}_{\mathfrak{A}}$, we can define the compound state $\varphi \otimes \gamma$ of $\mathfrak{A} \otimes \mathfrak{B}$ by

$$(\varphi \otimes \gamma)(A \otimes B) = (\varphi_{A^*} \circ \gamma)(B) = \varphi(A(\gamma B)), \qquad A \in \mathfrak{A}, B \in \mathfrak{B},$$

since

$$(\varphi \otimes \gamma)\left(\left(\sum_{i=1}^{n} A_i \otimes B_i\right)^*\left(\sum_{i=1}^{n} A_i \otimes B_i\right)\right) = \varphi\left(\sum_{i,j=1}^{n} A_i^*\gamma(B_i^*B_j)A_j\right) \geq 0.$$

Identifying $\mathfrak{A}$ and $\mathfrak{B}$ with $*$-subalgebras $\mathfrak{A} \otimes I$ and $I \otimes \mathfrak{B}$ of $\mathfrak{A} \otimes \mathfrak{B}$, we then have

THEOREM 2.3. (1) *$\mathfrak{A}$ is sufficient for $\{\varphi \otimes \gamma\colon \varphi \in \mathfrak{S}_{\mathfrak{A}}\}$.*

(2) *γ is weakly sufficient for $S \subset \mathfrak{S}_{\mathfrak{A}}$ if and only if $\mathfrak{B}$ is weakly sufficient for $\{\varphi \otimes \gamma\colon \varphi \in S\}$.*

Proof. (1) Define $\varepsilon\colon \mathfrak{A} \otimes \mathfrak{B} \to \mathfrak{A}$ by $\varepsilon(A \otimes B) = A\gamma(B), A \in \mathfrak{A}, B \in \mathfrak{B}$. Since γ is completely positive and $\mathfrak{A}$ is abelian, it follows that ε is a conditional expectation of $\mathfrak{A} \otimes \mathfrak{B}$ onto $\mathfrak{A}$. Hence (1) is seen from $(\varphi \otimes \gamma) \circ \varepsilon = \varphi \otimes \gamma$ for all $\varphi \in \mathfrak{S}_{\mathfrak{A}}$.

(2) For $\varphi \in \mathfrak{S}_{\mathfrak{A}}$, let $\{\mathcal{H}_{\tilde\varphi}, \pi_{\tilde\varphi}, \Omega_{\tilde\varphi}\}$ be the cyclic representation of $\mathfrak{A} \otimes \mathfrak{B}$ induced by $\tilde\varphi = \varphi \otimes \gamma$. Since $\tilde\varphi \restriction \mathfrak{B} = \varphi \circ \gamma$, the cyclic representation of $\mathfrak{B}$ induced by $\varphi \circ \gamma$ is given by $\{\overline{\mathfrak{B}_{\tilde\varphi}}, \pi_{\tilde\varphi} \restriction \mathfrak{B}, \Omega_{\tilde\varphi}\}$. Let $A \in \mathfrak{A}$,

$B \in \mathfrak{B}$ and $T = d(\varphi_A \circ \gamma)/d(\varphi \circ \gamma) \in \pi_{\tilde{\varphi}}(\mathfrak{B})^c$. It follows that

$$\begin{aligned} P_{\tilde{\varphi}}(A \otimes B \mid \mathfrak{B}) &= \left[d(\tilde{\varphi}_{A \otimes B} \restriction \mathfrak{B})/d(\tilde{\varphi} \restriction \mathfrak{B})\right]\Omega_{\tilde{\varphi}} \\ &= \left[d(\varphi_A \circ \gamma)_B/d(\varphi \circ \gamma)\right]\Omega_{\tilde{\varphi}} = T\pi_{\tilde{\varphi}}(B)\Omega_{\tilde{\varphi}}, \end{aligned}$$

where the first equality is a special case of the equation in the proof of Theorem 2.2 and the last equality is seen from

$$\begin{aligned} &\left\langle [d(\varphi_A \circ \gamma)_B/d(\varphi \circ \gamma)]\Omega_{\tilde{\varphi}}, \pi_{\tilde{\varphi}}(B_1)\Omega_{\tilde{\varphi}} \right\rangle \\ &\qquad = (\varphi_A \circ \gamma)(B^*B_1) = \left\langle T\pi_{\tilde{\varphi}}(B)\Omega_{\tilde{\varphi}}, \pi_{\tilde{\varphi}}(B_1)\Omega_{\tilde{\varphi}} \right\rangle, \qquad B_1 \in \mathfrak{B}. \end{aligned}$$

The "if" part of (2) is now immediate by taking $B = I$. Conversely if γ is weakly sufficient for S, then there exists a sequence $\{B_n\}$ in $\mathfrak{B}$ such that

$$T\Omega_{\tilde{\varphi}} = s\text{-}\lim \pi_{\tilde{\varphi}}(B_n)\Omega_{\tilde{\varphi}}, \qquad \varphi \in S.$$

Since $\pi_{\tilde{\varphi}}(B)$ is bounded from $B^*B \leq cI$, we have

$$P_{\tilde{\varphi}}(A \otimes B \mid \mathfrak{B}) = \pi_{\tilde{\varphi}}(B)T\Omega_{\tilde{\varphi}} = s\text{-}\lim \pi_{\tilde{\varphi}}(BB_n)\Omega_{\tilde{\varphi}}, \qquad \varphi \in S.$$

Hence $\mathfrak{B}$ is weakly sufficient for $\{\varphi \otimes \gamma \colon \varphi \in S\}$. □

EXAMPLE 2.4. Let $(X, \mathfrak{F})$ and $(Y, \mathfrak{G})$ be two measurable spaces and ν be a channel distribution from $(X, \mathfrak{F})$ to $(Y, \mathfrak{G})$, i.e., ν is a real-valued function on $X \times \mathfrak{G}$ such that for every $x \in X$, $\nu(x, \cdot)$ is a probability measure on $\mathfrak{G}$ and for every $B \in \mathfrak{G}$, $\nu(\cdot, B)$ is $\mathfrak{F}$-measurable on X. Let $\mathfrak{B}(X)$ and $\mathfrak{B}(Y)$ be the abelian C^*-algebras of bounded complex-valued measurable functions on X and Y. Define a positive linear map γ: $\mathfrak{B}(Y) \to \mathfrak{B}(X)$ by

$$(\gamma g)(x) = \int_Y g(y)\nu(x, dy), \qquad x \in X, g \in \mathfrak{B}(Y).$$

Let S be a set of probability measures on $\mathfrak{F}$. For each $\mu \in S$, $\mu \otimes \gamma$ is given by

$$(\mu \otimes \gamma)(f \otimes g) = \int_{X \times Y} f \otimes g \, d(\mu \otimes \nu), \qquad f \in \mathfrak{B}(X), g \in \mathfrak{B}(Y),$$

where $\mu \otimes \nu$ is the probability measure on $\mathfrak{F} \otimes \mathfrak{G}$ defined by $(\mu \otimes \nu)\times(A \times B) = \int_A \nu(x, B)\, d\mu$. Then we see in connection with Theorem 2.3(2) that γ is weakly sufficient for S if and only if the σ-subalgebra $X \times \mathfrak{G} = \{X \times B \colon B \in \mathfrak{G}\}$ of $\mathfrak{F} \otimes \mathfrak{G}$ is sufficient in the classical sense for $\{\mu \otimes \nu \colon \mu \in S\}$.

EXAMPLE 2.5. Let $\mathfrak{N}$ be a von Neumann algebra. An $\mathfrak{N}$-valued PO-measure M on a measurable space $(X, \mathfrak{F})$ is a map $M: \mathfrak{F} \to \mathfrak{N}$ such that $M(F) \geq 0$ for all $F \in \mathfrak{F}$ and $\sum_{n=1}^{\infty} M(F_n) = I$ (σ-weakly) for every countable measurable partition $\{F_n\}$ of X. Let $\mathfrak{B}(X)$ be the abelian C^*-algebra of bounded measurable functions on X. We define a positive linear map $\gamma: \mathfrak{B}(X) \to \mathfrak{N}$ with $\gamma(1) = I$ by

$$\varphi(\gamma(f)) = \int_X f\, d(\varphi \circ M), \qquad f \in \mathfrak{B}(X), \varphi \in \mathfrak{N}_*.$$

For each $\varphi \in \mathfrak{S}$, the cyclic representation $\{\mathcal{H}_{\varphi \circ \gamma}, \pi_{\varphi \circ \gamma}, \Omega_{\varphi \circ \gamma}\}$ of $\mathfrak{B}(X)$ induced by $\varphi \circ \gamma$ is given as follows $\mathcal{H}_{\varphi \circ \gamma} = L^2(X, \varphi \circ M)$, $\pi_{\varphi \circ \gamma}(f)$ is the multiplication operator by f, and $\Omega_{\varphi \circ \gamma} = 1$. For $A \in \mathfrak{N}$, $d(\varphi_A \circ \gamma)/d(\varphi \circ \gamma)$ is identical to the Radon-Nikodym derivative $d(\varphi_A \circ M)/d(\varphi \circ M)$ which is in $L^2(X, \varphi \circ M)$. Now assume that $\mathfrak{N}$ is σ-finite, so that $\mathfrak{N}$ has a faithful normal state. Then it is proved that γ is weakly sufficient for $S \subset \mathfrak{S}$ if and only if for every $A \in \mathfrak{N}$ there exists a measurable function f on X satisfying

$$d(\varphi_A \circ M)/d(\varphi \circ M) = f \quad \text{a.e. } [\varphi \circ M], \qquad \varphi \in S.$$

Further assume that M is pure, i.e., M is a spectral measure. Then γ is a $*$-homomorphism and $\gamma(\mathfrak{B}(X))$ is equal to the subalgebra $\mathfrak{M} = \{M(F): F \in \mathfrak{F}\}''$. Hence Theorem 2.2 shows that γ is weakly sufficient for S if and only if $\mathfrak{M}$ is weakly sufficient for S.

EXAMPLE 2.6. Let $\mathfrak{A}$ be a C^*-algebra and $C(\mathbb{S})$ be the abelian C^*-algebra of continuous functions on $\mathbb{S}$. Define a positive linear map $\gamma: \mathfrak{A} \to C(\mathbb{S})$ with $\gamma(I) = 1$ by $(\gamma A)(\omega) = \omega(A)$, $A \in \mathfrak{A}$, $\omega \in \mathbb{S}$. For each $\rho \in \mathbb{S}$ and each abelian von Neumann subalgebra $\mathfrak{B}$ of $\pi_\rho(\mathfrak{A})'$, we take the $\mathfrak{B}$-orthogonal measure λ of ρ (cf. [**30**, p. 241]). Now assume that $\mathfrak{A}$ is separable and $\mathfrak{B} \subset \mathfrak{Z}_\rho = \pi_\rho(\mathfrak{A})'' \cap \pi_\rho(\mathfrak{A})'$, i.e., λ is a subcentral measure of ρ, and let S be the set of all Borel probability measures μ on $\mathbb{S}$ with $\mu \ll \lambda$. Then γ is weakly sufficient for S. This is proved as follows. There is a $*$-isomorphism θ of $L^\infty(\mathbb{S}, \lambda)$ onto $\mathfrak{B}$ such that

$$\langle \Omega_\rho, \theta(f)\pi_\rho(A)\Omega_\rho \rangle = \int_{\mathbb{S}} f(\omega)\omega(A)\, d\lambda(\omega), \qquad A \in \mathfrak{A}, f \in L^\infty(\mathbb{S}, \lambda).$$

For each $\mu \in S$ and $f \in C(\mathbb{S})$, taking $g_{\mu n} = \min((d\mu/d\lambda)^{1/2}, n)$ we obtain

$$(\mu \circ \gamma)(A) = \int_{\mathbb{S}} \omega(A)\, d\mu(\omega) = \lim \langle \theta(g_{\mu n})\Omega_\rho, \pi_\rho(A)\theta(g_{\mu n})\Omega_\rho \rangle,$$

$$\begin{aligned}(\mu_f \circ \gamma)(A) &= \int_{\mathbb{S}} \bar{f}(\omega)\omega(A)\, d\mu(\omega) \\ &= \lim \langle \theta(f)\theta(g_{\mu n})\Omega_\rho, \pi_\rho(A)\theta(g_{\mu n})\Omega_\rho \rangle.\end{aligned}$$

Since $\{g_{\mu n}\}$ is Cauchy in $L^2(\mathbb{S}, \lambda)$, it follows that $\Phi_\mu = s\text{-lim}\, \theta(g_{\mu n})\Omega_\rho$ exists and

$$(\mu \circ \gamma)(A) = \langle \Phi_\mu, \pi_\rho(A)\Phi_\mu \rangle,$$

$$(\mu_f \circ \gamma)(A) = \langle \theta(f)\Phi_\mu, \pi_\rho(A)\Phi_\mu \rangle, \qquad A \in \mathcal{A}.$$

Hence the cyclic representation of $\mathcal{A}$ induced by $\mu \circ \gamma$ is given by

$$\left\{ \overline{\pi_\rho(\mathcal{A})\Phi_\mu}, \pi_\rho(\cdot) \restriction \overline{\pi_\rho(\mathcal{A})\Phi_\mu}, \Phi_\mu \right\}$$

and we have

$$[d(\mu_f \circ \gamma)/d(\mu \circ \gamma)]\Phi_\mu = \theta(f)\Phi_\mu.$$

Since $\mathcal{A}$ is separable, there exists a sequence $\{A_n\}$ in $\mathcal{A}$ such that $\pi_\rho(A_n) \to \theta(f)$ (strongly), and hence

$$[d(\mu_f \circ \gamma)/d(\mu \circ \gamma)]\Phi_\mu = s\text{-lim}\, \pi_\rho(A_n)\Phi_\mu, \qquad \mu \in S.$$

This shows that γ is weakly sufficient for S.

A linear map $\gamma: \mathcal{B} \to \mathcal{A}$ considered here describes more or less a quantum communication channel with the input space $\mathcal{A}$ and the output space $\mathcal{B}$ (cf. [**15, 21**]). Examples 2.4–2.6 provide classical-classical, quantum-classical and classical-quantum channels. Roughly speaking, the physical meaning of weak sufficiency of γ is that the indirect measurement through γ gives as much information (measured by the relative entropy) as the direct measurement of observables in $\mathcal{A}$ given a set S of input states (see §§3, 4).

3. Relative entropy of states of $*$-algebras. We begin with the definitions of Araki's relative entropy and Uhlmann's relative entropy.

(I) *Araki's relative entropy*. Let $(\mathfrak{N}, \mathcal{H}, J, \mathcal{P})$ be a standard form of a von Neumann algebra $\mathfrak{N}$ (cf. [**2, 12**]). Araki [**4, 5**] defined the relative entropy of normal positive linear functionals φ and ψ of $\mathfrak{N}$ as follows.

There exist unique vector representatives Φ and Ψ in $\mathcal{P}$ such that $\varphi(A) = \langle \Phi, A\Phi \rangle$ and $\psi(A) = \langle \Psi, A\Psi \rangle$ for all $A \in \mathfrak{N}$. The operator $S_{\Psi,\Phi}$ with the domain

$$D(S_{\Psi,\Phi}) = \mathfrak{N}\Phi + \left(I - s^{\mathfrak{N}'}(\Phi)\right)$$

is defined by

$$S_{\Psi,\Phi}(A\Phi + \Omega) = s^{\mathfrak{N}}(\Phi)A^*\Psi, \qquad A \in \mathfrak{N}, s^{\mathfrak{N}'}(\Phi)\Omega = 0,$$

where $s^{\mathfrak{N}}(\Phi)$ denotes the $\mathfrak{N}$-support of Φ. Then $S_{\Psi,\Phi}$ is a closable conjugate-linear operator and the relative modular operator $\Delta_{\Psi,\Phi}$ is defined by $\Delta_{\Psi,\Phi} = (S_{\Psi,\Phi})^*\overline{S_{\Psi,\Phi}}$. Let $\Delta_{\Psi,\Phi} = \int_0^\infty \lambda\, de_{\Psi,\Phi}(\lambda)$ be the spectral decomposition of $\Delta_{\Psi,\Phi}$. The Araki's relative entropy $S(\psi \mid \varphi)$ is now given by

$$S(\psi \mid \varphi) = \begin{cases} \int_{+0}^{\infty} \log \lambda\, d\langle \Psi, e_{\Psi,\Phi}(\lambda)\Psi \rangle & \text{if } \psi \ll \varphi, \\ +\infty & \text{otherwise.} \end{cases}$$

Note that the relative entropy $S(\psi \mid \varphi)$ is independent of the choice of a standard form of $\mathfrak{N}$ which is unique up to unitary equivalence. We used in [**14**] the notation $S(\varphi \mid \psi)$ instead of $S(\psi \mid \varphi)$.

(II) *Uhlmann's relative entropy.* Let $\mathfrak{L}$ be a complex linear space. Given two seminorms p and q on $\mathfrak{L}$, the quadratical mean $QM(p, q)$ is defined by

$$QM(p, q)(x) = \sup_{\alpha \in H} \alpha(x, x)^{1/2}, \qquad x \in \mathfrak{L},$$

where H is the set of all positive hermitian forms α on $\mathfrak{L}$ satisfying $|\alpha(x, y)| \le p(x)q(y)$ for all $x, y \in \mathfrak{L}$. A function $t \mapsto p_t$ on $[0, 1]$ whose values are seminorms on $\mathfrak{L}$ is called a quadratical interpolation from p to q if for every $x \in \mathfrak{L}$ the function $t \mapsto p_t(x)$ is continuous and if the following properties hold:

$$\begin{aligned} p_t &= QM(p_{t_1}, p_{t_2}), \qquad t = (t_1 + t_2)/2, t_1, t_2 \in [0, 1], \\ p_{1/2} &= QM(p, q), \\ p_{t/2} &= QM(p, p_t), \qquad t \in [0, 1], \\ p_{(1+t)/2} &= QM(q, p_t), \qquad t \in [0, 1]. \end{aligned}$$

Uhlmann [**32**] showed that for each positive hermitian forms α and β there exists a unique function $t \mapsto QF_t(\alpha, \beta)$ on $[0, 1]$ with values in the set of positive hermitian forms on $\mathfrak{L}$ such that the function p_t given by $p_t(x) = QF_t(\alpha, \beta)(x, x)^{1/2}$ is the quadratical interpolation from $\alpha(x, x)^{1/2}$ to $\beta(x, x)^{1/2}$, and defined the relative entropy functional $S(\alpha; \beta)(x)$ of α and β by

$$S(\alpha; \beta)(x) = -\liminf_{t \to +0} \frac{1}{t} \{QF_t(\alpha, \beta)(x, x) - \alpha(x, x)\}, \qquad x \in \mathfrak{L}.$$

Now let $\mathcal{A}$ be a $*$-algebra, and φ and ψ be positive linear functionals of $\mathcal{A}$. The Uhlmann's relative entropy $S(\psi \mid \varphi)$ is defined by

$$S(\psi \mid \varphi) = S(\psi^R; \varphi^L)(I),$$

where φ^L and ψ^R are the positive hermitian forms given by $\varphi^L(A, B) = \varphi(A^*B)$ and $\psi^R(A, B) = \psi(BA^*)$.

For each normal positive linear functionals φ and ψ of a von Neumann algebra $\mathfrak{N}$, the Uhlmann's relative entropy is equal to the Araki's relative entropy. We here contain the proof for completeness.

Let $\mathcal{K}$ be the domain of $(I + \Delta_{\Psi,\Phi})^{1/2}$, which becomes a Hilbert space with an inner product:

$$(\Omega_1, \Omega_2) = \left\langle (I + \Delta_{\Psi,\Phi})^{1/2}\Omega_1, (I + \Delta_{\Psi,\Phi})^{1/2}\Omega_2 \right\rangle, \qquad \Omega_1, \Omega_2 \in \mathcal{K}.$$

The operators $(I + \Delta_{\Psi,\Phi})^{-1}$ and $\Delta_{\Psi,\Phi}(I + \Delta_{\Psi,\Phi})^{-1}$ are positive bounded linear operators on $\mathcal{K}$. Define positive hermitian forms α and β on $\mathcal{K}$ by

$$\alpha(\Omega_1, \Omega_2) = \left(\Omega_1, \Delta_{\Psi,\Phi}(I + \Delta_{\Psi,\Phi})^{-1}\Omega_2\right),$$
$$\beta(\Omega_1, \Omega_2) = \left(\Omega_1, (I + \Delta_{\Psi,\Phi})^{-1}\Omega_2\right).$$

We then have (cf. [**24**], [**32**, Example 4])

$$QF_t(\alpha, \beta)(\Omega, \Omega) = \left(\Omega, \left[\Delta_{\Psi,\Phi}(I + \Delta_{\Psi,\Phi})^{-1}\right]^{1-t}\left[(I + \Delta_{\Psi,\Phi})^{-1}\right]^t\Omega\right)$$
$$= \left\langle \Omega, (\Delta_{\Psi,\Phi})^{1-t}\Omega \right\rangle, \qquad t \in (0, 1), \Omega \in \mathcal{K}.$$

Since $\mathfrak{N}\Phi \subset \mathcal{K}$ and

$$\psi^R(A, B) = \left(A\Phi, \Delta_{\Psi,\Phi}(I + \Delta_{\Psi,\Phi})^{-1}B\Phi\right),$$
$$\varphi^L(A, B) = \left(A\Phi, (I + \Delta_{\Psi,\Phi})^{-1}B\Phi\right),$$

it is easy to check that

$$QF_t(\psi^R, \varphi^L)(A, A) = QF_t(\alpha, \beta)(A\Phi, A\Phi), \qquad A \in \mathfrak{N}.$$

Take the spectral decomposition $\Delta_{\Psi,\Phi} = \int_0^\infty \lambda \, de_{\Psi,\Phi}(\lambda)$. If $\psi \ll \varphi$, then $(\Delta_{\Psi,\Phi})^{1/2}\Phi = JS_{\Psi,\Phi}\Phi = J\Psi = \Psi$ and hence $\psi^R(I, I) = \|(\Delta_{\Psi,\Phi})^{1/2}\Phi\|^2$.

We have

$$\begin{aligned} S(\psi^R; \varphi^L)(I) &= -\liminf_{t\to+0} \frac{1}{t}\left\{\left\langle \Phi, (\Delta_{\Psi,\Phi})^{1-t}\Phi\right\rangle - \left\langle \Phi, \Delta_{\Psi,\Phi}\Phi\right\rangle\right\} \\ &= -\liminf_{t\to+0} \int_{+0}^{\infty} \lambda \frac{\lambda^{-t}-1}{t} d\left\langle \Phi, e_{\Psi,\Phi}(\lambda)\Phi\right\rangle \\ &= \int_{+0}^{\infty} \lambda \log \lambda \, d\left\langle \Phi, e_{\Psi,\Phi}(\lambda)\Phi\right\rangle \\ &= \int_{+0}^{\infty} \log \lambda \, d\left\langle \Psi, e_{\Psi,\Phi}(\lambda)\Psi\right\rangle, \end{aligned}$$

because the function $(\lambda^{-t} - 1)/t$ converges decreasingly to $-\log \lambda$ as $t \to +0$. If $\psi \ll \varphi$ does not hold, then $\psi^R(I, I) < \|(\Delta_{\Psi,\Phi})^{1/2}\Phi\|^2$ and hence $S(\psi^R; \varphi^L)(I) = +\infty$. Thus the Uhlmann's relative entropy is equal to the Araki's one.

LEMMA 3.1. *Let $\mathcal{A}$ be a C^*-algebra and π be a nondegenerate representation of $\mathcal{A}$ on a Hilbert space. If φ and ψ are positive linear functionals of $\mathcal{A}$ having the normal extensions $\tilde{\varphi}$ and $\tilde{\psi}$ to $\pi(\mathcal{A})''$ such that $\varphi(A) = \tilde{\varphi}(\pi(A))$ and $\psi(A) = \tilde{\psi}(\pi(A))$, then $S(\psi \mid \varphi) = S(\tilde{\psi} \mid \tilde{\varphi})$.*

Proof. According to the Uhlmann's definition of relative entropy, it suffices to show that

$$QF_t(\psi^R, \varphi^L)(A, A) = QF_t(\tilde{\psi}^R, \tilde{\varphi}^L)(\pi(A), \pi(A)), \qquad t \in [0,1], A \in \mathcal{A}.$$

Let Γ be the set of $t \in [0, 1]$ for which the above equation holds for every $A \in \mathcal{A}$. Let H be the set of all positive hermitian forms α on $\mathcal{A}$ satisfying

$$|\alpha(A_1, A_2)| \le \psi^R(A_1, A_1)^{1/2}\varphi^L(A_2, A_2)^{1/2}, \qquad A_1, A_2 \in \mathcal{A},$$

and $\tilde{H}$ be the set of all positive hermitian forms $\tilde{\alpha}$ on $\pi(\mathcal{A})''$ satisfying

$$|\tilde{\alpha}(Q_1, Q_2)| \le \tilde{\psi}^R(Q_1, Q_2)^{1/2}\tilde{\varphi}^L(Q_2, Q_2)^{1/2}, \qquad Q_1, Q_2 \in \pi(\mathcal{A})''.$$

If $\tilde{\alpha} \in \tilde{H}$, then the form α on $\mathcal{A}$ defined by $\alpha(A_1, A_2) = \tilde{\alpha}(\pi(A_1), \pi(A_2))$ is in H. Conversely if $\alpha \in H$, then there exists a positive hermitian form $\hat{\alpha}$ on $\pi(\mathcal{A})$ such that $\alpha(A_1, A_2) = \hat{\alpha}(\pi(A_1), \pi(A_2))$ and hence

$$\begin{aligned} &|\hat{\alpha}(\pi(A_1), \pi(A_2))| \\ &\qquad \le \tilde{\psi}^R(\pi(A_1), \pi(A_2))^{1/2}\tilde{\varphi}^L(\pi(A_2), \pi(A_2))^{1/2}, \qquad A_1, A_2 \in \mathcal{A}. \end{aligned}$$

By the Kaplansky density theorem, $\hat{\alpha}$ can be uniquely extended to a positive hermitian form $\tilde{\alpha}$ on $\pi(\mathfrak{A})''$ which is in $\tilde{H}$. Therefore

$$\begin{aligned} QF_{1/2}(\psi^R, \varphi^L)(A, A) &= \sup_{\alpha \in H} \alpha(A, A) \\ &= \sup_{\tilde{\alpha} \in \tilde{H}} \tilde{\alpha}(\pi(A), \pi(A)) \\ &= QF_{1/2}(\tilde{\psi}^R, \tilde{\varphi}^L)(\pi(A), \pi(A)), \qquad A \in \mathfrak{A}. \end{aligned}$$

This implies $1/2 \in \Gamma$. Noting that

$$QF_t(\psi^R, \varphi^L)(A, A) \leq \psi^R(A, A)^{1-t} \varphi^L(A, A)^t, \qquad t \in [0, 1], A \in \mathfrak{A},$$

we can see by the similar arguments that $t \in \Gamma$ implies $t/2 \in \Gamma$ and $(1 + t)/2 \in \Gamma$, and that t_1, $t_2 \in \Gamma$ implies $(t_1 + t_2)/2 \in \Gamma$. Since Γ is closed, we deduce that $\Gamma = [0, 1]$. □

In the above lemma, we can take as π the cyclic representation induced by $\varphi + \psi$ or the universal representation of $\mathfrak{A}$.

We here remark that the relative entropy defined in (I) and (II) contains the usual relative entropies in the classical and quantum systems. Let $(X, \mathcal{F})$ be a measurable space, and μ and ν be probability measures on $\mathcal{F}$. Take a measure m on $\mathcal{F}$ with $\mu, \nu \ll m$. Then μ and ν are naturally regarded as normal states of the abelian von Neumann algebra $\mathfrak{N} = L^\infty(X, m)$ acting on $\mathcal{K} = L^2(X, m)$. Then the relative entropy $S(\nu \mid \mu)$ is equal to the classical relative entropy $I(\nu \mid \mu)$ (known as the Kullback-Leibler information):

$$I(\nu \mid \mu) = \begin{cases} \displaystyle\int \frac{d\nu}{d\mu} \log \frac{d\nu}{d\mu} d\mu & \text{if } \nu \ll \mu, \\ +\infty & \text{otherwise.} \end{cases}$$

Indeed, $\Phi = (d\mu/dm)^{1/2}$ and $\Psi = (d\nu/dm)^{1/2}$ are vector representatives for μ and ν, and $\Delta_{\Psi,\Phi}$ is the multiplication operator by $1_{\mathrm{supp}\Phi}(\Psi/\Phi)^2$ where $1_{\mathrm{supp}\Phi}$ is the characteristic function of the support of Φ. If $\nu \ll \mu$, then we have

$$\begin{aligned} S(\nu \mid \mu) &= \int \Psi^2 1_{\mathrm{supp}\Phi} \log(\Psi/\Phi)^2 \, dm \\ &= \int \frac{d\nu}{dm}\left(\log \frac{d\nu}{dm} - \log \frac{d\mu}{dm}\right) dm = I(\nu \mid \mu). \end{aligned}$$

Next let φ and ψ be normal states of the full von Neumann algebra $\mathfrak{N} = \mathbf{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$. Then φ and ψ are given by $\varphi(A) = \mathrm{Tr}(\rho_\varphi A)$ and $\psi(A) = \mathrm{Tr}(\rho_\psi A)$ with positive trace class operators ρ_φ and ρ_ψ on $\mathcal{H}$, and we obtain

$$S(\psi \mid \varphi) = \mathrm{Tr}\big(\rho_\psi \log \rho_\psi - \rho_\psi \log \rho_\varphi\big).$$

The relative entropy $S(\psi \mid \varphi)$ has several basic properties such as joint convexity, monotonicity, lower semicontinuity, etc. (cf. [**4**, **5**, **32**]). The monotonicity is stated as follows (cf. [**32**, Proposition 18]). Let $\mathcal{Q}$ and $\mathfrak{B}$ be $*$-algebras and γ: $\mathfrak{B} \to \mathcal{Q}$ be a linear map such that $\gamma(I) = I$, $\gamma(B^*) = \gamma(B)^*$ and $\gamma(B)^*\gamma(B) \le \gamma(B^*B)$ for all $B \in \mathfrak{B}$. If φ and ψ are positive linear functionals on $\mathcal{Q}$, then

$$S(\psi \circ \gamma \mid \varphi \circ \gamma) \le S(\psi \mid \varphi).$$

This monotonicity is applied to positive linear maps such as in Examples 2.4–2.6. Particularly if $\mathfrak{B}$ is a $*$-subalgebra of $\mathcal{Q}$, then we have $S_{\mathfrak{B}}(\psi \mid \varphi) \le S(\psi \mid \varphi)$ where $S_{\mathfrak{B}}(\psi \mid \varphi)$ denotes the relative entropy of the restrictions $\varphi \restriction \mathfrak{B}$ and $\psi \restriction \mathfrak{B}$.

In connection with Example 2.6, it is proved that the relative entropy of states of a C^*-algebra is equal to that of their decomposition measures in some cases.

THEOREM 3.2. *Let $\mathcal{Q}$ be a C^*-algebra and μ, ν be regular Borel probability measures on $\mathcal{S}$ with barycenters $\varphi, \psi \in \mathcal{S}$. If there is a subcentral measure λ on $\mathcal{S}$ such that $\mu, \nu \ll \lambda$, then $S(\psi \mid \varphi) = I(\nu \mid \mu)$.*

Proof. Let λ be the $\mathfrak{B}$-orthogonal measure of $\rho \in \mathcal{S}$ with an abelian von Neumann subalgebra $\mathfrak{B}$ of $\mathfrak{Z}_\rho = \pi_\rho(\mathcal{Q})'' \cap \pi_\rho(\mathcal{Q})'$, and θ be the $*$-isomorphism of $L^\infty(\mathcal{S}, \lambda)$ onto $\mathfrak{B}$ such that

$$\big\langle \Omega_\rho, \theta(f)\pi_\rho(A)\Omega_\rho \big\rangle = \int_{\mathcal{S}} f(\omega)\omega(A)\, d\lambda(\omega), \qquad A \in \mathcal{Q}, f \in L^\infty(\mathcal{S}, \lambda).$$

As is seen in Example 2.6, there exists a $\Phi_\mu \in \mathcal{H}_\rho$ such that $\varphi(A) = \langle \Phi_\mu, \pi_\rho(A)\Phi_\mu \rangle$ for all $A \in \mathcal{Q}$. Hence φ has the normal extension $\tilde{\varphi}$ to $\pi_\rho(\mathcal{Q})''$ and it is easily checked that

$$\tilde{\varphi}(\theta(f)) = \int_{\mathcal{S}} f\, d\mu, \qquad f \in L^\infty(\mathcal{S}, \lambda).$$

Analogously ψ has the normal extension $\tilde{\psi}$ to $\pi_\rho(\mathcal{Q})''$ satisfying

$$\tilde{\psi}(\theta(f)) = \int_{\mathcal{S}} f\, d\nu, \qquad f \in L^\infty(\mathcal{S}, \lambda).$$

Using Lemma 3.1, we have

$$S(\psi \mid \varphi) = S(\tilde{\psi} \mid \tilde{\varphi}) \geq S_{\mathfrak{B}}(\tilde{\psi} \mid \tilde{\varphi}) = I(\nu \mid \mu).$$

The inverse inequality always holds by the monotonicity. □

COROLLARY 3.3. (1) *Let $\varphi, \psi \in \mathcal{S}$ which satisfy the* KMS *condition with respect to a strongly continuous one-parameter automorphism group α_t of $\mathfrak{A}$. If μ and ν are the central measures of φ and ψ, then $S(\psi \mid \varphi) = I(\nu \mid \mu)$.*

(2) *Let $\{\mathfrak{A}, G, \alpha\}$ be a C^*-dynamical system such that α_G is a large group of automorphisms of $\mathfrak{A}$, and $\varphi, \psi \in \mathcal{S}$ be α-invariant. If μ and ν are the ergodic decomposition measures of φ and ψ, then $S(\psi \mid \varphi) = I(\nu \mid \mu)$.*

Proof. (1) Let K be the set of all states satisfying the KMS condition with respect to α_t. Then K is a Choquet simplex and the central measure of $\rho \in K$ is identical to the unique maximal measure on K representing ρ (cf. [**8**, p. 121]). Hence it follows that $\lambda = (\mu + \nu)/2$ is the central measure of $\rho = (\varphi + \psi)/2$, so that Theorem 3.2 gives the desired equality.

(2) First note that the set $\mathcal{S}_\alpha$ of all α-invariant states becomes a Choquet simplex, because the condition of large group implies the G-abelianness (cf. [**10**]). Hence $\lambda = (\mu + \nu)/2$ is the ergodic decomposition measure of $\rho = (\varphi + \psi)/2$. It follows (cf. [**26**, Theorem 3.6], [**27**, Theorem 3.1]) that λ is the $\mathfrak{B}$-orthogonal measure of ρ with $\mathfrak{B} = (\pi_\rho(\mathfrak{A}) \cup U_\rho(G))' = \mathfrak{Z}_\rho \cap U_\rho(G)'$ where $g \mapsto U_\rho(g)$ is the unitary representation of G on $\mathcal{H}_\rho$ such that $\pi_\rho(\alpha_g(A)) = U_\rho(g)\pi_\rho(A)U_\rho(g)^*$ and $U_\rho(g)\Omega_\rho = \Omega_\rho$. Thus we have the desired equality. □

4. Relative entropy, sufficiency and KMS condition. In this section, we establish some relations between the relative entropy, the sufficiency and the KMS condition in W^*-dynamical systems and C^*-dynamical systems. The following theorem is obvious from Definition 1.1 and the monotonicity of relative entropy.

THEOREM 4.1. *If a $*$-subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ is sufficient for $\{\varphi, \psi\}$ in $\mathcal{S}$, then $S(\psi \mid \varphi) = S_{\mathfrak{B}}(\psi \mid \varphi)$.*

THEOREM 4.2. *Let $\mathfrak{N}$ be a von Neumann algebra and $\mathfrak{S}$ be the set of all normal states of $\mathfrak{N}$.*

(1) *Let $\{\mathfrak{N}, G, \alpha\}$ be a W^*-dynamical system. If $\varphi, \psi \in \mathfrak{S}$ are α-invariant, then $S(\psi \mid \varphi) = S_{\mathfrak{N}^\alpha}(\psi \mid \varphi)$ where $\mathfrak{N}^\alpha$ is the fixed point subalgebra of α.*

(2) *Let* α_t *be a strongly continuous one-parameter automorphism group of* $\mathfrak{N}$. *If* $\varphi, \psi \in \mathfrak{S}$ *satisfy the* KMS *condition with respect to* α_t, *then* $S(\psi \mid \varphi) = S_{\mathfrak{Z}}(\psi \mid \varphi)$ *where* $\mathfrak{Z} = \mathfrak{N} \cap \mathfrak{N}'$.

Proof. (1) Let $s(\varphi)$ and $s(\psi)$ be the support projections of φ and ψ, which are in $\mathfrak{N}^\alpha$ from the α-invariance of φ and ψ. Since $S(\psi \mid \varphi) = S_{\mathfrak{N}^\alpha}(\psi \mid \varphi) = +\infty$ if $s(\psi) \leq s(\varphi)$ does not hold, we assume that $s(\psi) \leq s(\varphi)$. Letting $e = s(\varphi)$, we can define a W^*-dynamical system $\{\hat{\mathfrak{N}}, G, \hat{\alpha}\}$ by $\hat{\mathfrak{N}} = e\mathfrak{N}e$ and $\hat{\alpha}_g = \alpha_g \restriction \hat{\mathfrak{N}}$. Then $\hat{\varphi} = \varphi \restriction \hat{\mathfrak{N}}$ and $\hat{\psi} = \psi \restriction \hat{\mathfrak{N}}$ are $\hat{\alpha}$-invariant. Since $\hat{\varphi}$ is faithful, it follows (see Example 1.4) that $\hat{\mathfrak{N}}^{\hat{\alpha}} = e\mathfrak{N}^\alpha e$ is sufficient for $\{\hat{\varphi}, \hat{\psi}\}$. Hence we have $S(\hat{\psi} \mid \hat{\varphi}) = S_{e\mathfrak{N}^\alpha e}(\hat{\psi} \mid \hat{\varphi})$ by Theorem 4.1. It now suffices to show the equations:

$$S(\psi \mid \varphi) = S(\hat{\psi} \mid \hat{\varphi}) \quad \text{and} \quad S_{\mathfrak{N}^\alpha}(\psi \mid \varphi) = S_{e\mathfrak{N}^\alpha e}(\hat{\psi} \mid \hat{\varphi}).$$

Define a linear map $\gamma\colon \mathfrak{N} \to \hat{\mathfrak{N}}$ by $\gamma(A) = eAe$. Then we have $\gamma(I) = e$, $\gamma(A^*) = \gamma(A)^*$ and $\gamma(A)^*\gamma(A) \leq \gamma(A^*A)$ for all $A \in \mathfrak{N}$. Since $\varphi = \hat{\varphi} \circ \gamma$ and $\psi = \hat{\psi} \circ \gamma$, the monotonicity gives $S(\psi \mid \varphi) \leq S(\hat{\psi} \mid \hat{\varphi})$. Next define a linear map $\hat{\gamma}\colon \hat{\mathfrak{N}} \to \mathfrak{N}$ by $\hat{\gamma}(B) = B + \hat{\varphi}(B)(I - e)$. Then we have $\hat{\gamma}(e) = I$, $\hat{\gamma}(B^*) = \hat{\gamma}(B)^*$ and

$$\begin{aligned}\hat{\gamma}(B)^*\hat{\gamma}(B) &= B^*B + |\hat{\varphi}(B)|^2 (I - e) \\ &\leq B^*B + \hat{\varphi}(B^*B)(I - e) = \hat{\gamma}(B^*B), \qquad B \in \hat{\mathfrak{N}}.\end{aligned}$$

Since $\hat{\varphi} = \varphi \circ \hat{\gamma}$ and $\hat{\psi} = \psi \circ \hat{\gamma}$, the monotonicity again gives $S(\hat{\psi} \mid \hat{\varphi}) \leq S(\psi \mid \varphi)$. We hence obtain the first equation and analogously the second equation.

(2) By the KMS condition, the support projections $s(\varphi)$ and $s(\psi)$ are in $\mathfrak{Z}$ (cf. [**22**, Lemma 5.1]). Letting $s(\psi) \leq s(\varphi) = e$, we define $\hat{\mathfrak{N}} = \mathfrak{N}e$ and $\hat{\alpha}_t = \alpha_t \restriction \hat{\mathfrak{N}}$. Then $\hat{\varphi} = \varphi \restriction \hat{\mathfrak{N}}$ and $\hat{\psi} = \psi \restriction \hat{\mathfrak{N}}$ satisfy the KMS condition with respect to $\hat{\alpha}_t$. Since $\hat{\varphi}$ is faithful and hence $\hat{\alpha}_t = \sigma_t^{\hat{\varphi}}$ the modular automorphism group of $\hat{\varphi}$, it follows (see Example 1.2) that $\hat{\mathfrak{Z}} = \mathfrak{Z}e$ is sufficient for $\{\hat{\varphi}, \hat{\psi}\}$. As in the proof of (1), we thus have

$$S(\psi \mid \varphi) = S(\hat{\psi} \mid \hat{\varphi}) = S_{\mathfrak{Z}e}(\hat{\psi} \mid \hat{\varphi}) = S_{\mathfrak{Z}}(\psi \mid \varphi). \qquad \square$$

THEOREM 4.3. *Let* α_t *be a strongly continuous one-parameter automorphism group of* $\mathfrak{N}$ *and* $\varphi, \psi \in \mathfrak{S}$. *Assume that* φ *satisfies the* KMS *condition with respect to* α_t.

(1) *If* $S(\psi \mid \varphi) = S_{\mathfrak{N}^\alpha}(\psi \mid \varphi) < +\infty$, *then* ψ *is* α_t-*invariant.*

(2) *If* $S(\psi \mid \varphi) = S_{\mathfrak{Z}}(\psi \mid \varphi) < +\infty$, *then* ψ *satisfies the* KMS *condition with respect to* α_t.

Proof. By the assumptions, we have $s(\varphi) \in \mathfrak{Z} \cap \mathfrak{N}^\alpha$ and $s(\psi) \le s(\varphi)$. As is seen from the proof of Theorem 4.2, we may suppose that φ is faithful, so that $\alpha_t = \sigma_t^\varphi$ the modular automorphism group and $\mathfrak{N}^\alpha = Z_\varphi$ the centralizer of φ (cf. [**28**, Lemma 15.8]). In [**14**, Corollaries 4.2 and 4.3], we proved (1) and (2) for the case when also ψ is faithful. Now let ψ be not faithful and $p = s(\psi)$.

(1) We first show that $p \in Z_\varphi$. Let $\hat{\mathfrak{N}} = (Z_\varphi \cup \{p\})''$, $\hat{\varphi} = \varphi \restriction \hat{\mathfrak{N}}$ and $\hat{\psi} = \psi \restriction \hat{\mathfrak{N}}$. Then $\hat{\varphi}$ is a trace of $\hat{\mathfrak{N}}$ and we have $S(\hat{\psi} \mid \hat{\varphi}) = S_{Z_\varphi}(\hat{\psi} \mid \hat{\varphi}) < +\infty$ by the assumption. Let ε be the conditional expectation of $\hat{\mathfrak{N}}$ onto Z_φ with $\hat{\varphi} \circ \varepsilon = \hat{\varphi}$. Define $\hat{\psi}' = \hat{\psi} \circ \varepsilon$, $\hat{\psi}_t = (1-t)\hat{\psi} + t\hat{\varphi}$ and $\hat{\psi}_t' = \hat{\psi}_t \circ \varepsilon = (1-t)\hat{\psi}' + t\hat{\varphi}$ for $0 < t < 1$. Since $\hat{\psi}_t$ is faithful, it follows by [**14**, Theorem 3.3] that

$$(*) \qquad \|\hat{\psi}_t' - \hat{\psi}_t\| \le \left\{2\left(S(\hat{\psi}_t \mid \hat{\varphi}) - S_{Z_\varphi}(\hat{\psi}_t \mid \hat{\varphi})\right)\right\}^{1/2}, \qquad 0 < t < 1.$$

Since $\hat{\varphi}$ is a trace, there exists a positive self-adjoint operator h affiliated with $\hat{\mathfrak{N}}$ such that $\hat{\psi}(A) = \hat{\varphi}(hA)$ for all $A \in \hat{\mathfrak{N}}$. Take the spectral decomposition $h = \int_0^\infty \lambda\, de(\lambda)$. Noting that $\Delta_{\hat{\Psi},\hat{\Phi}} = h$ and $\Delta_{\hat{\Psi}_t,\hat{\Phi}} = (1-t)h + tI$ where $\hat{\Phi}$, $\hat{\Psi}$, and $\hat{\Psi}_t$ are vector representatives of $\hat{\varphi}$, $\hat{\psi}$ and $\hat{\psi}_t$ in the standard form of $\hat{\mathfrak{N}}$, we have

$$S(\hat{\psi} \mid \hat{\varphi}) = \int_0^\infty \lambda \log \lambda \, d\hat{\varphi}(e(\lambda)),$$

$$S(\hat{\psi}_t \mid \hat{\varphi}) = \int_0^\infty [(1-t)\lambda + t]\log[(1-t)\lambda + t]\, d\hat{\varphi}(e(\lambda)).$$

Since

$$-\frac{1}{e} \le [(1-t)\lambda + t]\log[(1-t)\lambda + t] \le (1-t)\lambda \log \lambda,$$

it follows from the Lebesgue's convergence theorem that

$$S(\hat{\psi} \mid \hat{\varphi}) = \lim_{t \to +0} S(\hat{\psi}_t \mid \hat{\varphi}),$$

and analogously

$$S_{Z_\varphi}(\hat{\psi} \mid \hat{\varphi}) = \lim_{t \to +0} S_{Z_\varphi}(\hat{\psi}_t \mid \hat{\varphi}).$$

By letting $t \to +0$ in (*), we obtain $\hat{\psi}' = \hat{\psi}$, which implies that Z_φ is sufficient for $\{\hat{\varphi}, \hat{\psi}\}$. Then it is easy to see that h is affiliated with Z_φ, so that $p = s(h) \in Z_\varphi$. Now define a faithful state $\bar{\psi} = c\psi + (1-c)\bar{\varphi}$ where

$c = \varphi(p) < 1$ and $\bar{\varphi} = (1-c)^{-1}\varphi_{I-p}$. Since $s(\psi) \perp s(\bar{\varphi})$, by [**5**, Theorem 3.6] we have

$$S(\bar{\psi} \mid \varphi) = cS(\psi \mid \varphi) + (1-c)S(\bar{\varphi} \mid \varphi) + c\log c + (1-c)\log(1-c),$$

$$S_{Z_\varphi}(\bar{\psi} \mid \varphi) = cS_{Z_\varphi}(\psi \mid \varphi) + (1-c)S_{Z_\varphi}(\bar{\varphi} \mid \varphi) + c\log c + (1-c)\log(1-c).$$

Since $\bar{\varphi}$ is σ_t^φ-invariant and $\bar{\varphi} \le (1-c)^{-1}\varphi$, it follows from Theorem 4.2 (1) that $S(\bar{\varphi} \mid \varphi) = S_{Z_\varphi}(\bar{\varphi} \mid \varphi) < +\infty$, and hence $S(\bar{\psi} \mid \varphi) = S_{Z_\varphi}(\bar{\psi} \mid \varphi) < +\infty$. This implies by [**14**, Corollary 4.2] that $\bar{\psi}$ is σ_t^φ-invariant. Thus ψ is σ_t^φ-invariant.

(2) Substituting $\mathfrak{Z}$ for Z_φ in the proof of (1), we can show that $p \in \mathfrak{Z}$ and a faithful state $\bar{\psi}$ defined as above satisfies the KMS condition with respect to σ_t^φ, and thus ψ satisfies the same condition. □

Let $\mathfrak{A}$ be a C^*-algebra and α_t be a strongly continuous one-parameter automorphism group of $\mathfrak{A}$. Let $\varphi \in \mathfrak{S}$ and $\{\mathcal{H}_\varphi, \pi_\varphi, \Omega_\varphi\}$ be the cyclic representation of $\mathfrak{A}$ induced by φ. Suppose that φ satisfies the KMS condition with respect to α_t. Since φ is α_t-invariant, there is a strongly continuous one-parameter unitary group $U_\varphi(t)$ on $\mathcal{H}_\varphi$ such that $U_\varphi(t)\Omega_\varphi = \Omega_\varphi$ and

$$\pi_\varphi(\alpha_t(A)) = U_\varphi(t)\pi_\varphi(A)U_\varphi(t)^*, \qquad t \in \mathbf{R}, A \in \mathfrak{A}.$$

The normal extensions $\tilde{\varphi}$ and $\tilde{\alpha}_t$ of φ and α_t to $\pi_\varphi(\mathfrak{A})''$ are given by

$$\tilde{\varphi}(Q) = \langle \Omega_\varphi, Q\Omega_\varphi \rangle, \qquad Q \in \pi_\varphi(\mathfrak{A})'',$$
$$\tilde{\alpha}_t(Q) = U_\varphi(t)QU_\varphi(t)^*, \quad t \in \mathbf{R}, Q \in \pi_\varphi(\mathfrak{A})'',$$

and it is known (cf. [**1**, Lemma 2.4]) that $\tilde{\varphi}$ satisfies the KMS condition with respect to $\tilde{\alpha}_t$, i.e., $\tilde{\alpha}_t = \sigma_t^{\tilde{\varphi}}$ the modular automorphism group of $\tilde{\varphi}$. Then we have

THEOREM 4.4. *Let $\mathfrak{A}$, α_t and φ be as above. For each $\psi \in \mathfrak{S}$ with $\psi \prec \varphi$, let $\tilde{\psi}$ be the normal extension of ψ to $\pi_\varphi(\mathfrak{A})''$. Then the following conditions are equivalent*:

(i) *ψ satisfies the* KMS *condition with respect to α_t*;

(ii) *$\mathfrak{Z}_\varphi = \pi_\varphi(\mathfrak{A})'' \cap \pi_\varphi(\mathfrak{A})'$ is sufficient for $\{\tilde{\varphi}, \tilde{\psi}\}$*;

(iii) *$\mathfrak{Z}_\varphi$ is weakly sufficient for $\{\tilde{\varphi}, \tilde{\psi}\}$*;

(iv) $(d\psi/d\varphi)\Omega_\varphi \in \overline{\mathfrak{Z}_\varphi\Omega_\varphi}$;

(v) $(D\tilde{\varphi}: D(\tilde{\varphi} + \tilde{\psi}))_t \in \mathfrak{Z}_\varphi$ *for all* $t \in \mathbf{R}$ *where* $(D\tilde{\varphi}: D(\tilde{\varphi} + \tilde{\psi}))_t$ *is the Connes Radon-Nikodym derivative* (cf. [**9**]);

(vi) $S(\psi \mid \varphi) = S_{\mathfrak{Z}_\varphi}(\tilde{\psi} \mid \tilde{\varphi})$.

Proof. Note that $\tilde{\psi}$ is given by

$$\tilde{\psi}(Q) = \left\langle (d\psi/d\varphi)\Omega_\varphi, Q\Omega_\varphi \right\rangle, \qquad Q \in \pi_\varphi(\mathfrak{A})'',$$

and hence $\tilde{\psi} \prec \tilde{\varphi}$. Since there exists a conditional expectation $\varepsilon_{\tilde{\varphi}}$ of $\pi_\varphi(\mathfrak{A})''$ onto $\mathfrak{Z}_\varphi$ with $\tilde{\varphi} = \tilde{\varphi} \circ \varepsilon_{\tilde{\varphi}}$, the equivalence of (ii), (iii) and (iv) follows from Theorem 1.6 (Remark) and the proof of Theorem 1.8. Because the KMS condition of ψ with respect to α_t and the same of $\tilde{\psi}$ with respect to $\tilde{\alpha}_t$ are equivalent, it follows from [**14**, Theorem 2.3] that (i) and (ii) are equivalent. Since $(D\tilde{\varphi}: D(\tilde{\varphi} + \tilde{\psi}))_t = (D(\tilde{\varphi} + \tilde{\psi}): D\tilde{\varphi})_t^*$, we see by [**14**, Lemma 2.1] the equivalence of (ii) and (v). Finally the equivalence of (i) and (vi) follows from Theorems 4.2 (2) and 4.3 (2) and Lemma 3.1 if we prove $S_{\mathfrak{Z}_\varphi}(\tilde{\psi} \mid \tilde{\varphi}) < +\infty$. There exists a positive self-adjoint operator h affiliated with $\mathfrak{Z}_\varphi$ such that $\tilde{\psi}(Q) = \tilde{\varphi}(hQ)$ for all $Q \in \mathfrak{Z}_\varphi$. Take the spectral decomposition $h = \int_0^\infty \lambda \, de(\lambda)$. Then the condition $\tilde{\psi} \prec \tilde{\varphi}$ gives rise to $\tilde{\varphi}(h^2) = \int_0^\infty \lambda^2 \, d\tilde{\varphi}(e(\lambda)) < +\infty$. Hence we have

$$\begin{aligned} S_{\mathfrak{Z}_\varphi}(\tilde{\psi} \mid \tilde{\varphi}) &= \int_0^\infty \lambda \log \lambda \, d\tilde{\varphi}(e(\lambda)) \\ &\leq \int_0^\infty \lambda^2 \, d\tilde{\varphi}(e(\lambda)) < +\infty. \end{aligned} \qquad \square$$

REMARK. Assuming only that ψ has the normal extension $\tilde{\psi}$ to $\pi_\varphi(\mathfrak{A})''$ (which is necessarily a vector state), we obtain the equivalence of the conditions (i), (ii) and (v) in Theorem 4.4, which imply (vi) and are implied by the equality (vi) with a finite value. For the case of ψ being dominated by φ, the condition (iv) can be replaced by $d\psi/d\varphi \in \mathfrak{Z}_\varphi$ (see e.g. [**17**, p. 104]). Also for the α_t-invariance of $\psi \in \mathfrak{S}$ with $\psi \prec \varphi$, we can obtain the similar equivalent conditions by substituting $Z_{\tilde{\varphi}} = \pi_\varphi(\mathfrak{A})'' \cap U_\varphi(\mathbf{R})'$ for $\mathfrak{Z}_\varphi$ in the above conditions (ii)–(vi).

Theorem 4.4 finds an application in quantum lattice systems. Let L be a countable set and $\mathcal{H}_0$ a finite-dimensional Hilbert space. For each nonempty finite set $\Lambda \subset L$, let $\mathcal{H}_\Lambda = \otimes_{x \in \Lambda} \mathcal{H}_x$ with $\mathcal{H}_x = \mathcal{H}_0$ and $\mathfrak{A}_\Lambda = \mathbf{B}(\mathcal{H}_\Lambda)$. Then the quantum lattice system on L is described by the quasi-local C^*-algebra $\mathfrak{A} = \overline{\bigcup_{\Lambda \subset L} \mathfrak{A}_\Lambda}$. An interaction Φ is defined as a function from finite subsets $\Lambda \subset L$ into the self-adjoint elements of $\mathfrak{A}$ such that $\Phi(\Lambda) \in \mathfrak{A}_\Lambda$. Let φ be a state of $\mathfrak{A}$ satisfying the Gibbs condition with respect to Φ (see [8] for the definition). Now assume that Φ satisfies

$$\sum_{n=0}^{\infty} e^{rn}\left(\sup_{x\in L} \sum_{\substack{\Lambda\ni x\\ |\Lambda|=n+1}} \|\Phi(\Lambda)\|\right) < +\infty$$

for some $r > 0$. Then the strongly continuous one-parameter automorphism group α_t^Φ of $\mathcal{A}$ can be given by

$$\alpha_t^\Phi(A) = \lim_{\Lambda\to L} e^{itH_\Phi(\Lambda)}Ae^{-itH_\Phi(\Lambda)}, \qquad A\in\mathcal{A},\, t\in\mathbf{R},$$

where $H_\Phi(\Lambda) = \Sigma_{X\subset\Lambda}\Phi(X)$. It is known (cf. [8, p. 268]) that $\psi\in\mathfrak{S}$ satisfies the Gibbs condition with respect to Φ if and only if ψ satisfies the KMS condition with respect to α_t^Φ. Then we have

COROLLARY 4.5. *Let $\mathcal{A}$, Φ, $\alpha_t = \alpha_t^\Phi$ and φ be as above, and let $\psi\in\mathfrak{S}$ with $\psi \prec \varphi$. Then the Gibbs condition for ψ with respect to Φ is equivalent to each of the conditions* (i)–(vi) *in Theorem* 4.4, *and these conditions imply the following*:

(vii) *for each $\Lambda\subset L$, $\mathcal{A}_{\Lambda^c} = \overline{\bigcup_{X\subset\Lambda^c}\mathcal{A}_X}$ is weakly sufficient for $\{\varphi,\psi\}$*;

(viii) *for each $\Lambda\subset L$, $S(\psi\,|\,\varphi) = S_{\mathcal{A}_{\Lambda^c}}(\psi\,|\,\varphi)$.*

Further if $\bigcup_{\Lambda\subset L}\pi_\varphi(\mathcal{A}_\Lambda)\Omega_\varphi$ is a core for the modular operator Δ_{Ω_φ} associated with Ω_φ, the condition (vii) *conversely implies the Gibbs condition for ψ with respect Φ.*

The main part of the corollary is immediate from Theorem 4.4 and the fact that $\mathfrak{Z}_\varphi$ is identical to $\bigcap_{\Lambda\subset L}\pi_\varphi(\mathcal{A}_{\Lambda^c})''$ the algebra of observables at infinity. The last part follows by [6, Lemma 3].

We finally give some notes on the translationally invariant case of $L = \mathbf{Z}^d$. Let τ be the automorphism group of translations on $\mathbf{Z}^d$. Let Φ be a τ-invariant interaction satisfying $\Sigma_{\Lambda\ni 0}e^{r|\Lambda|}\|\Phi(\Lambda)\| < +\infty$ for some $r > 0$. A τ-invariant state φ is said to be equilibrium with respect to Φ if the following variational equality holds (see [17, 25]): $P(\Phi) = s(\varphi) - \varphi(A_\Phi)$ where $s(\varphi)$ is the mean entropy of φ and

$$P(\Phi) = \lim_{\substack{\Lambda\to\infty\\ \text{(van Hove)}}} |\Lambda|^{-1}\log \mathrm{tr}_\Lambda(e^{-H_\Phi(\Lambda)}),$$

$$A_\Phi = \sum_{\Lambda\ni 0} |\Lambda|^{-1}\Phi(\Lambda).$$

Then the equilibrium condition with respect to Φ, the Gibbs condition with respect to Φ and the KMS condition with respect to α_t^Φ are all equivalent for τ-invariant states of $\mathcal{A}$ (cf. [3, 8, 20]). Let $\varphi,\psi\in\mathfrak{S}$ be

τ-invariant. Since $\{\mathcal{Q}, \mathbf{Z}^d, \tau\}$ is asymptotically abelian, we obtain $S(\psi \mid \varphi) = I(\nu \mid \mu)$ by Corollary 3.3 (2) where μ and ν are the ergodic decomposition measures of φ and ψ. If φ is equilibrium and $\psi \prec \varphi$ (or more weakly ψ has the normal extension to $\pi_\varphi(\mathcal{Q})''$), then it can be proved that $\nu \ll \mu$, so that ψ is automatically equilibrium because μ is supported on the set of equilibrium states.

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