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PRODUCTS OF POSITIVE REFLECTIONS IN REAL ORTHOGONAL GROUPS

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Let $O(f)$ be the orthogonal group of a symmetric bilinear form f defined on a finite-dimensional real vector space V. If f is indefinite then $O(f)$ has two conjugacy classes of reflections, one of which consists of so called positive reflections. We denote by G^+ the subgroup of $O(f)$ generated by all positive reflections. In this paper we describe this subgroup and solve the length problem in G^+ with respect to the distinguished set of generators. When f is non-degenerate this problem was solved by J. Malzan. Our proof (in the case of arbitrary f) is shorter and completely different from his proof.

Introduction. Let $O(f)$ be the orthogonal group of a symmetric bilinear form f defined on a finite-dimensional real vector space V. If f is indefinite then $O(f)$ has two conjugacy classes of reflections, one of which consists of so called positive reflections. We denote by G^+ the subgroup of $O(f)$ generated by all positive reflections. In this paper we solve the length problem in G^+ with respect to the distinguished set of generators. When f is non-degenerate this problem was solved by J. Malzan. Our prooof (in the case of arbitrary f) is shorter and completely different from his proof.

A non-isotropic vector a determines a unique orthogonal reflection R_a and we say that R_a is positive if $f(a, a) > 0$. The weak orthogonal group $O^*(f)$ consists of all isometries which fix every vector in Rad V. To avoid trivial and known cases let us assume that f is indefinite, i.e., that $f(x, x)$ takes both positive and negative values. Then $O^*(f) \supset G^+ \supset O_1^*(f)$ where $O_1^*(f)$ denotes the identity component of $O^*(f)$. Moreover $O^*(f)/O_1^*(f) \cong Z_2 \times Z_2$ and $G^+/O_1^*(f) \cong Z_2$.

Our main theorem (Theorem 2) gives explicit formulas for the length of any $u \in G^+$ with respect to the generating set consisting of all positive reflections. When f is nondegenerate this result is due to J. Malzan [5]. The proof is based on some earlier results of M. Götzky [3] on $O^*(f)$. One should point out that Götzky considers also weak unitary groups and his underlying field F is arbitrary (char $F \neq 2$ in the case of $O^*(f)$).

The main idea of the proof is to take a shortest representation of $u \in G^+$ as a product of reflections and then try to convert all reflections into positive ones. This method is effective in the generic case; the exceptional cases are treated separately.

1. Weak orthogonal groups in general. Let V be a finite-dimensional vector space over a field F, char $F \neq 2$, and let f be a symmetric bilinear form on V. An automorphism u of V is called an *isometry* if $f(u(x), u(y)) = f(x, y)$ for all $x, y \in V$. The group of all isometries will be denoted by $O(f)$ and we refer to it as the *orthogonal group* of the form f. (Note that we allow f to be degenerate.)

The *weak orthogonal group* $O^*(f)$ is the subgroup of $O(f)$ consisting of all isometries which fix every vector in the radical $\operatorname{Rad} V = \{x \in V: f(x, y) = 0, \forall y \in V\}$.

For $u \in O(f)$ we define its *fixed space* Fix u and its *residual space* Res u by

$$\operatorname{Fix} u = \operatorname{Ker}(u - 1), \qquad \operatorname{Res} u = \operatorname{Im}(u - 1).$$

We also define the *residue* $r(u)$ and the *radical residue* $r_0(u)$ of u to be

$$r(u) = \dim \operatorname{Res} u, \qquad r_0(u) = \dim(\operatorname{Res} u \cap \operatorname{Rad} V).$$

If a is a non-isotropic vector, i.e., $f(a, a) \neq 0$, then the transformation $R_a: V \to V$ defined by

$$R_a(x) = x - 2f(a, x)f(a, a)^{-1}a$$

belongs to $O^*(f)$ and is called a *reflection*. We have

$$\operatorname{Fix} R_a = \langle a \rangle^{\perp}, \qquad \operatorname{Res} R_a = \langle a \rangle$$

and $R_a(a) = -a$. (For any subspace W of V we denote by $W^{\perp}$ the orthogonal complement of W with respect to the form f.)

We shall now state some results of M. Götzky **[3]** concerning the group $O^*(f)$. (In his paper he also treats the weak unitary groups but we shall not need those results.) For further results and generalizations we refer the reader to a paper of E. Ellers **[2]**.

Every $u \in O^*(f)$ can be expressed as a product of reflections

$$u = R_{a_1} R_{a_2} \cdots R_{a_m}. \tag{1}$$

Since $\det R_a = -1$ for every reflection R_a, it follows that $\det u = \pm 1$ for all $u \in O^*(f)$. Moreover the subgroup

$$SO^*(f) = \{u \in O^*(f): \det u = 1\}$$

has index 2 in $O^*(f)$.

For $u \in O^*(f)$ we shall denote by $l(u)$ the length of u with respect to the generating set consisting of all reflections. Thus $l(u)$ is the smallest integer $m(\geq 0)$ for which a factorization (1) exists.

THEOREM 1. (*M. Götzky*) *For* $u \in O^*(f)$ *we have* $l(u) = r(u) + r_0(u)$ *except when* $(\text{Fix } u)^\perp$ *is totally isotropic and* $u \neq 1$. *In the exceptional case we have* $l(u) = r(u) + r_0(u) + 2$.

When f is non-degenerate, i.e., $\text{Rad } V = 0$; this theorem is due to P. Scherk [**6**].

2. Real case and the statement of the main result. From now on we shall assume that F is the real field R. A vector x is called *positive* (resp. *negative*) if $f(x, x) > 0$ (resp. $f(x, x) < 0$). We shall denote by n the dimension of V and by (p, q, s) the signature of f. This means that every orthogonal basis of V consists of p positive vectors, q negative vectors, and s isotropic vectors.

A reflection R_a is *positive* (resp. *negative*) if a is positive (resp. negative). It follows from Witt's theorem that all positive (resp. negative) reflections are conjugate in $O^*(f)$. We shall denote by G^+ (resp. G^-) the subgroup of $O^*(f)$ generated by all positive (resp. negative) reflections. If $p = 0$, i.e., f is negative semidefinite then there are no positive reflections and we have $G^+ = \{1\}$ and $G^- = O^*(f)$. If $q = 0$ then $G^+ = O^*(f)$ and $G^- = \{1\}$.

In view of these remarks and Theorem 1 *we shall assume throughout that f is indefinite, i.e., $p \geq 1$ and $q \geq 1$*. Clearly $O(f)$ and $O^*(f)$ are real algebraic groups and so Lie groups. Let $O_1^*(f)$ be the identity component of $O^*(f)$ viewed as a Lie group.

Let $V = V_1 \oplus \text{Rad } V$ and let f_1 be the restriction of f to $V_1 \times V_1$. Clearly f_1 is a non-degenerate symmetric bilinear form on V_1 of signature $(p, q, 0)$. Then the elements u of $O(f)$ are represented by matrices

$$u = \begin{pmatrix} u_1 & 0 \\ v & u_0 \end{pmatrix}$$

where $u_1 \in O(f_1)$, u_0 is an automorphism of $\text{Rad } V$ and $v\colon V_1 \to \text{Rad } V$ is an arbitrary linear map. We have $u \in O^*(f)$ if and only if $u_0 = 1$.

LEMMA 1. $O^*(f)/O_1^*(f) \cong Z_2 \times Z_2$.

Proof. If $s = 0$ this is well known, see e.g. [**4**, Lemma 2.4(b), p. 451]. In general the assertion follows from this special case and the above matrix description of elements of $O^*(f)$.

COROLLARY. *$G^+ \cdot O_1^*(f)/O_1^*(f)$ and $G^- \cdot O_1^*(f)/O_1^*(f)$ are cyclic groups of order two. The three subgroups $G^+ O_1^*(f)$, $G^- \cdot O_1^*(f)$, and $SO^*(f)$ are distinct.*

Proof. Since all positive (resp. negative) reflections are conjugate in $O^*(f)$, they lie in a single connected component of $O^*(f)$. This implies the first assertion. We have $G^+ O_1^*(f) \neq G^- O_1^*(f)$ because $O^*(f)$ is generated by reflections. These two groups are different from $SO^*(f)$ because $\det R = -1$ for each reflection R.

For $u \in G^+$ we shall denote by $l^+(u)$ the length of u with respect to the generating set consisting of all positive reflections. We can now state our main result.

THEOREM 2. *We have $G^+ \supset O_1^*(f)$. For $u \in G^+$ we have $l^+(u) = r(u) + r_0(u)$ except in the following cases:*

(i) *The subspace* $(\text{Fix } u)^\perp$ *is negative semidefinite and $u \neq 1$,*

(ii) *$u^2 = 1$ and $u(x) = -x$ for some negative vector x.*

In the exceptional cases we have $l^+(u) = r(u) + r_0(u) + 2$.

When f is non-degenerate this theorem is due to J. Malzan [**5**]. Our proof below even in the more general case is simpler and more elementary than his. For instance we do not need the detailed knowledge of the conjugacy classes of $O(f)$, which is heavily used in [**5**] in the case when f is non-degenerate.

3. Proofs. We shall assume that the reader is familiar with Götzky's paper [**3**] and we shall use some of his technical lemmas in addition to Theorem 1. The main tool in our proof is the following technical lemma.

LEMMA 2. *Let a, b, c be linearly independent vectors with a positive and b and c negative. If the sequence a, b, c is not orthogonal then the isometry $u = R_a R_b R_c$ can be written as a product of three positive reflections.*

Proof. Without any loss of generality we may assume that $f(a, a) = 1$ and $f(b, b) = f(c, c) = -1$. Set $f(a, b) = \alpha$, $f(a, c) = \beta$, and $f(b, c) = \gamma$. By hypothesis at least one of α, β, γ is non-zero. Since $R_b R_c = R_d R_b$

where $d = R_b(c)$, we may assume that in fact β or γ is non-zero. Then for $e = (\eta - \alpha\xi)a + \xi b$ we have

$$f(e, e) = (\eta - \alpha\xi)^2 - \xi^2 + 2\alpha\xi(\eta - \alpha\xi) = \eta^2 - (1 + \alpha^2)\xi^2,$$

and

$$\Delta = \begin{vmatrix} f(c, c) & f(c, e) \\ f(e, c) & f(e, e) \end{vmatrix} = (1 + \alpha^2)\xi^2 - \eta^2 - (\beta\eta + (\gamma - \alpha\beta)\xi)^2.$$

Since β or γ is not zero, we can choose ξ and η so that $f(e, e) = -1$ and $\Delta < 0$. By Dreispiegelungssatz [**1**, Proposition 6.1] the product $R = R_a R_b R_e$ is a reflection. Since b and e are negative vectors, we have $R_b R_e \in O_1^*(f)$ and so R must be a positive reflection by Lemma 1, Cor. We have $u = RR_e R_c$ where R_e and R_c are negative reflections. Since $\Delta < 0$ the space $W = \langle c, e \rangle$ is a hyperbolic plane. We claim that $R_e R_c$ is a product of two positive reflections. To prove this it suffices to consider the restrictions of R_e and R_c to W. Then in W the operators $-R_e$ and $-R_c$ are positive reflections whose product is $R_e R_c$. This completes the proof.

Proof of Theorem 2. Let $u \in G^+ \cdot O_1^*(f)$.

Case 1. u is not exceptional, i.e., neither (i) nor (ii) holds.

Clearly $l^+(u) \geq l(u)$ and by Theorem 1, $l(u) = r(u) + r_0(u)$. Write $m = l(u)$ and let (1) be a factorization of u into a product of m reflections containing a maximal number, say k, of positive reflections. We have to prove that $k = m$.

This is clear if $m = 0$, i.e., $u = 1$. Otherwise we prove first that $k \geq 1$. Since (i) does not hold there exists a positive vector $a \in (\operatorname{Fix} u)^\perp$. It follows from [**3**, Hilfssatz 2.1, p. 385] that for $v = R_a u$ we have $r(v) = r(u)$ and $r_0(v) = r_0(u) - 1$. By Theorem 1 $l(v) = m - 1$ and since $u = R_a v$ we have $k \geq 1$. We may assume that the vectors a_i are positive for $1 \leq i \leq k$ and negative for $k < i \leq m$.

Now assume that $k < m$. By Lemma 1, Cor. $m - k$ must be even, and so $k \leq m - 2$. Assume that for every pair of indices (i, j) such that $1 \leq i < j \leq m$ and $j > k$ we have $a_i \perp a_j$. Since (ii) does not hold there must exist a pair of indices (i, j) such that $1 \leq i < j \leq k$ and $f(a_i, a_j) \neq 0$. Without any loss of generality we may assume that $f(a_{k-1}, a_k) \neq 0$. Let

$b \in \langle a_k, a_{k+1} \rangle$ be a positive vector such that $b \notin \langle a_k \rangle$. By Dreispiegelungssatz the product $R_b R_{a_k} R_{a_{k+1}}$ is a reflection, say R_c, and by Lemma 1, Cor. it is a negative reflection. Thus we can replace in (1) the product $R_{a_k} R_{a_{k+1}}$ by $R_b R_c$. Note that $f(a_{k-1}, c) \neq 0$. This shows that we may assume that there exists a pair of indices (i, j) such that $1 \leq i < j \leq m$, $j > k$ and $f(a_i, a_j) \neq 0$. Without any loss of generality we may in fact assume that the sequence a_k, a_{k+1}, a_{k+2} is not orthogonal. By Lemma 2 the product $R_{a_k} R_{a_{k+1}} R_{a_{k+2}}$ can be replaced by a product of three positive reflections. This contradicts the maximality of k.

Hence we have shown that $k = m$, and in particular $u \in G^+$.

Case 2. (i) or (ii) holds. Let $m = r(u) + r_0(u)$. We prove first that $l^+(u) \geq m + 2$. This is clear if $l(u) = m + 2$. Otherwise we have $l(u) = m$ and since $\det u = (-1)^m$, it suffices to show that u cannot be written as a product of m positive reflections. Assume that it can and let (1) be such a factorization.

We claim that $a_k \in (\operatorname{Fix} u)^\perp$ for all k. It suffices to prove this for $k = 1$. Thus let us assume that $a_1 \notin (\operatorname{Fix} u)^\perp$. Then by [3, Proposition 2.1.3] for $v = R_{a_1} u$ we have $\operatorname{Res} v = \operatorname{Res} u \oplus \langle a_1 \rangle$, and consequently $r(v) = r(u) + 1$ and $r_0(v) = r_0(u)$. It follows that

$$l(v) = r(v) + r_0(v) = r(u) + r_0(u) + 1 = m + 1.$$

This is a contradiction since v is a product of $m - 1$ reflections. Hence our claim is proved.

If (i) holds then since $a_k \in (\operatorname{Fix} u)^\perp$ for all k, we conclude that all reflections in (1) are negative, contrary to our hypothesis. Thus if (i) holds then $l^+(u) \geq m + 2$.

Now assume that (ii) holds. Since $u^2 = 1$ we have $V = \operatorname{Fix} u \oplus \operatorname{Res} u$ and $\operatorname{Fix} u \perp \operatorname{Res} u$. Since $\operatorname{Rad} V \subset \operatorname{Fix} u$, it follows that $\operatorname{Res} u$ is non-degenerate, $r_0(u) = 0$, and so $m = r(u)$. From (1) it follows that $\operatorname{Res} u \subset \langle a_1, \ldots, a_m \rangle$, see e.g. [2, §3]. Since $r(u) = m$, we conclude that $a_1, \ldots, a_m$ is a basis of $\operatorname{Res} u$.

We claim that this basis is orthogonal. It suffices to show that $a_1 \perp a_i$ for $2 \leq i \leq m$. Let b be a non-zero vector in $\operatorname{Res} u$ such that $b \perp a_i$ for $2 \leq i \leq m$. Since u is -1 on $\operatorname{Res} u$, we have $u(b) = -b$. On the other hand it follows from (1) that $u(b) = R_{a_1}(b)$. Hence we have $R_{a_1}(b) = -b$ and so $a_1 \in \langle b \rangle$. This proves our claim.

Since the basis $a_1, \ldots, a_m$ of $\operatorname{Res} u$ is orthogonal and each of these vectors is positive, we conclude that $\operatorname{Res} u$ is a positive definite subspace.

This contradicts (ii). Hence also in the case (ii) we must have $l^+(u) \geq m + 2$.

It remains to show that $l^+(u) \leq m + 2$, i.e., that u can be written as a product of $m + 2$ positive reflections.

Assume first that (i) holds. Since the positive vectors form an open set in V, we can choose a positive vector a such that $a \notin \operatorname{Fix} u$. Since (i) holds we have also $a \notin (\operatorname{Fix} u)^\perp$. Therefore $\operatorname{Fix} u$ is not invariant under R_a. Hence we can choose $x \in \operatorname{Fix} u$ such that $R_a(x) \notin \operatorname{Fix} u$. Let $v = R_a u$ and note that

$$v^2(x) = R_a u R_a(x) \neq R_a R_a(x) = x,$$

and so $v^2 \neq 1$. By [**3**, Proposition 2.1.3] we have $\operatorname{Res} v = \operatorname{Res} u \oplus \langle a \rangle$, and so $r(v) = r(u) + 1$ and $r_0(v) = r_0(u)$. Thus v is non-exceptional and by the result of Case 1 we have

$$l^+(v) = l(v) = r(v) + r_0(v) = m + 1.$$

Since $u = R_a v$, u is a product of $m + 2$ positive reflections.

Now assume that (ii) holds. Choose an orthogonal basis $a_1, \ldots, a_m$ of $\operatorname{Res} u$ such that $a_1, \ldots, a_k$ are positive and $a_{k+1}, \ldots, a_m$ are negative vectors. It follows from (ii) that $k < m$. Let

$$v = R_{a_1} \cdots R_{a_k} u.$$

This v satisfies (i) and we have $l(v) = m - k$. Hence $l^+(v) = m - k + 2$ by the result just proved above, and so $l^+(u) \leq m + 2$.

This completes the proof of Theorem 2.

Remark. It is easy to modify Theorem 2 so that it applies to the case when V is infinite-dimensional. Clearly if $u \in G^+$ then $r(u) < \infty$. The length formulas of Theorem 2 remain valid.

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