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THERE ARE NO PHANTOM COHOMOLOGY OPERATIONS IN K-THEORY

D. W. ANDERSON

Let h_1^*, h_2^* be cohomology theories defined on the category of finite CW-complexes, and suppose that h_1^*(point), h_2^*(point) are both countable. Then by Brown's [5] Representability Theorem, there are Ω-spectra $\mathbf{Z}_1$, $\mathbf{Z}_2$ such that $h_i^*(X) = [X, \mathbf{Z}_i]_*$, the graded group of homotopy classes of maps of X into the terms of the spectrum. If we exercise some care in the choice of $\mathbf{Z}_1$, we shall see that every stable cohomology operation φ: $h_1^*(X) \to h_2^*(X)$ defined for X finite extends to a map φ: $\mathbf{Z}_1 \to \mathbf{Z}_2$ of spectra. We shall examine the question: How many choices, up to homotopy, are there for φ, given φ? As an intermediate question, we shall also investigate: How many extensions are there to infinite CW-complexes are there of φ?

In the case when h_1^*, h_2^* are the connected forms of K-theory (real, symplectic, or complex), we shall show that every cohomology operation extends uniquely from finite complexes to all complexes and that the spectral homotopy class of the representing map is unique. Since a cohomology class which vanishes on all finite subcomplexes of a complex is called a phantom class, we shall call a cohomology operation which vanishes on all finite complexes a phantom operation. We shall call a spectral map of Ω-spectra a completely phantom cohomology operation if it vanishes on all CW-complexes. Our main theorem is as follows.

THEOREM 1. *There are no phantom or completely phantom cohomology operations other than the zero operations between the connected forms of complex, real, or symplectic K-theory, and every stable cohomology operation defined between these theories on finite complexes is represented by a spectral map which is unique up to homotopy.*

In the course of the proof of Theorem 1, we prove another theorem. Let E_{**}^r be the classical to general cohomology spectral sequence (known to the K-theorists as the Atiyah-Hirzebruch spectral sequence and to homotopy theorists by many names).

THEOREM 2. *Let E_{**}^r be the classical to general cohomology spectral sequence for computing the stable cohomology operations from connected complex K-theory to itself. Then $E_{p,q}^r$ is finite if p is odd, zero if q is odd or* *positive, and for any prime l, the l-index of* $Z^{\infty}_{2p,2q}$ *in* $E^2_{2p,2q}$ *for* $q \leq 0$ *is no greater than the l-primary part of*:

$$\begin{array}{ll} 2^p p & \text{if } l = 2 \\ [p/l - 1] & \text{if } l \neq 2, \end{array}$$

(*the l-index is the l-primary part of the index and the l-primary part of a number is the highest power of l which divides the number*). *The symbol* [ ] *indicates the greatest integer part of a number.*

The bound on the l-index given in Theorem 2 is probably the best possible — this is indicated by preliminary calculations beyond the scope of this paper.

Since, up to a factor of 2, real and symplectic K-theory are retracts of complex K-theory, the bounds on the l-index for the spectral sequences involving any of the three forms of K-theory do not exceed the estimates given above for complex K-theory except in the 2-primary part, where a factor of 4 may occur in addition to the bound given. As we shall see, this means that in all cases, $Z^{\infty}_{p,q}$ is of finite index in $E^2_{p,q}$, so that there are no phantom or completely phantom cohomology operations. It is not too difficult to show that there is a nontrivial phantom map $K(Z, 3) \to BSU$ which is an infinite loop map all of whose deloopings are phanton maps. On the other hand, it is not difficult to show that there are no phantom or completely phantom maps from complex bordism theory to any connected K-theory.

The first part of this paper is devoted to the derived functors of inverse limit, and the treatment is mostly derived from Grothendieck's work. Part 2 consists of some elementary applications of this via Milnor's classification of phantom cohomology classes to relate the classical to general spectral sequence to the classification of phantom cohomology classes. Part 3 is devoted to a development of a homotopy theory of spectra in the setting of Quillen's model categories for homotopy theory. This generality is used because a map of spectra which is a homotopy equivalence in each degree need not be a strict homotopy equivalence of spectra. Also, this allows us to take as weak equivalences of topological spaces either homotopy equivalence or CW-equivalence as we like. For the reader who believes that there is such a thing as a homotopy theory of spectra, this section may be skipped. The thorough treatment of the homotopy theory of spectra in this way does allow us to perform the standard arguments of homotopy theory on spectra. The model structure which we place on spectra is not equivalent to the model structure on semisimplicial spectra in Ken Brown's MIT thesis, since in our setting a map between spectra which induces an isomorphism on (stable) homotopy groups will not be a weak equivalence unless the spectra are Ω-spectra. The last section is devoted to connected K-theory, where the Adams operations are used to show that there are no phantom operations for the l-localizations of the connected K-theory, and sufficient bounds are obtained on the localized spectral sequence to prove Theorems 1 and 2 for the unlocalized case. While the Adams operations give explicit operations in the localized cases, they cannot be used in the same way to give operations in the unlocalized case. Perhaps it is possible to construct the operations whose existence are implied by Theorem 2 using Adams operations cleverly.

E. Thomas has called to my attention the ETH thesis "Phantomabbildungen und Klassifizierende Räume" of W. Meier. This thesis shows the existence of many phantom maps between spaces using techniques closely related to ours.

1. The functors $\lim^i$. Milnor [**8**] has proved a theorem which characterizes the phantom cohomology classes defined on a countable CW-complex. If H^* is a cohomology theory, call H^* additive if it takes disjoint unions of spaces to products of groups (all representable theories are additive). Milnor's theorem states that if X is a CW-complex, $X_1 \subset X_2 \subset \cdots \subset X$ is an increasing sequence of subcomplexes whose union is X, then for all n there is a natural short exact sequence:

$$0 \to \lim{}^1 H^{n-1}(X_i) \to H^n(X) \to \lim{}^0 H^n(X_i) \to 0. \tag{1.1}$$

In this exact sequence, $\lim^0$ is the usual inverse limit functor, and $\lim^1$ is a second functor defined by Milnor for inverse sequences of groups. Notice that $\lim^1 H^{n-1}(X_i)$ consists of exactly those cohomology classes $\alpha \in H^n(X)$ which vanish on all X_i. Thus, if X is a countable complex, we can choose the X_i to be a cofinal subset of the finite subcomplexes of X, and observe that the phantom cohomology classes of X are the elements of $\lim^1 H^*(X_i)$.

We now turn our attention to the functor $\lim^1$. As Milnor points out, it is the first right derived functor of $\lim^0$. It is defined for an inverse system $\cdots \xrightarrow{\mu_2} M_1 \xrightarrow{\mu_1} M_0$ of abelian groups to be the co-kernel of the map $\delta\colon \Pi M_i \to \Pi M_i$, where $\delta(m_0, m_1, m_2,\ldots) = (m_0 - \mu_1(m_1),\ m_1 - \mu_2(m_2),\ldots)$. Notice that the kernel of δ is the group $\lim^0 M_i$.

It is unnatural to restrict ourselves to increasing sequences of subcomplexes of a CW-complex. Milnor's theorem has a variant in which the X_i can be taken to run over all finite subcomplexes of X. This variant is an algebraic consequence of the theorem above. We shall now develop enough of the theory of the derived functors of the limit functor for our purposes.

Suppose that $\mathcal{C}$, $\mathcal{D}$ are two categories. If C is an object of $\mathcal{C}$, let $\kappa(C)$: $\mathcal{D} \to \mathcal{C}$ be the constant functor whose value is C. Recall that if Φ: $\mathcal{D} \to \mathcal{C}$, C is called a limit for Φ if there is a natural transformation $\kappa(C) \to \Phi$ such that for all C' in $\mathcal{C}$, the map $\mathrm{Hom}(C', C) \to \mathrm{Hom}(\kappa(C'), \Phi)$ is an isomorphism, where the first Hom refers to $\mathcal{C}$, the second to the category of functors from $\mathcal{D}$ to $\mathcal{C}$, and the map sends a morphism γ: $C' \to C$ to the composition $\kappa(C') \xrightarrow{\kappa(\gamma)} \kappa(C) \to \Phi$. $\mathcal{C}$ is said to be complete if for all $\mathcal{D}$ and for all Φ, Φ has a limit. Recall that limits, if they exist, are unique up to unique isomorphism.

Let $\mathcal{F}^0(\mathcal{D}, \mathcal{C})$ be the category of contravariant functors from $\mathcal{D}$ to $\mathcal{C}$ and natural transformations between functors. Notice that if $\mathcal{C}$ is complete, by making choices, one can define a functor lim: $\mathcal{F}^0(\mathcal{D}, \mathcal{C}) \to \mathcal{C}$ which assigns to each Φ its limit. Observe that lim can be characterized as a right adjoint to κ: $\mathcal{C} \to \mathcal{F}^0(\mathcal{D}, \mathcal{C})$.

Suppose now that $\mathcal{C}$ is a complete abelian category and that $\mathcal{D}$ is a small category. Notice that $\mathcal{F}^0(\mathcal{D}, \mathcal{C})$ is an abelian category and that lim is a left exact additive functor. In $\mathcal{F}^0(\mathcal{D}, \mathcal{C})$, call a monomorphism $\Phi \to \Psi$ relatively split if for all D in $\mathcal{D}$ the monomorphism $\Phi(D) \to \Psi(D)$ has a right inverse. We will say that Ξ is relatively injective if for all relatively split monomorphisms $\Phi \to \Psi$ $\mathrm{Hom}(\Psi, \Xi) \to \mathrm{Hom}(\Phi, \Xi)$ is epimorphic.

LEMMA 1.1. *There is a functor* Q: $\mathcal{F}^0(\mathcal{D}, \mathcal{C}) \to \mathcal{F}^0(\mathcal{D}, \mathcal{C})$ *and a natural transformation* η: $1 \to Q$ *such that*:

(i) *for all* Φ, $Q(\Phi)$ *is a relative injective and* $\eta(\Phi)$: $\Phi \to Q(\Phi)$ *is a relatively split monomorphism.*

(ii) Q *preserves the relatively exact sequences in* $\mathcal{F}^0(\mathcal{D}, \mathcal{C})$.

(iii) *If* $\mathcal{C}$ *has sufficiently many projectives,* Q *preserves the exact sequences in* $\mathcal{F}^0(\mathcal{D}, \mathcal{C})$.

Proof. Construct Q by letting $(Q(\Phi))(D)$ be the product over all objects D' and maps δ: $D' \to D$ in $\mathcal{D}$ by $\Phi_\delta = \Phi(D')$. If α: $D_1 \to D_2$, $(Q(\Phi))(\alpha)$ is defined by letting its projection onto Φ_δ be the projection onto $\Phi_{\alpha\delta} = \Phi_\delta$. Clearly, Q preserves the class of relatively exact sequences. Let $(\eta(\Phi))(D)$ be defined by letting its projection to Φ_δ be $\Phi(\delta)$.

If $\mathcal{C}$ has sufficiently many projectives, then in $\mathcal{C}$ and thus in $\mathcal{F}^0(\mathcal{D}, \mathcal{C})$, products preserve exact sequences. Thus Q preserves exact sequences.

Finally, to see that Q is relatively injective, observe that for any Ψ, a natural transformation $\xi: \Psi \to Q(\Phi)$ amounts to a choice for each D in $\mathcal{D}$ of a map $\gamma(D)$: $\Psi(D) \to \Phi(D)$, and that any γ defines such a ξ. The correspondence is given by letting $\gamma(D)$ be the composition of $\xi(D)$ with the projection of $(Q(\Phi))(D)$ onto $\Phi_1 = \Phi(D)$, where 1 is the identity map of D.

We now define $\lim^i$: $\mathcal{F}^0(\mathcal{D}, \mathcal{C}) \to \mathcal{C}$ to be the relatively right derived functors of lim. Since lim is left exact, $\lim = \lim^0$. Since (1.1) says that there are sufficiently many relative injectives, we can do this.

LEMMA 1.2. *If $\mathcal{C}$ has sufficiently many projectives, every short exact sequence $0 \to \Phi'' \to \Phi \to \Phi' \to 0$ in $\mathcal{F}^0(\mathcal{D}, \mathcal{C})$ has associated to it a long exact sequence*:

$$\cdots \to \lim^i(\Phi'') \to \lim^i(\Phi) \to \lim^i(\Phi') \to \lim^{i+1}(\Phi'') \to \cdots.$$

Proof. Q preserves short exact sequences.

Recall that a category $\mathcal{D}$ is called a partially ordered set if for any two objects D, D' which are distinct, Hom(D, D') and Hom(D', D) have at most one element between them. We write $D' > D$ if there is a morphism $D \to D'$. $\mathcal{D}$ is said to be filtered if for all D, D' there is a D'' with $D'' \geq D$, $D'' \geq D'$. By an inverse system we shall mean a contravariant functor Φ: $\mathcal{D} \to \mathcal{C}$ for a filtered partially ordered set $\mathcal{D}$. We call Φ an inverse sequence if $\mathcal{D}$ is the partially ordered set of the non-negative integers.

If Φ is an inverse system, Grothendieck [7] says that Φ satisfies the Mittag-Leffler (M-L) condition if for all D in $\mathcal{D}$ there is a D' with $D' \geq D$, with $\mathrm{Im}(\Phi(D') \to \Phi(D)) = \mathrm{Im}(\Phi(D'') \to \Phi(D))$ for all $D'' \geq D'$. Grothendieck shows (Proposition 13.2.2) that if $\mathcal{C}$ is the category of abelian groups and if $0 \to \Phi'' \to \Phi \to \Phi' \to 0$ is an exact sequence of inverse systems such that Φ'' satisfies M-L, then $0 \to \lim^0\Phi'' \to \lim^0\Phi \to \lim^0\Phi' \to 0$ is exact. Since there are sufficiently many relative injectives, this shows that the M-L condition implies the vanishing of $\lim^1$. While M-L is not always equivalent to the vanishing of $\lim^1$, we shall see that these two conditions are equivalent for countable inverse systems of finitely generated modules over certain rings. The following lemma is a first step toward proving this.

LEMMA 1.4. *If A is a subring of the rational numbers Q, and if Φ is an inverse sequence of finitely generated A-modules for which all $\Phi(i+j) \otimes_A Q \to \Phi(i) \otimes_A Q$ are isomorphisms, then either Φ satisfies* M-L *or* $\lim^1\Phi$ *is an uncountable group.*

Proof. If $M \to N$ is a map of finitely generated A-modules such that $M \otimes_A Q \to N \otimes_A Q$ is an isomorphism, both the kernel and the cokernel of this map are finite groups. Notice that inverse systems of finite groups satisfy M-L. Let $\Phi''(i)$ be the kernel of $\Phi(i) \to \Phi(0)$, $\Phi'(i)$ be the cokernel of this map, and let $\Psi(i)$ be the image of this map. Since Φ'' satisfies M-L, it is easy to show that Φ satisfies M-L if and only if Ψ does, since $\Psi = \Phi/\Phi''$. The constant functor with value $\Phi(0)$ clearly satisfies M-L, so we have an exact sequence $0 \to \lim^0\Psi \to \Phi(0) \to \lim^0\Phi' \to \lim^1\Psi \to 0$. If Ψ does not satisfy M-L, the index of $\Psi(i)$ in $\Phi(0)$ must be an unbounded function of i, so that Φ' becomes an inverse system of epimorphisms of finite groups of unbounded order. Thus $\lim^0\Phi'$ is uncountable. Since $\Phi(0)$ is a finitely generated A-module, it is countable. Thus $\lim^1\Psi$ is uncountable. However, we have an exact sequence $0 \to \lim^1\Phi \to \lim^1\Psi \to \lim^2\Phi''$, and as we shall show in our next lemma, $\lim^2$ vanishes for inverse sequences. Thus $\lim^1\Phi = \lim^1\Psi$ is uncountable.

Notice that an inverse system Φ of monomorphisms satisfies M-L if and only if there is a D such that for $D' \geq D$, $\Phi(D') \to \Phi(D)$ is an isomorphism. If Φ is an arbitrary inverse system, we call Φ eventually constant if such a D exists. Clearly all eventually constant inverse systems satisfy M-L.

LEMMA 1.5. *If Φ is an inverse sequence,* $\lim^i(\Phi) = 0$ *for $i \geq 2$, and Milnor's definition of* $\lim^i\Phi$ *agree with ours for $i = 0, 1$.*

Proof. Recall that $Q(\Phi)(i) = \Pi\{\Phi(j) \mid j \leq i\}$. Thus if $P(\Phi) = \lim^0 Q(\Phi)$, $P(\Phi) = \Pi\{\Phi(j) \mid 0 \leq j\}$. The short exact sequence $0 \to \Phi \to Q(\Phi) \to Q'(\Phi) \to 0$, where $Q'(\Phi) = Q(\Phi)/\Phi$, gives us an exact sequence $0 \to \lim^0 \Phi \to P(\Phi) \to \lim^0 Q'(\Phi) \to \lim^1\Phi \to 0$ and isomorphisms $\lim^{i+1}\Phi \cong \lim^i Q'(\Phi)$ for $i \geq 1$. Notice that $Q'(\Phi)(i) = \Pi\{\Phi(j) \mid j < i\}$, so $Q'(\Phi) = Q(\Phi')$, where $\Phi'(i) = \Phi(i-1)$ for $i \neq 0$, $\Phi'(0) = 0$. Thus $\lim^i Q'(\Phi) = 0$ for $i \geq 1$, or $\lim^i(\Phi) = 0$ for $i \geq 2$. Clearly $\lim^0 Q(\Phi') = P(\Phi)$, and it is elementary to verify that under this identification, the map $P(\Phi) \to \lim^0 Q(\Phi') = P(\Phi)$ is the same as Milnor's.

We next turn our attention to two related problems. First, when can the $\lim^i$ functors be calculated by restricting first to a subpartially ordered set? Second, given an inverse system over a product of two partially ordered sets, what is the relationship between the $\lim^i$ calculated with respect to both variables simultaneously and those $\lim^i \lim^j$ terms calculated with respect to the variables separately?

To begin, we must consider a more general situation. Suppose that Γ: $\mathcal{D}_1 \to \mathcal{D}_2$ is a functor. Then Γ^*: $\mathcal{F}^0(\mathcal{D}_2, \mathcal{C}) \to \mathcal{F}^0(\mathcal{D}_1, \mathcal{C})$ is defined by composition with Γ. If $\mathcal{D}_2$ consists of only one object and morphism, $\mathcal{F}^0(\mathcal{D}_2, \mathcal{C})$ is naturally isomorphic to $\mathcal{C}$, and Γ^* is the functor κ introduced before. We call any functor R_Γ: $\mathcal{F}^0(\mathcal{D}_1, \mathcal{C}) \to \mathcal{F}^0(\mathcal{D}_2, \mathcal{C})$ a right Kan extension functor if it is right adjoint to Γ^*. When $\Gamma^* = \kappa$ as above, R_Γ is just the limit functor. Since adjoints are unique up to homomorphism, if they exist, R_Γ is well defined if it exists and is simply called right Kan extension along Γ.

LEMMA 1.6. *If $\mathcal{C}$ has limits and if $\mathcal{D}_1$ and $\mathcal{D}_2$ are small, R_Γ exists for all Γ.*

Proof. For Φ: $\mathcal{D}_1 \to \mathcal{C}$, let $R_\Gamma(\Phi)(D_2)$ be the limit over the category whose objects are maps δ: $\Gamma(D_1) \to D_2$ (the comma category Γ/D_2) of the functor Φ_{D_2} which assigns to δ the value $\Phi(D_1)$ (here D_1 is in $\mathcal{D}_1$, D_2 is in $\mathcal{D}_2$). Elementary arguments show that R_Γ is right adjoint to Γ^*.

Suppose that R_Γ exists. We then have functors R^i_Γ: $\mathcal{F}^0(\mathcal{D}_1, \mathcal{C}) \to \mathcal{F}^0(\mathcal{D}_2, \mathcal{C})$ which are the relatively right derived functors of R_Γ. Since R_Γ has a left adjoint, it is right exact, so $R^0_\Gamma = R_\Gamma$. When $\Gamma = \kappa$ as above, $R^i_\Gamma = \lim^i$.

LEMMA 1.7. *If Γ has a right adjoint Δ, $R_\Gamma = \Delta^*$ and $R^i_\Gamma = 0$ for $i \neq 0$.*

Proof. Recall that Δ is right adjoint to Γ if and only if there are natural transformations η: $1 \to \Delta\Gamma$, ε: $\Gamma\Delta \to 1$ such that $(\varepsilon\Gamma)(\Gamma\eta) = 1$, $(\eta\Gamma)(\Delta\varepsilon) = 1$. However, η induces a natural transformation ε': $\Gamma^*\Delta^* \to 1$, since $(\Delta\Gamma)^* = \Gamma^*\Delta^*$, and ε induces a natural transformation η': $1 \to \Delta^*\Gamma^*$. These clearly satisfy the conditions which exhibit Δ^* as a right adjoint to Γ^*. Since Δ^* is exact, R_Γ is exact, so it is relatively exact. Thus $R^i_\Gamma = 0$ for $i \neq 0$.

If Ξ: $\mathfrak{D}_2 \to \mathfrak{D}_3$, $(\Xi\Gamma)^* = \Gamma^*\Xi^*$, so if R_Γ and R_Ξ exist, $R_{\Xi\Gamma}$ exists and is equal to $R_\Xi R_\Gamma$. Also, R_Γ preserves sufficiently many relative injectives, since it is elementary to verify that for all Φ, $R_\Gamma(Q(\Phi)) = Q(R_\Gamma\Phi)$. From this we see that in the usual manner, we have a spectral sequence with $E_2^{p,q}(\Phi) = R_\Xi^p R_\Gamma^q(\Phi)$ converging to the graded group associated to a finite filtration of $R_{\Xi\Gamma}^{p+q}(\Phi)$. In particular, when $\mathfrak{D}_3$ contains only one object and one morphism, we see that there is a spectral sequence for all Φ with $E_2^{p,q} = \lim^p R_\Gamma^q(\Phi)$ converging to the associated graded group of $\lim^{p+q}(\Phi)$.

If Ξ: $\mathfrak{D}_2 \to \mathfrak{D}_1$ is a monomorphism of partially ordered sets, Ξ has a left adjoint if and only if every D in $\mathfrak{D}_1$ has a least upper bound in $\mathfrak{D}_2$. We shall call $\mathfrak{D}_2$ completely cofinal in $\mathfrak{D}_1$ if Ξ has a left adjoint. If Γ is a left adjoint to Ξ, notice that for any inverse system Φ on $\mathfrak{D}_1$, $\lim^i\Phi = \lim^i(R_\Gamma\Phi) = \lim^i(\Xi_*\Phi) = \lim^i(\Phi \mid \mathfrak{D}_2)$. Thus the $\lim^i(\Phi)$ can be computed from the restriction of Φ to $\mathfrak{D}_2$ if $\mathfrak{D}_2$ is completely cofinal.

LEMMA 1.8. *If Φ is a countable filtered inverse system*, $\lim^i(\Phi) = 0$ *for* $i \neq 0, 1$.

Proof. Every countable filtered partially ordered set clearly has a completely cofinal sequence.

Let $\mathfrak{D}_1$, $\mathfrak{D}_2$ be two partially ordered sets, and let π_i: $\mathfrak{D}_1 \times \mathfrak{D}_2 \to \mathfrak{D}_i$ be projection onto the ith factor. If Φ: $\mathfrak{D}_1 \times \mathfrak{D}_2 \to \mathcal{C}$, for $D_2 \in \mathfrak{D}_2$, $R^i_{\pi_1}(\Phi)(D_2) = \lim^i(-, D_2)$. This follows from the observation that the comma category π_1/D_2 is a partially ordered set in which $\mathfrak{D}_1 \times [D_2]$ is completely cofinal. We shall write $\lim^i_{D_1}(\Phi)$: $\mathfrak{D}_2 \to \mathcal{C}$ for $R^i_{\pi_1}(\Phi)$. Notice that the standard spectral sequence above now has $E_2^{p,q} = \lim^p_{D_2}\lim^q_{D_1}\Phi$ and abuts to $\lim^{p+q}_{D_1,D_2}(\Phi)$.

Let Z^+ denote the set of integers i with $i \geq 0$.

LEMMA 1.9. *Suppose that* Φ: $Z^+ \times Z^+ \to \mathcal{C}$ *is an inverse system defined on pairs* (i, j) *of integers. Then* $\lim^1_i\lim^1_i\Phi = 0$, $\lim^0_{i,j}\Phi = \lim^0_i\lim^0_j\Phi$, *and there is a short exact sequence*

$$0 \to \lim^1_i\lim^0_j\Phi \to \lim^1_{i,j}\Phi \to \lim^0_i\lim^1_j\Phi \to 0.$$

Proof. $\lim^p_{i,j}\Phi = 0$ for $p \geq 2$ since $Z^+ \times Z^+$ is a filtered countable partially ordered set.

LEMMA 1.10. *If $\mathfrak{D}$ is a partially ordered set in which every pair of elements has a least upper bound, and if $\Phi\colon \mathfrak{D} \times \mathfrak{D} \to \mathcal{C}$ is an inverse system in two variables,* $\lim^i_{(D_1, D_2)}\Phi(D_1, D_2) = \lim^i_{\mathfrak{D}}\Phi(D, D)$ *for all i (that is, the* $\lim^i$ *over two variables agree with the* $\lim^i$ *taken over the diagonal).*

Proof. The diagonal $\mathfrak{D} \to \mathfrak{D} \times \mathfrak{D}$ has a left adjoint which sends (D_1, D_2) to l.u.b.(D_1, D_2).

LEMMA 1.11. *If A is a subring of the rational numbers Q, and if Φ is an inverse sequence of finitely generated A-modules, the following are equivalent.*

(a) *ϕ satisfies M-L.*

(b) $\lim^1\Phi = 0$.

(c) $\lim^1\Phi$ *is countably generated over A.*

Proof. Grothendieck showed a $\Rightarrow$ b, and clearly b $\Rightarrow$ c. Thus it suffices to show that if Φ does not satisfy M-L, $\lim^1\Phi$ is uncountable. Suppose first that Φ is an inverse sequence of monomorphisms. Since the dimensions of the $\Phi(i) \otimes_A Q$ are finite and monotone decreasing, they eventually stop. Thus there is a completely cofinal subsequence for which the $\Phi(i) \otimes_A Q \to \Phi(i-1) \otimes_A Q$ are all isomorphisms, so that we can apply (1.4) to see that $\lim^1\Phi$ is uncountable in this case.

Let $\Psi(i, j)$ be $\Phi(i)$ if $i \leq j$, and the image of $\Phi(i) \to \Phi(j)$ if $i \leq j$. By (1.10), $\lim^1_i\Phi(i) = \lim^1_i\Psi(i, i) = \lim^1_{i,j}\Psi(i, j)$. Since $\Psi(i, j+1) \to \Psi(i, j)$ is always an epimorphism, the vanishing of $\lim^2$ for sequences implies that $\lim^1_i\Psi(i, j+1) \to \lim^1_j\Psi(i, j)$ is an epimorphism for all j. If we fix j, the $\Psi(i, j)$ eventually are monomorphisms, and if Φ does not satisfy M-L, for some j the $\Psi(i, j)$ do not satisfy M-L. By our previous argument, if Φ does not satisfy M-L, there is a j for which $\lim^1_i\Psi(i, j)$ is uncountable. Since $\lim^1_i\Psi(i, j+1) \to \lim^1_i\Psi(i, j)$ is an epimorphism, $\lim^0_j\lim^1_i\Psi(i, j) \to \lim^1_i\Psi(i, j)$ is an epimorphism for all j, so $\lim^0_j\lim^1_i\Psi(i, j)$ is uncountable. By (1.9), $\lim^1_{i,j}\Psi(i, j) = \lim^1_i\Phi(i)$ is uncountable.

We next have two results on inverse systems of filtered objects. Suppose that $F_i\Phi(\alpha)$ is an inverse system where i runs over integers, α over some filtered partially ordered set, and that $F_{i+1}\Phi(\alpha) \to F_i\Phi(\alpha)$ is always a monomorphism. In this case we speak of $F_i\Phi(\alpha)$ as a filtered inverse system, and we call the cokernels $G_i\Phi(\alpha)$ of the $F_{i+1}\Phi(\alpha) \to F_i\Phi(\alpha)$ the associated graded inverse systems.

LEMMA 1.12. *If $F_i\Phi(\alpha)$ is a countable filtered inverse system of finitely generated abelian groups such that for each α there is some $i(\alpha)$ with $F_{i(\alpha)}\Phi(\alpha) = 0$, then for all i the $F_i\Phi(\alpha)$ satisfy* M-L *if the $G_i\Phi(\alpha)$ all satisfy* M-L. *Further, if $F_i\lim^0\Phi(\alpha)$ is the kernel of the map $\lim^0\Phi(\alpha) \to \lim^0(\Phi(\alpha)/F_i\Phi(\alpha))$, $G_i\lim^0\Phi(\alpha) = \lim^0 G_i\Phi(\alpha)$, $F_i\lim^0_\alpha\Phi(\alpha) \to \lim^0_\alpha F_i\Phi(\alpha)$ is an isomorphism, $\lim^0_i F_i\lim^0_\alpha\Phi(\alpha) = 0 = \lim^1_i F_i\lim^0_\alpha\Phi(\alpha)$, and $\lim^0_\alpha\Phi(\alpha) \to \lim^0_i(\lim^0_\alpha\Phi(\alpha)/F_i\lim^0_\alpha\Phi(\alpha))$ is an isomorphism.*

Proof. Notice that $\Phi(\alpha) = \Phi(\alpha)/F_{i(\alpha)}\Phi(\alpha)$, so to show that the $\Phi(\alpha)$ satisfy M-L, it suffices to show that the $\Phi(\alpha)/F_i\Phi(\alpha)$ do. Since these are finite extensions of inverse systems satisfying M-L, so do they.

The exact sequences $0 \to F_i\Phi(\alpha) \to \Phi(\alpha) \to \Phi(\alpha)/F_i\Phi(\alpha) \to 0$ give us exact sequences $0 \to \lim^0_\alpha F_i\Phi(\alpha) \to \lim^0_\alpha\Phi(\alpha) \to \lim^0_\alpha(\Phi(\alpha)/F_i\Phi(\alpha)) \to 0$, since the $F_i\Phi(\alpha)$ and the $\Phi(\alpha)$ satisfy M-L, and $\lim^1_\alpha(\Phi(\alpha)/F_i\Phi(\alpha)) = 0$. Thus $F_i\lim^0_\alpha\Phi(\alpha) = \lim^0_\alpha F_i\Phi(\alpha)$. Since $\lim^0_i F_i\Phi(\alpha) = 0 = \lim^1_i F_i\Phi(\alpha)$, we have $\lim^0_i\lim^0_\alpha F_i\Phi(\alpha) = \lim^0_\alpha\lim^0_i F_i\Phi(\alpha) = 0$, and $\lim^1_{\alpha,i} F_i\Phi(\alpha) = \lim^1_\alpha\lim^0 F_i\Phi(\alpha) = 0$. Thus $\lim^1_i F_i\lim^0_\alpha\Phi(\alpha) = \lim^1_i\lim^0_\alpha F_i\Phi(\alpha) = 0$. The exact sequences $0 \to F_{i+1}\Phi(\alpha) \to F_i\Phi(\alpha) \to G_i\Phi(\alpha) \to 0$ give us $0 \to \lim^0_\alpha F_{i+1}\Phi(\alpha) \to \lim^0_\alpha\Phi(\alpha) \to \lim^0_\alpha G_i\Phi(\alpha) \to 0$, or $\lim^0_\alpha G_i\Phi(\alpha) = G_i\lim^0_\alpha\Phi(\alpha)$. The vanishing of $\lim^0_i F_i\lim^0_\alpha\Phi(\alpha)$ and of $\lim^1_i F_i\lim^0_\alpha\Phi(\alpha)$ give us the isomorphism $\lim^0_\alpha\Phi(\alpha) \to \lim^0_i(\lim^0_\alpha\Phi(\alpha)/F_i\lim^0_\alpha\Phi(\alpha))$.

Notice that if M is an abelian group with a decreasing filtration F_iM, the F_iM define a topology on M. This topology is Hausdorff if and only if $\lim^0_i F_iM = 0$, or equivalently, $M \to \lim^0_i M/F_iM$ is a monomorphism. M is complete if and only if $M \to \lim^0_i M/F_iM$ is an epimorphism — that is, $\lim^1 F_iM = 0$. Notice that if M is a complete Hausdorff group topologized by a sequence F_iM of subgroups, the F_iM will not satisfy M-L unless $F_iM = \{0\}$ for some i, or in other words, M is discrete. Thus if M is a complete Hausdorff group topologized by subgroups F_iM, if M is not discrete the F_iM do not satisfy M-L even though $\lim^1_i F_iM = 0$. Thus something like the finite generation hypothesis is needed in proving that (1.11b) implies (1.11a).

LEMMA 1.13. *Suppose that $F_i\Phi(\alpha)$ is a filtered countable inverse system of filtered objects. Suppose that*

(a) $\lim^1_\alpha G_i\Phi(\alpha) = 0$ *all i,*

(b) *every $\Phi(\alpha)$ is a complete Hausdorff group in the filtration topology. Then $\lim^1_\alpha\Phi(\alpha) = 0$, and if $F_i\lim^0_\alpha\Phi(\alpha)$ is the kernel of $\lim^0_\alpha\Phi(\alpha) \to \lim^0_\alpha(\Phi(\alpha)/F_i\Phi(\alpha))$, $\lim^0_\alpha\Phi(\alpha)$ is a complete Hausdorff group with $G_i\lim^0_\alpha\Phi(\alpha) = \lim^0_\alpha G_i\Phi(\alpha)$.*

Proof. Since $\Phi(\alpha) = \lim_i^0(\Phi(\alpha)/F_i\Phi(\alpha))$, to show that $\lim^1_\alpha \Phi(\alpha) = 0$ it suffices to show that $\lim^1_{i,\alpha}(\Phi(\alpha)/F_i\Phi(\alpha)) = 0$. From the exact sequences $0 \to G_i\Phi(\alpha) \to \Phi(\alpha)/F_{i+1}\Phi(\alpha) \to \Phi(\alpha)/F_i\Phi(\alpha) \to 0$, condition (a) shows that $0 \to \lim^0_\alpha G_i\Phi(\alpha) \to \lim^0_\alpha(\Phi(\alpha)/F_{i+1}\Phi(\alpha)) \to \lim^0_\alpha(\Phi(\alpha)/F_i\Phi(\alpha)) \to 0$ is exact, and that $\lim^1_\alpha(\Phi(\alpha)/F_{i+1}\Phi(\alpha)) = \lim^1_\alpha(\Phi(\alpha)/F_i\Phi(\alpha))$. Since $\Phi(\alpha)/F_0\Phi(\alpha) = 0$, $\lim^1_\alpha(\Phi(\alpha)/F_i\Phi(\alpha)) = 0$ all i, so $\lim_i^0\lim^1_\alpha(\Phi(\alpha)/F_i\Phi(\alpha)) = 0$. Since $\lim^0_\alpha(\Phi(\alpha)/F_{i+1}\Phi(\alpha)) \to \lim^0_\alpha(\Phi(\alpha)/F_i\Phi(\alpha))$ is always an epimorphism, $\lim^1_i\lim^0_\alpha(\Phi(\alpha)/F_i\Phi(\alpha)) = 0$. Thus $\lim^1_{i,\alpha}(\Phi(\alpha)/F_i\Phi(\alpha)) = 0$.

Each $F_i\Phi(\alpha)$ satisfies the same conditions as $\Phi(\alpha)$, and thus $\lim^1_\alpha F_i\Phi(\alpha) = 0$ for all i. Thus the sequence $0 \to \lim^0_\alpha F_i\Phi(\alpha) \to \lim^0_\alpha\Phi(\alpha) \to \lim^0_\alpha(\Phi(\alpha)/F_i\Phi(\alpha)) \to 0$ is exact, so that $F_i\lim^0_\alpha\Phi(\alpha) = \lim^0_\alpha F_i\Phi(\alpha)$. Similarly, the vanishing of the $\lim^1_\alpha F_{i+1}\Phi(\alpha)$ implies that the $0 \to F_{i+1}\lim^0_\alpha\Phi(\alpha) \to F_i\lim^0_\alpha\Phi(\alpha) \to \lim^0 G_i\Phi(\alpha) \to 0$ are exact, so $G_i\lim^0_\alpha\Phi(\alpha) = \lim^0_\alpha G_i\Phi(\alpha)$.

2. Filtrations, spectral sequences, and limits. This section will be devoted to two theorems which state that certain $\lim^1$ terms vanish if a suitable spectral sequence is locally eventually constant.

Suppose that $\phi = X_{-1} \to X_0 \to \cdots$ is a sequence of cofibrations whose colimit is X. Then for any additive cohomology theory h^*, we obtain in the manner of Dold [6] a spectral sequence with $E_1^{p,q}(X) = h^{p+q}(X_p, X_{p-1})$. If $X = X_n$ for some n, $E_\infty^{p,q}(X)$ is the pth graded group associated with a finite filtration of $h^{p+q}(X)$. We will say that the filtration $\{X_p\}$ of X is an M-L filtration (with respect to h^*) if for all p, q the inverse system $E_{r+p}^{p,q}(X)$ is eventually constant.

THEOREM 2.1. *Suppose that $\{X_p\}$ is an* M-L *filtration of X. Then $\lim^1 h^*(X_p) = 0$ and $h^*(X) \to \lim^0 h^*(X_p)$ is an isomorphism. If $F_p h^*(X)$ is the kernel of $h^*(X) \to h^*(X_{p-1})$, $h^*(X) = \lim^0(h^*(X)/F_p h^*(X))$, and $E_\infty^{p,q}(X)$ is the pth graded group of $h^{p+q}(X)$ associated to this filtration.*

Proof. Since the X_n are finitely filtered, each spectral sequence $E_r^{**}(X_n)$ is eventually constant, with $E_\infty^{p,q}(X_n) = G_p H^{p+q}(X_n)$. By (1.12), it suffices to show that $\lim E_\infty^{p,q}(X_n) = 0$ for all p, q.

The map $E_1^{p,q}(X) \to E_1^{p,q}(X_n)$ is an isomorphism for $p \leq n$. Since $E_1^{p,q}(X_n) = 0$ for $n > p$, we see that $E_r^{p,q}(X) \to E_r^{p,q}(X_n)$ is an isomorphism for $p \leq n - r + 1$. Thus, if $r = r(p, q)$ is such that $E_r^{p,q} = E_\infty^{p,q}(X)$,

$E_\infty^{p,q}(X) \to E_\infty^{p,q}(X_n)$ is an isomorphism for $n \geq p + r - 1$, and thus $E_\infty^{p,q}(X_n)$ is independent of n for n sufficiently large. Thus $\lim^1 E_\infty^{p,q}(X_n) = 0$.

REMARK 2.2. If the $h^p(X_n)$ are all finitely generated A-modules, it is not difficult to show that $\lim^1 h^*(X_n) = 0$ implies that the filtration $\{X_n\}$ is M-L. As we shall not need this, we leave the proof to the reader.

THEOREM 2.3. *Suppose that* $X = \operatorname{colim} X_{\alpha,n}$ *where* n *runs over the integers and* α *runs over some countable filtered partially ordered set. Let* Z_n *be the colimit over* α *of the* $X_{\alpha,n}$ *and* Y_α *be the colimit over* n *of the* $X_{\alpha,n}$. *Suppose*:

(a) *for all* α, $\{X_{\alpha,n}\}$ *is an* M-L *filtration of* Y_α,

(b) $\{Z_n\}$ *is an* M-L *filtration of* X,

(c) *for all* p, q, $E_1^{p,q}(X) = \lim^0 E_1^{p,q}(Y_\alpha)$ *and* $\lim^1 E_1^{p,q}(Y_\alpha) = 0$,

(d) *for all* p, q, α, $E_2^{p,q}(X)$ *is a finitely generated* A-*module and* $E_2^{p,q}(X_\alpha)$ *is a finitely generated* A-*module*,

(e) *for* $\alpha \leq \beta$, $Y_\alpha \to Y_\beta$ *is a cofibration.*

Then $h^*(X) = \lim^0 H^*(Y_\alpha)$, $\lim^1 h^*(Y_\alpha) = 0$, *and for all* p, q, r, $E_r^{p,q}(X) = \lim^0 E_r^{p,q}(Y_\alpha)$, $\lim E_r^{p,q}(Y_\alpha) = 0$.

Proof. If C_α^* is an inverse system of cochain complexes with $\lim^1_\alpha C_\alpha^* = 0$, standard homological algebra gives us exact sequences $0 \to \lim^1 H^{n-1}(C_\alpha^*) \to H^n(\lim^0 C_\alpha^*) \to \lim^0 H^n(C_\alpha^*) \to 0$. Inductively, this gives us short exact sequences $0 \to \lim^1 E_r^{p-r,q+r-1}(X_\alpha) \to E_r^{p,q}(X) \to \lim^0 E_r^{p,q}(X_\alpha) \to 0$. Since $E_r^{p,q}(X)$ is finitely generated, $\lim^1 E_r^{p-r,q+r-1}(X_\alpha)$ is finitely generated and thus vanishes by (1.11). Thus, inductively, $E_r^{p,q}(X) = \lim^0_\alpha E_r^{p,q}(X_\alpha)$. By (1.13), to show that $h^*(X) = \lim^0 h^*(Y_\alpha)$ and that $\lim^1 h^*(Y_\alpha) = 0$, it suffices to show that $\lim^1_\alpha E_\infty^{p,q}(Y_\alpha) = 0$ for all p, q. Since $\lim^1_\alpha E_\infty^{p,q}(Y_\alpha) = \lim^1_\alpha \lim^0_r E_{r+p}^{p,q}(Y_\alpha)$, it suffices by (1.9) to show that $\lim^1_{\alpha,r} E_{r+p}^{p,q}(Y_\alpha) = 0$. Since $\lim^0_\alpha E_{r+p}^{p,q}(Y_\alpha) = E_{r+p}^{p,q}(X)$, we see that hypothesis (b) above implies that $\lim^1_r \lim^0_\alpha E_{r+p}^{p,q}(Y_\alpha) = 0$. Since $\lim^1_\alpha E_{r+p}^{p,q}(Y_\alpha) = 0$, $\lim^0_r \lim^1_\alpha E_{r+p}^{p,q}(Y_\alpha) = 0$. Thus $\lim^1_{r,\alpha} E_{r+p}^{p,q}(Y_\alpha) = 0$.

COROLLARY 2.4. *If* X *is a countable* CW-*complex, and if the following two conditions are satisfied, then* $h^*(X) = \lim^0 h^*(Y_\alpha)$ *where* Y_α *runs over the finite subcomplexes of* X, *and* $\lim^1 h^*(Y_\alpha) = 0$:

(a) $H^p(X; h^q(\text{point}))$ *is a finitely generated* A-*module for all* p, q.

(b) *The skeletal filtration of* X *is* M-L.

Proof. In this case, $E_1^{p,q}(Y_\alpha)$ is the p-cochain $C^p(Y_\alpha; h^q(\text{point}))$, and so the $E_1^{p,q}(Y_\alpha)$ form an inverse system of epimorphisms, and thus satisfy M-L. Thus (2.3c) is satisfied. Each Y_α is of finite dimension, so (2.3a) is satisfied. By hypothesis, (2.3b) and (2.3d) are satisfied, and since the inclusion of a subcomplex in a CW-complex is a cofibration, we see that (2.3e) is satisfied.

If h^* is a multiplicative cohomology theory with $A \subset h^0(\text{point})$, it is not difficult to show that the product of two A-locally finite complexes for which the skeletal filtration is M-L again has an M-L skeletal filtration. This follows from the observation that the image of the map

$$\bigoplus_{p'+p''=p} E_2^{p',0}(X) \otimes_{h^0(\text{point})} E_2^{p'',q}(Y) \to E_2^{p,q}(X \times Y)$$

has finite cokernel. Thus if $Z_\infty^{p',0}(X)$, $Z_\infty^{p'',q}(Y)$ have finite index in $E_2^{p',0}(X)$, $E_2^{p'',q}(Y)$ respectively, we see that $Z_\infty^{p,q}(X \times Y)$ has finite index in $E_2^{p,q}(X \times Y)$. Since all groups are finitely generated A-modules, the condition that the skeletal filtration be M-L is the same as the condition that the index of the Z_∞-terms in the E_2-terms be finite in every degree. Thus we see that the category of locally A-finite complexes for which the skeletal filtration is M-L is closed under finite Cartesian product (and, obviously, also smash product). It seems that this should be true whether or not h^* is multiplicative, but I see no way to prove it.

3. Spectra. In order to compute the stable cohomology operations from one representable spectrum to another, one computes the inverse limit of a sequence of cohomology groups. More precisely, if **X**, **Y** are two Ω-spectra whose terms are $\{X_n\}$, $\{Y_n\}$, the stable cohomology operations of degree t from $H^*(-,\mathbf{X})$ to $H^*(-; Y)$ are the elements of $\lim^0 H^{n+t}(X_n; \mathbf{Y})$. Since inverse limit is not exact, these functors do not behave well with respect to various constructions such as mapping cones and localizations in the **X** variable. We shall show how to define groups $H^t(\mathbf{X}; \mathbf{Y})$ which do behave well with respect to the usual constructions (in particular, for which the Mayer-Vietoris sequence is exact). Further, these are related to the cohomology operations above by a short exact sequence, which turns out to be closely related to Milnor's:

(3.1) $\quad 0 \to \lim^1 H^{n+t-1}(X_n; \mathbf{Y}) \to H^t(\mathbf{X}; \mathbf{Y}) \to \lim^0 H^{n+t}(X_n; \mathbf{Y}) \to 0.$

The obvious definition of $H^0(\mathbf{X}; \mathbf{Y})$ should be, in some sense, the set of homotopy classes of maps from **X** to **Y**. This will be essentially our definition, which will allow us to define $H^{-n}(\mathbf{X}; \mathbf{Y}) = H^0(S^n\mathbf{X}; \mathbf{Y})$ for

$n \geq 0$, where suspension is done degreewise. Since $\mathbf{Y}$ is an Ω-spectrum, it is an infinite loop space in the category of spectra, since $\mathbf{Y} = \Omega^n \mathbf{Y}_n$, where $\mathbf{Y}_n$ in degree m is $\Omega^n Y_{n+m}$.

Unfortunately, the homotopy theory of spectra is somewhat complicated. We have elected to describe it here in terms of Quillen's axiomatic homotopy theory because of the generality of the results, the fact that in Quillen's formulation all of the standard apparatus of homotopy theory is available, and because this approach makes many things easier to prove, since typically we only need prove that a map between spectra is a homotopy equivalence in every degree to know that it is an isomorphism in the homotopy category, rather than prove that it is a homotopy equivalence in the category of spectra, which might be true only if we have been extremely careful in our constructions. The experts will notice that we are describing the homotopy theory of what Kan calls prespectra rather than the Kan-Boardman homotopy theory of spectra. We avoid the truly "stable" homotopy theory of Kan and Boardman because it is more complicated than our theory and because it has few advantages for us.

We begin with an abstract description of spectra which will allow us to talk about both spectra in the homotopy category and the homotopy category of spectra. Suppose that $\mathfrak{T}$ is a category with a pair of functors Σ, Ω: $\mathfrak{T} \to \mathfrak{T}$ so that Σ is left adjoint to Ω. By a spectrum $\mathbf{X}$ over $\mathfrak{T}$ we mean a collection of objects X_n out of $\mathfrak{T}$ ($n \geq 0$) together with morphisms x_n: $\Sigma X_n \to X_{n+1}$ (or, equivalently, x_n^*: $X_n \to \Omega X_{n+1}$). We call the X_n the terms of $\mathbf{X}$ and the x_n the structure maps of $\mathbf{X}$. By a map f: $\mathbf{X} \to Y$ of spectra we mean a collection of maps f_n: $X_n \to Y_n$ such that for all $n, f_{n+1}x_n = y_n\Sigma(f_n)$. We write Spect($\mathfrak{T}$) for the category of spectra over $\mathfrak{T}$.

In Spect($\mathfrak{T}$), limits and colimits are formed termwise. There are functors T_n: Spect($\mathfrak{T}$) $\to \mathfrak{T}$, $\mathbf{S}^n$: $\mathfrak{T} \to$ Spect($\mathfrak{T}$) given by $T_n(\mathbf{X}) = X_n$, $T_nS^n(X) = \Sigma^{n+m}(X)$. Clearly, $\mathbf{S}^n$ is left adjoint to T_n.

If $\mathfrak{T}$ is the category of basepointed topological spaces, the suspension and loop space functors are a pair of adjoint functors Σ, Ω: $\mathfrak{T} \to \mathfrak{T}$. These induce a pair of adjoint functors σ, ω: Ho($\mathfrak{T}$) $\to$ Ho($\mathfrak{T}$). A spectrum over Ho($\mathfrak{T}$) is represented by a spectrum over $\mathfrak{T}$ which in each term has a nondegenerate basepoint. There are more maps between two spectra over Ho($\mathfrak{T}$) than there are between their representatives in Spect($\mathfrak{T}$), for all that is required in Ho($\mathfrak{T}$) is that the representing maps $f_{n+1}x_n$ and $y_n\Sigma(f_n)$ be homotopic. If each x_n: $\Sigma X_n \to X_{n+1}$ is a cofibration, then, beginning with f_1, we can modify the f_n by a homotopy so that they define a map of spectra. Thus, if we are careful in our choice of representatives, every object and every map in Spect Ho($\mathfrak{T}$) can be represented in Spect($\mathfrak{T}$).

Recall that a category $\mathfrak{T}$ is called a closed model category if certain of the morphisms have been distinguished with the terms "cofibration", "fibration", and "weak equivalences" so that certain axioms are satisfied (see Quillen **[10]** or **[9]** or Bousfield-Kan **[4]**). The homotopy category Ho($\mathfrak{T}$) is obtained from $\mathfrak{T}$ by localizing with respect to the weak equivalences. If $\tau\colon \mathfrak{T} \to \mathrm{Ho}(\mathfrak{T})$ is the localizing functor, for X, Y objects in $\mathfrak{T}$, $\mathrm{Hom}(\tau(X), \tau(Y))$ is called the set of homotopy classes of maps from X to Y.

On Ho($\mathfrak{T}$), suspension and loop space functors can be defined (see Quillen **[9]**), but these are not induced by functors on $\mathfrak{T}$ unless $\mathfrak{T}$ has further structure. This extra structure, described below, will always exist for what Quillen calls a closed simplicial model category.

The axioms which Quillen gives are as follows:

(CM 1) $\mathfrak{T}$ has finite limits and finite colimits.

(CM 2) If f and g are maps such that gf is defined, then if two of f, g, gf are weak equivalences, so is the third.

(CM 3) Every retract of a cofibration, weak equivalence, or fibration is a cofibration, weak equivalence, or fibration, respectively.

(CM 4) (Lifting) Given a solid arrow diagram

$$\begin{array}{ccc} A & \rightarrow & X \\ \downarrow i & \nearrow & \downarrow p \\ B & \rightarrow & Y \end{array}$$

the dotted arrow exists if i is a cofibration, p is a fibration, and either is a weak equivalence.

(CM 5) (Factorization) Any map f can be factored in as $f = pi$ where i is a cofibration, p is a fibration, and either p or i can be required to be also a weak equivalence.

An object A is called cofibrant if the initial map $\phi \to A$ is a cofibration. An object X is called fibrant if the terminal map $X \to *$ is a fibration. A cofibration or a fibration is called trivial if it is also a weak equivalence.

The composition of two cofibrations or of two fibrations is again a cofibration or a fibration, respectively. The base extension of a fibration or of a trivial fibration is a fibration or a trivial fibration, respectively. The cobase extension of a cofibration or of a trivial cofibration is a cofibration or a trivial cofibration, respectively. Every isomorphism is a cofibration, a weak equivalence, and a fibration.

If $\mathfrak{S}$, $\mathfrak{T}$ are two model categories, and if F: $\mathfrak{S} \to \mathfrak{T}$ is a functor which preserves weak equivalences between cofibrant objects, there is an induced functor LF: $\mathrm{Ho}(\mathfrak{S}) \to \mathrm{Ho}(\mathfrak{T})$ defined as follows. If S is an object of $\mathfrak{S}$, choose a factorization $\phi \xrightarrow{i} S' \xrightarrow{p} S$ where i is a cofibration and p is a trivial fibration. Then $(LF)(S) = \tau F(S')$, where τ: $\mathfrak{T} \to \mathrm{Ho}(\mathfrak{T})$ is the localization. It is not difficult to extend LF to maps, once it is remembered that τ carries weak equivalences to isomorphisms. Similarly, if G: $\mathfrak{S} \to \mathfrak{T}$ preserves weak equivalences between fibrant objects, one can similarly construct RG: $\mathrm{Ho}(\mathfrak{S}) \to \mathrm{Ho}(\mathfrak{T})$.

We will call $\mathfrak{T}$ a spectral category if $\mathfrak{T}$ is a pointed closed model category, and there are functors Σ, Ω: $\mathfrak{T} \to \mathfrak{T}$ with Σ left adjoint to Ω, such that

(S 1) Σ preserves cofibrations, trivial cofibrations, and weak equivalences between cofibrant objects.

(S 2) Ω preserves fibrations, trivial fibrations, and weak equivalences between fibrant objects.

According to Quillen ([**9**] Theorem I.4.3), $L\Sigma$: $\mathrm{Ho}(\mathfrak{T}) \to \mathrm{Ho}(\mathfrak{T})$ is left adjoint to $R\Omega$.

Every pointed closed simplicial model category in Quillen's sense is a spectral category. Thus the category of topological spaces is a spectral category if the weak equivalences are taken to be the usual homotopy equivalences (see Strøm [**11**]) or if they are taken to be the maps inducing homotopy equivalences on all homotopy groups with all basepoints (see Quillen [**9**]).

Suppose now that $\mathfrak{T}$ is a spectral category. In Spect($\mathfrak{T}$), we define a map $\mathbf{f}$: $\mathbf{X} \to \mathbf{Y}$ to be:

(a) a cofibration if all the maps $\Sigma(Y_n) \sqcup_{(X_n)} X_{n+1} \to Y_{n+1}$ for $n \geq 0$ and $X_0 \to Y_0$ are cofibrations,

(b) a weak equivalence if all $X_n \to Y_n$ are weak equivalences,

(c) a fibration if all $X_n \to Y_n$ are fibrations.

THEOREM 3.2. Spect($\mathfrak{T}$) *is a closed model category if* $\mathfrak{T}$ *is a spectral category.*

Proof. Since Σ preserves colimits and Ω preserves limits, (CM 1) is satisfied. Clearly (CM 2) is satisfied. It is easy to prove (CM 4) by induction.

The only part of (CM 3) which is nontrivial is to show that a retract of a cofibration is a cofibration. However, this follows directly from the fact that Σ and the cofiber sum construction are functors and that (CM 3) holds for $\mathfrak{T}$. To show the factorization axiom, suppose that f: $\mathbf{X} \to Y$ is a map of spectra. Factor $f_0 = p_0 i_0$ so that i_0 is a trivial cofibration, p_0: $X_0' \to Y_0$ is a fibration. Let $X_1'' = X_1 \coprod_{\Sigma X_0} \Sigma X_1'$. Since $\Sigma X_0 \to \Sigma X_1'$ is a trivial cofibration, its cobase extension $X_1 \to X_1''$ is a trivial cofibration. Factor $X_1'' \to Y_1$ into a trivial cofibration $X_1'' \to X_1'$ followed by a fibration p_0: $X_1' \to Y_1$, and let i_1: $X_1 \to X_1'$ be the composition of the given maps. Continue on by induction. The other half of (CM 5) is similarly proved.

Let σ: Spect($\mathfrak{T}$) $\to$ Ho(Spect($\mathfrak{T}$)) be the localization functor. Notice that τ induces a functor which we denote by τ^*: Ho(Spect($\mathfrak{T}$)) $\to$ Spect($H_0(\mathfrak{T})$), which sends $\sigma(\mathbf{X})$ to the spectrum whose nth term is $\tau(X_n)$.

THEOREM 3.3. *τ^* is epimorphic on objects, and for all* $\mathbf{X}$, $\mathbf{Y}$, τ^*: Hom($\sigma(\mathbf{X})$, $\sigma(\mathbf{Y})$) $\to$ Hom($\tau^*\sigma(\mathbf{X})$, $\tau^*\sigma(\mathbf{Y})$) *is an epimorphism.*

Proof. Suppose that $\mathbf{Z}'$ is a spectrum over Ho($\mathfrak{T}$). Then choose Z_n cofibrant — fibrant in $\mathfrak{T}$ so that $\tau(Z_n)$ is the nth term Z_n' of $\mathbf{Z}'$. Since ΣZ_n is cofibrant and Z_{n+1} is fibrant,

$$\mathrm{Hom}(\Sigma Z_n, Z_{n+1}) \to \mathrm{Hom}(\tau(\Sigma Z_n), \tau(Z_{n+1}))$$

is an epimorphism (see Quillen [9]), we can choose a map z_n: $\Sigma Z_n \to Z_{n+1}$ such that $\tau(z_n)$ is the structure map z_n': $\Sigma Z_n' \to Z_{n+1}'$.

To prove the second part, we can choose $\mathbf{X}$ cofibrant and $\mathbf{Y}$ fibrant without changing the isomorphism classes of $\sigma(X)$ and $\sigma(Y)$. If we choose representatives f: $X_n \to Y_n$ for the maps $\tau^*\sigma(\mathbf{X})_n \to \tau^*\sigma(\mathbf{Y})_n$, the two maps $\Sigma x_n \to Y_{n+1}$ defined by f_n and f_{n+1} agree in the homotopy category. Since ΣX_n is cofibrant and Y_{n+1} is fibrant, these two maps are left homotopic (see Quillen [9], §I.1). Cofibrations between cofibrant objects have the left HEP for maps into fibrant objects, so since $\Sigma X_n \to X_{n+1}$ is a cofibration, f_{n+1} could have been chosen so that f_{n+1} and Σf_n define the same maps $\Sigma X_n \to Y_{n+1}$. Proceeding inductively, we obtain a spectrum map f: $\mathbf{X} \to \mathbf{Y}$ which is carried to the given map $\tau^*\sigma(\mathbf{X}) \to \tau^*\sigma(\mathbf{Y})$ by $\tau^*\sigma$. Thus Hom($\mathbf{X}, \mathbf{Y}$) $\to$ Hom($\tau^*\sigma(\mathbf{X})$, $\tau^*\sigma(\mathbf{Y})$) is an epimorphism.

From now on, we shall assume that on Ho($\mathfrak{T}$), $L\Sigma$ is naturally isomorphic to suspension, or, equivalently that $R\Omega$ is naturally isomorphic to the loop space functor. The next lemma is somewhat tedious to prove, so we defer the proof until we have investigated its applications.

LEMMA 3.4. *Define* $\mathbf{\Sigma}$, $\mathbf{\Omega}$: Spect($\mathfrak{T}$) $\to$ Spect($\mathfrak{T}$) *by* $(\mathbf{\Sigma}(\mathbf{X}))_n = \Sigma X_n$, $(\mathbf{\Omega}(\mathbf{X}))_n = \Sigma X_n$. *Then for all* $\mathbf{X}$, *there is a natural isomorphism between* $L\mathbf{\Sigma}(\sigma(\mathbf{X}))$ *and the suspension of* $\sigma(\mathbf{X})$ *and a natural isomorphism between* $R\mathbf{\Omega}(\sigma(\mathbf{X}))$ *and the loop space of* $\sigma(\mathbf{X})$. *Further, these isomorphisms can be chosen to form natural isomorphisms of functors.*

Suppose now that $\mathbf{Y}$ is an Ω-spectrum; that is, each $Y_n \to \Omega Y_{n+1}$ is a weak equivalence and $\mathbf{Y}$ is fibrant. Define $(B\mathbf{Y})_n = Y_{n+1}$. Then the natural transformation $\mathbf{Y} \to \mathbf{\Omega B Y}$ will be a weak equivalence if Y is an Ω-spectrum and $\mathbf{\Omega B Y}$ will also be an Ω-spectrum. By the lemma above, there is a cofibrant spectrum $\mathbf{Z}$ which is a loop space for $\mathbf{BY}$, together with a weak equivalence $\mathbf{Z} \to \mathbf{\Omega B Y}$. Consequently, for all $\mathbf{A}$ there is a natural isomorphism

$$(3.5) \qquad \operatorname{Hom}(\tau(\mathbf{A}), \tau(Y)) \cong \operatorname{Hom}(\tau(A), R\mathbf{\Omega}(\mathbf{B}Y)).$$

Since $\mathbf{\Omega}$ preserves fibrations, $R(\mathbf{\Omega}^n) = (R\mathbf{\Omega})^n$, so we see that we have for all $\mathbf{A}$ and all $n \geq 0$ a natural isomorphism

$$(3.6) \qquad \operatorname{Hom}(\tau(A), \tau(Y)) \cong \operatorname{Hom}(\tau(A), R\mathbf{\Omega}^n\tau(\mathbf{B}^n Y)).$$

We now define, for $\mathbf{X}$ any spectrum, $\mathbf{Y}$ an $\mathbf{\Omega}$ spectrum, and n any integer, functors $H^n(\mathbf{X}; \mathbf{Y}) = \operatorname{Hom}(L\mathbf{\Sigma}^t\sigma(\mathbf{X}), R\mathbf{\Omega}^{n+t}\sigma(\mathbf{B}^{n+t}(\mathbf{Y}))$ for $t \geq 0$ chosen so that $n + t \geq 0$. Since for any model category, the iterated loop functors on the homotopy category take their values in the abelian group valued objects after the second loop space functor (see Quillen [**9**]), the $H^n(\mathbf{X}; \mathbf{Y})$ are abelian groups in a natural manner.

If X is in $\mathfrak{T}$, we define the cohomology groups of T with values in the Ω-spectrum $\mathbf{Y}$ by $H^n(X; \mathbf{Y}) = \operatorname{Hom}(L\Sigma^t\tau(X); \tau(Y_{n+t}))$ for $t \geq 0$ such that $n + t \geq 0$. Notice that since $\mathfrak{T}$ was assumed pointed, these cohomology groups are what are usually called the reduced cohomology groups. Notice that $H^n(X; \mathbf{Y}) = H^n(\mathbf{S}^0(X); \mathbf{Y})$, where $\mathbf{S}^0(X)$ is the spectrum whose nth term is $\Sigma^m X$.

If $\mathbf{X}$ is any cofibrant spectrum, let $\mathbf{X}_m$ be the spectrum whose nth term is X_n if $n \leq m$, and is $\Sigma^{n-m}X_m$ if $n \geq m$. Then if X_{-1} is the initial spectrum, $\mathbf{X}_{-1} \to \mathbf{X}_0 \to \mathbf{X}_1 \to \cdots$ is a sequence of cofibrations whose colimit is $\mathbf{X}$. If, for a moment, we assume that the theorem of Milnor's generalizes, from the observation that $H^h(\mathbf{X}_m; \mathbf{Y}) = H^{n+m}(X_m; \mathbf{Y})$ for any Ω-spectrum $\mathbf{Y}$, we obtain a short exact sequence:

$$(3.7) \quad 0 \to \lim{}^1 H^{n+m-1}(X_m; \mathbf{Y}) \to H^n(\mathbf{X}; \mathbf{Y}) \to \lim{}^0 H^{n+m}(X_m; \mathbf{Y}) \to 0.$$

Notice that for topological spaces in the Serre homotopy theory, if $\mathbf{Y}$ is an Eilenberg-MacLane spectrum and if $\mathbf{X}$ has the property that the X_n are $(n-1)$ connected and also that the maps $X_n \to \Omega X_{n+1}$ induce an isomorphism on homotopy groups up to dimension $n + \varphi(n)$, where $\varphi(n)$ is an unbounded nondecreasing function, then for all n, $H^n(\mathbf{X}; \mathbf{Y}) = \lim^0 H^{n+m}(X_m; \mathbf{Y})$, and $\lim^1 H^{n+m}(X_m; \mathbf{Y}) = 0$. In particular, if $\mathbf{X}$ is a connected Ω-spectrum, this will be true.

Before we prove the generalization of Milnor's theorem needed to prove the exactness of (3.7), we investigate some of its consequences. First, suppose that $A \subset Q$ is a subring of the rationals. Notice that an abelian group M has at most one A-module structure, and it has an A-module structure if and only if for every prime p which is a unit in A, M is uniquely p-divisible. Thus all abelian group extensions of A-modules are also A-modules in a well defined manner. Since the forgetful functor from A-modules to abelian groups is exact and preserves limits, it is compatible with the functors $\lim^i$.

THEOREM 3.8. *If* $\mathbf{Y}$ *is an* Ω*-spectrum of topological spaces such that every* $H^n(S^0; \mathbf{Y})$ *is a finitely generated* A*-module, then for all spectra* $\mathbf{X}$, $H^n(\mathbf{X}; \mathbf{Y})$ *is an* A*-module for all* n.

Proof. We may assume that each Y_n is a countable CW-complex without changing the weak homotopy type of $\mathbf{Y}$, since the homotopy groups of each Y_n are all countable. For any finite CW-complex K, the groups $H^n(K; \mathbf{Y})$ are all finite extensions of A-modules, and thus are A-modules. From Milnor's theorem, the groups $H^{n+m}(Y_m; \mathbf{Y})$ are extensions of A-modules and thus are A-modules. Finally, by (3.7), the groups $H^n(\mathbf{Y}; \mathbf{Y})$ are A-modules. Thus, there is in $H^0(\mathbf{Y}; \mathbf{Y})$ an A-submodule M generated by the identity map. Thus, if i is the residue class of the identity, it suffices to show that for all integers p which are units in A, $(pi) \cdot (p^{-1}i) = i = (p^{-1}i) \cdot (pi)$, where $\cdot$ denotes composition of cohomology operations. It is a standard exercise to show that for all α: $\mathbf{Y} \to \mathbf{Y}$, $(pi) \cdot \alpha = p\alpha$, so the relation $(pi) \cdot (p^{-1}i) = i$ is immediate. The relation $\alpha \cdot (pi) = p\alpha$ for all α follows from the observation that for any two cofibrant spectra $\mathbf{X}'$, $\mathbf{X}''$, $H^n(\mathbf{X}' \vee \mathbf{X}''; \mathbf{Y}) = H^n(\mathbf{X}'; \mathbf{Y}) \oplus H^n(\mathbf{X}''; \mathbf{Y})$ for all n, where $\vee$ denotes one point union. This fact follows from the usual exact sequence for a cofibration $\mathbf{X}' \to \mathbf{X}' \vee \mathbf{X}''$.

We now show that there is an exact sequence as described in (3.7) above. To do this, it suffices to show that Milnor's theorem can be generalized to additive cohomology theories on pointed model categories. The proof which Milnor gives is entirely general except in one point. This concerns a construction which begins with a sequence of inclusions $X_0 \to X_1 \to X_2 \to \cdots$ of subcomplexes of a CW-complex X, whose union is X, and forming the mapping cylinders $M_0 \to M_1 \to \cdots$. Milnor's proof that the map $M \to X$ from the colimit M of the mapping cylinders to X relies on the fact that this map induces an isomorphism on homotopy groups. This fact can be replaced by a lemma of Reedy's (see Anderson [3]) which states that if one has a map between two sequences of cofibrations with cofibrant initial objects, if that map is a weak equivalence at every stage, the colimit of that map is a weak equivalence. With this modification, Milnor's proof and the usual "Puppe sequence" types of arguments yield the following.

LEMMA 3.9. *If $\mathfrak{T}$ is a pointed closed model category with countable colimits, and if* $\mathbf{Y}$ *is any Ω-spectrum over $\mathfrak{T}$, then for any sequence $\phi \to X_0 \to X_1 \to \cdots$ of cofibrations, if X is the colimit of the sequence, there is for all n a natural short exact sequence*:

$$0 \to \lim_i^1 H^{n-1}(X_i; \mathbf{Y}) \to H^n(X; \mathbf{Y}) \to \lim_i^0 H^n(X_i, \mathbf{Y}) \to 0.$$

LEMMA 3.10. *Suppose that* $\mathbf{Y}$ *is an Ω-spectrum, and that* $\mathbf{X}$ *is any spectrum such that each X_n is a countable* CW*-complex and each ΣX_n is a subcomplex of the corresponding X_{n+1}. Let $H^*(-; \mathbf{X})$ be defined on finite* CW*-complexes in the usual manner. Then every stable cohomology operation φ: $H^*(-; \mathbf{X}) \to H^*(-; \mathbf{Y})$ defined on finite* CW*-complexes is induced by a map* $\mathbf{X} \to \mathbf{Y}$ *of spectra.*

Proof. Let $\mathbf{X}^\alpha$ run over those spectra such that each X_n^α is a finite subcomplex of X_n, and for some $m = m(\alpha)$, the maps $\Sigma X_{m+t}^\alpha \to X_{m+t+1}^\alpha$ are isomorphisms for $t \geq 0$. Notice that the $\mathbf{X}^\alpha$, ordered by inclusion, form a countable filtered system whose colimit is $\mathbf{X}$, and the inclusions are all cofibrations. Thus by (3.9), the map $H^*(\mathbf{X}; \mathbf{Y}) \to \lim^0 H^*(\mathbf{X}^\alpha; \mathbf{Y})$ is an epimorphism. However, $H^n(\mathbf{X}^\alpha; \mathbf{Y}) = H^{n+m(\alpha)}(X_{m(\alpha)}^\alpha; \mathbf{Y})$ by (3.7), so we see that for all n, $H^n(\mathbf{X}; \mathbf{Y}) \to \lim^0 H^{n+m(\alpha)}(X_{m(\alpha)}^\alpha; \mathbf{Y})$ is an epimorphism. The cohomology operation φ determines a class in $\lim^0 H^{n+m(\alpha)}(X_{m(\alpha)}^\alpha; Y)$, which is the image of some class in $H^n(\mathbf{X}; \mathbf{Y})$, where n is the degree of φ. Since $\mathbf{X}$ is cofibrant and $\mathbf{Y}$ is fibrant, this is represented by a map $\mathbf{X} \to \mathbf{Y}$.

Suppose now that $\mathbf{X}$ is any spectrum of topological spaces. We call $\mathbf{X}$ a CW-spectrum if each $\mathbf{X}_n$ is a CW-complex and if for all n, $\Sigma X_n \to X_{n+1}$ is the inclusion of a subcomplex. Notice that a CW-spectrum is cofibrant in the Serre model structure, and for any $\mathbf{X}$ there is an $\mathbf{X}'$ together with a weak equivalence $\mathbf{X}' \to \mathbf{X}$ such that $\mathbf{X}'$ is a CW-spectrum.

Suppose that $\mathbf{Y}$ is both a CW-spectrum and an Ω-spectrum so that each Y_n is countable. If $H^*(-; \mathbf{Y})$ has a ring structure on the category of finite CW-complexes, according to (3.10), we can find a spectrum map $\mathbf{Y} \wedge \mathbf{Y} \to \mathbf{Y}$ which induces this product, where $\mathbf{Y} \wedge \mathbf{Y}$ is, in degree $2n$ the CW-complex $Y_n \wedge Y_n$, and in degree $2n + 1$ is $\Sigma(Y_n \wedge Y_n)$. If $\mathbf{X}$ is any spectrum, let $\mathbf{DX}$ be $\Sigma^n X_n$ in degree $2n$, $\Sigma^{n+1} X_n$ in degree $2n + 1$. Then, by (3.7), we see that the obvious map $\mathbf{DX} \to \mathbf{X}$ induces an isomorphism $H^*(\mathbf{X}; \mathbf{Y}) \to H^*(\mathbf{DX}; \mathbf{Y})$. Thus, the pairing $\mathbf{Y} \wedge \mathbf{Y}$ defines a pairing, using the maps $\mathbf{DX} \to \mathbf{X}$, $\mathbf{DX} \to \mathbf{X} \wedge \mathbf{X}$ (diagonal map), of the form $H^i(\mathbf{X}; \mathbf{Y}) \oplus H^j(\mathbf{X}; \mathbf{Y}) \to H^{i+j}(\mathbf{X}; \mathbf{Y})$. Unfortunately, there is no reason to expect this pairing to be associative or commutative even if the original pairing was on finite complexes.

If $\mathbf{X}$ is a CW-spectrum, the filtration $* = sk_{-1}\mathbf{X} \to sk_0\mathbf{X} \to \cdots$ is a sequence of cofibrations with colimit $\mathbf{X}$, where $(sk_m\mathbf{X})_n = sk_{m+n}X_n$. Thus, for any Ω-spectrum $\mathbf{Y}$, we can mimic Dold's procedure **[6]** and produce a spectral sequence with $E_1^{p,q} = H^{p+q}(sk_p\mathbf{X}/sk_{p-1}\mathbf{X}; \mathbf{Y})$. If $\mathbf{Y}$ admits a multiplication, then by Dold's arguments, this spectral sequence will be a spectral sequence of bigraded rings, and each differential will be a derivation from the E_2-level on. Notice that since the maps $\Sigma(sk_{p+n}X_n/sk_{p+n-1}X_n) \to sk_{p+n+1}X_{n+1}/sk_{p+n}X_{n+1}$ are split monomorphisms, since each space is a bouquet of spheres of dimension $n + p + 1$ and the map is an inclusion of a subbouquet. Thus we see that $\lim^1 E_1^{p+n,q}(X_n) = 0$ for all p, q, and therefore that $E_1^{p,q}(\mathbf{X}) = \lim^0 E_1^{p+n,q}(X_n)$. This gives us a short exact sequence by standard homological algebra: $0 \to \lim^1 E_2^{p+n-1,q}(X_n) \to E_2^{p,q}(\mathbf{X}) \to \lim^0 E_2^{p+n,q}(X_n) \to 0$. By the standard 5-lemma argument and (3.7) we obtain the following:

LEMMA 3.11. *If* $\mathbf{X}$ *is any* CW-*spectrum, there is a natural isomorphism*

$$E_2^{p,q}(\mathbf{X}) \cong H^p\big(\mathbf{X}; \mathbf{K}\big(H^q(S^0; \mathbf{Y})\big)\big),$$

where $\mathbf{K}$ *is the Eilenberg-MacLane spectrum functor which associates to a group* A *the* Ω*-spectrum which in degree* n *is the Eilenberg-MacLane space* $K(A, n)$.

We shall simply write $H^p(\mathbf{X}; A)$ for $H^p(\mathbf{X}; \mathbf{K}(A))$.

Notice that we now have all of the structure needed to extend Theorems (2.1) and (2.2) from spaces to spectra. This gives us the following as a special case.

THEOREM 3.12. *Let* $\mathbf{X}$, $\mathbf{Y}$ *be two connected spectra*, A *be a subring of the rationals. Suppose that* $\mathbf{Y}$ *is an* Ω*-spectrum and that*:

(1) $H_*(\mathbf{X}; A)$ *is finitely generated in each degree.*

(2) $H^*(S^0; \mathbf{Y})$ *is a finitely generated* A*-module in each degree.*

(3) *In the spectral sequence for* $H^*(-; \mathbf{Y})$, *the terms* $E_r^{p,q}(X_n)$ *are eventually constant for all* n, p, q.

(4) *The terms* $E_\infty^{p+n,q}(X_n)$ *are eventually constant for all* p, q.

Then $H^*(\mathbf{X}; \mathbf{Y}) = \lim^0 H^*(X_n; \mathbf{Y}) = \lim^0 H^*(X_n^\alpha; \mathbf{Y})$ *where* X_n^α *runs over the finite subcomplexes of* X_n.

For connected Ω-spectra, there is a further phenomenon which can be exploited to prove the nonexistence of phantom maps. Suppose that $\mathbf{X}$ is a connected Ω-spectrum. Then for $m < n$, $\pi_{n+m}(X_n) \otimes Q = H_{n+m}(X_n) \otimes Q$, since rational homology agrees with rational stable homotopy, and we are in the stable range for $\pi_*(X_n)$. By repeated application of the Serre spectral sequence and the Zeeman comparison theorem, we see that for all $k < n$, $H^*(X_k; Q)$ is the free graded commutative algebra on $\mathrm{Hom}(\pi_*(X_k), Q)$ in dimensions $< k + n$ if $\pi_*(X_k) \otimes Q$ is finitely generated in each degree. Thus, we see, since n is arbitrary, that each $H^*(X_k; Q)$ is the free commutative algebra on $\mathrm{Hom}(\pi_*(X_k), Q)$.

THEOREM 3.13. *Suppose that* $\mathbf{X}$, $\mathbf{Y}$ *are two connected* Ω*-spectra which are countable* CW*-spectra, and that for finite complexes* $H^*(-; \mathbf{Y})$ *takes its values in the category of graded* A*-algebras which are finitely generated over* A *in each degree, where* A *is a subring of the rational numbers. Suppose also that* $H_*(\mathbf{X}; A)$ *is finitely generated over* A *in each degree. Then if for all* p, *the map* $H^p(\mathbf{X}; \mathbf{Y}) \to \mathrm{Hom}(\pi_p(\mathbf{X}), \pi_0(\mathbf{Y}))$ *has finite cokernel, there are no phantom elements in* $H^*(X_n; \mathbf{Y})$ *for any* n, *and there are no completely phantom elements or phantom elements in* $H^*(\mathbf{X}; \mathbf{Y})$.

Proof. Suppose that each $E_r^{p,0}(\mathbf{X})$ is eventually constant. Then $Z_\infty^{p,0}(\mathbf{X})$ has finite index in $E_2^{p,0}(\mathbf{X})$. Since the image of $E_2^{p,0}(\mathbf{X}) \to E_2^{p+n,0}(X_n)$ maps onto a subgroup of the indecomposables of $H^{p+n}(X_n; \pi_0(\mathbf{Y}))$ of finite index. Thus $Z_\infty^{*,0}(X_n)$ is of finite index in each degree in $E_2^{*,0}(X_n)$. Since

$Z_\infty^{*,0}(X_n) \otimes_{\pi_0(\mathbf{Y})} \pi_q(\mathbf{Y}) \subset Z_\infty^{*,-q}(X_n)$, and since $E_2^{*,0}(X_n) \otimes_{\pi_0(Y)} \pi_q(\mathbf{Y})$ is of finite index in $E_2^{*,-q}(X_n)$, we see that $Z_\infty^{*,-q}(X_n)$ is of finite index in $E_2^{*,-q}(X_n)$. (Notice that up to finite groups, $E_2^{*,0}(X_n) = H^*(X_n; A) \otimes_A \pi_0(\mathbf{Y})$, so that $E_2^{*,0}(X_n) \otimes_{\pi_0(\mathbf{Y})} \pi_q(\mathbf{Y}) = H^*(X_n; A) \otimes_A \pi_q(\mathbf{Y})$.) Thus the $E_r^{p,-q}(X_n)$ are eventually constant for all p, q. Thus part (3) of (3.12) is satisfied. Since $Z_\infty^{p,-q}(\mathbf{X})$ clearly has finite index in $E_2^{p,-q}(\mathbf{X})$ for all p, q by arguments similar to those above, we see that in the stable range, $Z_\infty^{p,-q}(\mathbf{X})$ has finite index in $Z_\infty^{p+n,-q}(X_n)$. Thus part (4) of (3.12) is satisfied. Thus (3.12) implies the desired result once we prove that each $E_r^{p,0}(\mathbf{X})$ is eventually constant.

Up to finite groups,

$$E_2^{p,0}(\mathbf{X}) = \operatorname{Hom}\big(H_p(\mathbf{X}), \pi_0(\mathbf{Y})\big) = \operatorname{Hom}\big(\pi_p(\mathbf{X}), \pi_0(\mathbf{Y})\big).$$

Thus if the image of $H^p(\mathbf{X}; \mathbf{Y})$ in $\operatorname{Hom}(\pi_p(\mathbf{X}), \pi_0(\mathbf{Y}))$ has finite cokernel, so does $Z_\infty^{p,0}(\mathbf{X})$ in $E_2^{p,0}(\mathbf{X})$, since the image of $H^p(\mathbf{X}; \mathbf{Y})$ in $E_{2+p}^{p,0}(\mathbf{X})$ is just the image of $Z_\infty^{p,0}(\mathbf{X})$.

4. Operations in K-theory.

We shall prove in this section that if k^* is the connected cohomology theory associated to complex K-theory, the spectral sequence for the stable cohomology operations from k^* to itself satisfies M-L, so that the stable cohomology operations are the same on finite complexes, spaces, or spectra. We shall do this by studying the cohomology operations on the localization of the theory at various primes where one can describe operations in terms of the periodicity map and the Adams operations. Over the integers, no such description is possible, as the only Adams operations which are stable operations are ψ^1 and ψ^{-1}.

Let $A \subset Q$ be the subring of the rationals consisting of those fractions with denominator not divisible by some prime l. Then $-\otimes A$ is exact, so that $k^*(-) \otimes A$ is a cohomology theory. Unfortunately, it is not additive. However, by E. H. Brown's theorem, since $k^*(\text{point}) \otimes A$ is countable, $k^*(-) \otimes A$ agrees with a representable cohomology theory on finite CW-complexes. We denote this theory by $k^*(-; A)$, and observe that there is a map $k^*(-) \otimes A \to k^*(-; A)$ which is an isomorphism on finite complexes.

Let $E_r^{**}(\mathbf{X})$ be the spectral sequence for $k^*(\mathbf{X})$, and let $E_r^{**}(X; A)$ be the spectral sequence for $k^*(X; A)$. Then if $\mathbf{X}$ is locally finite ($H^*(\mathbf{X}; Z)$ is finitely generated in each degree), the map $E_r^{**}(\mathbf{X}) \otimes A \to E_s^{**}(\mathbf{X}; A)$ is an isomorphism for $r = 2$, and by the exactness of $-\otimes A$, for $r < \infty$. If $E_r^{p,q}(\mathbf{X})$ satisfies M-L, clearly so does $E_r^{p,q}(\mathbf{X}) \otimes A$.

LEMMA 4.1. *$E_r^{**}(\mathbf{X})$ satisfies* M-L *in each bidegree if and only if $E_r^{**}(\mathbf{X}) \otimes A$ does for all l, and $E_r^{**}(X) \otimes A = E_2^{**}(\mathbf{X}) \otimes A$ for all r for all but a finite number of primes l. (We assume* $\mathbf{X}$ *locally finite.)*

Proof. $E_r^{p,q}(X)$ satisfies M-L if and only if $Q_r^{p,q}(X) = E_2^{p,q}(X)/Z_r^{p,q}(X)$ is eventually constant. Since this quotient is torsion and finitely generated, $Q_r^{p,q}(X) \otimes A$ is zero for all but a finite number of l. Thus if $Q_r^{p,q}(X)$ is eventually constant, so are all $Q_r^{p,q}(X; A) = Q_r^{p,q}(X) \otimes A$, and for all but a finite number of l they are all zero. Conversely, if for all l the $Q_r^{p,q}(X; A)$ are all eventually constant and all for all but a finite number of l they are zero, then there is an r independent of l such that $Q_{r+s}^{p,q}(X; A)$ is constant for all $s \geq 0$ and all A. Thus for all primes l the l-primary part $Q_{r+s}^{p,q}(X) \otimes A$ of $Q_{r+s}^{p,q}(X)$ is constant for $s \geq 0$, so $Q_{r+s}^{p,q}(X)$ is constant. Thus $E_r^{p,q}(X)$ satisfies M-L.

Let K^* be the usual periodic K-theory defined with complex vector bundles, so that $K^n \cong K^{n-2}$ for all n. Let $\pi: K^n \to K^{n-2}$ be the periodicity map. Then according to Adams [2], there are multiplicative cohomology operations $\psi^k: K^0 \to K^0$, such that $\psi^k\pi = k\pi\psi^k$ where $\psi^k: K^{-2} \to K^{-2}$ is defined by the suspension isomorphism. If l is a prime, A the localization of the integers at l, then for k not divisible by l the operation $\psi^k \otimes A$: $K^0(X; A) \to K^0(X; A)$ defined for finite complexes A can be made into a stable cohomology operation by the rule that ψ^k: $K^{2n}(X; A) \to K^{2n}(X; A)$ is $k^{-n}\pi^{-n}(\psi^k \otimes A)\pi^n$ (we write ψ^k for this stable operation for simplicity). The Chern characters ch^n: $K^0(X) \to H^{2n}(X; Q)$ can also be extended to stable cohomology operators ch^n: $K^i(X; A) \to H^{2n+i}(X; A)$ since $\mathrm{ch}^n\pi = \mathrm{ch}^{n-1}$ if we extend the Chern characters to $K^i(X)$ for $i \leq 0$ by suspension. Notice that the Adams operations are related to the Chern characters by $\mathrm{ch}^n\psi^k = k^n\mathrm{ch}^n$.

For X a CW-complex, let $sk_n X$ denote the n-skeleton of X. Let $k^n(X)$ be the kernel of the map $K^n(X, sk_{n-2}X) \to K^n(sk_{n-1}X, sk_{n-2}X)$, It is now an elementary exercise in the manipulation of exact sequences to show that k^* is a cohomology theory and that the map $k^n(X) \to K^n(X)$ is a stable cohomology operation. Since $sk_{m+n-1}(X \wedge Y)$ lies inside $((sk_{m-1}(X)) \wedge Y) \cup (X \wedge sk_{n-1}(Y))$, k^* is a multiplicative theory, and $k^* \to K^*$ is multiplicative. Finally, for every stable cohomology operation φ: $K^* \to K$ of degree ≤ 0, there is an induced cohomology operation $\bar{\varphi}$: $k^* \to k^*$ compatible with φ and $k^* \to K^*$. Further, for any $\varphi, \psi, \overline{\varphi\psi} = \bar{\varphi}\bar{\psi}$.

The construction of k^* from K^* above clearly works for any cohomology theory defined on CW-pairs. Further, it is clearly compatible with localization, since localization is an exact functor. Notice that if X is n-dimensional, $k^i(X) = 0$ for $i > n$. If X is $(n-1)$-connected, $\tilde{k}^n(X) \to \tilde{K}^n(X)$ is an isomorphism, as one can find a model for X such that $sk_{n-1}(X)$ is a point. Finally, the image of $k^n(X) \to K^n(X)$ is just the kernel of $K^n(X) \to K^n(sk_{n-1}X)$, or the nth filtration $F_n K^n(X)$ in the skeletal filtration.

Consider the map $k^n(X) \to k^n(sk_n X)$. Since $k^n(sk_n X) = k^n(sk_n X, sk_{n-1}X) = C^n(X;\ k^0(\text{point}))$ (the n-dimensional CW-chains), and since the image of $k^n(X)$ in $C^n(X;\ k^0(\text{point}))$ clearly lies in the kernel of δ: $k^n(sk_n X, sk_{n-1}X) \to k^{n+1}(sk_{n+1}X, sk_n X)$ we see that there is a well-defined stable cohomology operation γ_0: $k^n(X) \to H^n(X;\ k^0(\text{point}))$. Further, it is elementary to verify that the following diagram is commutative:

$$\begin{array}{ccc} k^n(X) & \xrightarrow{\gamma_0} & H^n(X;Z) \\ \downarrow & & \downarrow -\otimes Q \\ K^n(X) & \xrightarrow{\mathrm{ch}_0} & H^n(X;Q). \end{array}$$

It is possibly a simple diagram chase to see that $\gamma_0\bar{\pi} = 0$, where $\bar{\pi}$: $k^{n+2}(X) \to k^n(X)$ is induced by periodicity. Let **bu** be the classifying spectrum for k^*. Then each bu_n has the homotopy type of a countable CW-complex, so that the map $bu_{n+2} \to bu_n$ which induces $\bar{\pi}$ factors through the fiber of $bu_n \to K(Z, n)$. The exact sequence of homotopy groups of a fibration shows that the map is a homotopy equivalence. If we choose **bu** to be cofibrant, the obstruction to extending the map of Σbu_{n+2} into the fiber of $bu_{n+1} \to K(Z, n+1)$ lies in $H^{n+2}(bu_{n+3}, \Sigma bu_{n+2}) = 0$, so there is a spectral map of degree -2 of **bu** to the fiber of $\mathbf{bu} \to \mathbf{K}(Z, 0)$ which is a weak equivalence of spectra.

The exact sequence of homotopy classes of maps into a fibration (the dual of the "Puppe sequence") holds in any closed model category. Thus from the remarks above, there is an exact sequence

$$\cdots \to k^{n+2}(X) \xrightarrow{\bar{\pi}} k^n(X) \xrightarrow{\gamma_0} H^n(X) \xrightarrow{k} k^{n+1}(X) \to \cdots$$

which holds if X is a spectrum or a space.

If $A \subset Q$, we write $\mathbf{BU} \otimes A$ for the spectrum representing $K^*(-;A)$, and $\mathbf{bu} \otimes A$ for the spectrum representing $k^*(-;A)$. Adams [1] computed the cohomolgy of **bu** with mod p coefficients for all primes and since $\mathbf{bu} \to \mathbf{bu} \otimes A$ is a localization, we can determine from the calculations of

Adams the mod p cohomology of the **bu** $\otimes A$ for all A. For now, we need only two facts: $H^1(\mathbf{bu} \otimes A; A) = 0$, $H^0(\mathbf{bu} \otimes A; A) = A$.

Let k be an integer which is invertible in A. Then $\gamma_0(\bar{\psi}^k - 1) = \mathrm{ch}_0(\psi^k - 1) = 0$, so there exists a unique $\varphi^k \in k^2(\mathbf{bu} \otimes A; A)$ with $\bar{\pi}\varphi^k = \bar{\psi}^k - 1$ ($\bar{\pi}$: $k^2(\mathbf{bu} \otimes A;\ A) \to k^0(\mathbf{bu} \otimes A;\ A)$ is a monomorphism since $H^1(\mathbf{bu} \otimes A;\ A) = 0$). Notice that $\bar{\pi}\varphi^k\bar{\pi}^n = \bar{\pi}k^n\bar{\pi}^n\varphi^k + \bar{\pi}(k^n - 1)\bar{\pi}^{n-1}$. Thus $\varphi^k\bar{\pi}^n = k^n\bar{\pi}^n\varphi^k + (k^n - 1)\bar{\pi}^{n-1}$.

PROPOSITION 4.2. φ^k: $k^{-2n}(\text{point}; A) \to k^{2-2n}(\text{point}; A)$ *is multiplication by* $k^n - 1$.

Proof. $\bar{\pi}^n$: $k^0(\text{point}; A) \to k^{-2n}(\text{point}; A)$ is an isomorphism and φ^k: $k^0(\text{point}; A) \to k^2(\text{point}; A) = 0$ is trivial.

THEOREM 4.3. *The spectral sequence for* $k^*(\mathbf{bu} \otimes A;\ A)$ *satisfies* M-L, *so that stable cohomology operations on finite complexes on* $k^*(-;\ A)$ *have unique extensions to spaces and to spectra.*

Proof. Over the rationals, φ^k and $\bar{\pi}$ generate all maps $k^i(\text{point}; A)$ to $k^j(\text{point}; A)$ for all i, j.

We now turn our attention to the details of the spectral sequence for $k^*(\mathbf{bu}; A)$. According to Adams [**1**] there are cohomology classes $\mathrm{ch}_{q,r}$ in $H^{2q+2r}(bu_{2q}; Z)$ which have the following properties:

1. If $\mathrm{ch} \in H^{2q+2r}(bu_{2q};\ Q)$ is the representative of the stable rth Chern character, $\mathrm{ch}_{q,r} \otimes Q = m(r)\mathrm{ch}_r$, where $m(r)$ is the product over all primes p of $p^{[r/p-1]}$, and where [ ] denotes the greatest integer function.
2. Under the map $H^{2q+2r}(bu_{2q};\ Z) \to H^{2q+2r-2}(bu_{2q-2};\ Z)$ $\mathrm{ch}_{q,r}$ is carried to $\mathrm{ch}_{q-1,r}$.

Unfortunately, Adams does not state his results quite this way in his Theorem 2, but it is not difficult to see that his statement implies ours. Let the Adams-Chern character $\mathrm{ac}_r \in H^{2r}(\mathbf{bu};\ Z)$ be the class determined by the $\mathrm{ch}_{q,r}$ for $q \geq 0$. Then $\mathrm{ac}_r \otimes Q = m(r)\mathrm{ch}_r$ in $H^{2r}(\mathbf{bu};\ Q)$.

Since $\pi\varphi^k = \psi^k - 1$, $\mathrm{ch}_r\pi\varphi^k = (k^r - 1)\mathrm{ch}_r$, or $\mathrm{ch}_{r-1}\varphi^k = (k^r - 1)\mathrm{ch}_r$. In particular, $\mathrm{ch}_0(\varphi^k) = (k^r - 1) \cdots (k - 1)\mathrm{ch}_r$. Thus

$$\mathrm{ac}_0(\varphi^k)^r = m(r)^{-1}(k^r - 1) \cdots (k - 1)\mathrm{ac}_r$$

modulo torsion. Consequently, in the spectral sequence for $k^*(\mathbf{bu}; A)$ modulo torsion,

$$m(r)^{-1}(k^r - 1) \cdots (k - 1)\mathrm{ac}_r \otimes 1 \in H^{2r}(bu, k^0(\text{point}; A))$$

is an infinite cycle representing $(\varphi^k)^r$.

Next, let $v_l(n)$ be the number of times which l divides n for any integer n. If k is chosen so that $k^{p-1} - 1$ is not zero modulo l^2, $v_l(k^r - 1)$ is given as follows:

$l = 2$		$l \neq 2$	
1	if $2 \nmid r$	0	if $l - 1 \nmid r$
$2 + v_l(r)$	if $2 \mid r$	$1 + v_l(r)$	if $l - 1 \mid r$

It is now easy to verify that $v_l(m(r)^{-1}(k^r - 1) \cdots (k - 1))$ is $v_l(|r/(l-1)|!)$ if $l \neq 2$, and is $r + v_l(|r/(l-1)|!)$ if $l = 2$.

THEOREM 4.4. *Let* $n(r) = 2^r \prod_l l^{v_l(|r/(l-1)|!)}$. *Then in the spectral sequence for* $k^*(\mathbf{bu})$, *for all* $r \geq 0$, $n(r)\mathrm{ac}_r$ *is, modulo torsion, an infinite cycle.*

Proof. We have just seen that the l-localization of this class is an infinite cycle for all primes l.

Finally, we observe that real, complex, and symplectic K-theory all behave in essentially the same manner insofar as $\lim^1$ problems are concerned. To see this, first recall that these maps $c: KO^0(x) \to KU^0(x)$, c': $KSp^0(x) \to KU^0(x)$, r: $KU^0(x) \to KO^0(x)$, r': $KU^0(x) \to KSp^0(x)$ which are extensions of scalars along the usual inclusion maps $R \to C$, $H \to C(2)$, $C \to R(2)$, $C \to H$ where R is the ring of real numbers, C is the ring of complex numbers, H is the ring of quaternions, and $R(2)$, $C(2)$ denote the ring of rank 2 matrices over R and C respectively. The composition rc: $KO^0(x) \to KO^0(x)$ is multiplication by 2, as is the composition $r'c'$: $KSp^0(x) \to KSp^0(x)$. Thus, up to a group of exponent 2, the spectral sequence for $ko^*(x)$ will always be a retract of the spectral sequence for $k^*(x)$, where ko^* is the connected cohomology theory associated to KO^*, and similarly for $ksp^*(x)$. Thus, if the spectral sequence for $k^*(x)$ satisfies the Mittag-Leffler condition, so do the spectral sequences for $ko^*(x)$ and $ksp^*(x)$. Similarly, modulo finite cohomology groups, the spectra for ko^* and ksp^* are retracts of the spectrum for k^*. Thus if for some cohomology theory h^* which is of finite type over some ring $A \subset Q$, if the spectral sequence for operations

$k^* \to h^*$ satisfies the Mittag-Leffler condition, so do the spectral sequences for cohomology operations $ko^* \to h^*$ and $ksp^* \to h^*$. Thus the theorems which we have proved for complex K-theory all extend to real and symplectic K-theory easily.

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UNIVERSITY OF CALIFORNIA-SAN DIEGO
LA JOLLA, CA 92037