PACIFIC JOURNAL OF MATHEMATICS
Vol. 107, No. 2, 1983

ON THE BEHAVIOR NEAR A TORUS OF FUNCTIONS HOLOMORPHIC IN THE BALL

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If f is bounded and holomorphic in the unit ball in $\mathbf{C}^n$ then it has radial limits at almost all points of the boundary of the ball. More is true; for example, f will have limits almost everywhere with respect to arclength on any arc that forms part of the boundary of an anlaytic disc. Motivated by these considerations we consider an n-dimensional torus in the boundary of the ball and ask if there are growth conditions less restrictive than boundedness that imply the existence of radial limits on this torus. We show that the answer is no for some of the standard function classes. For example, we show that there is holomorphic function of bounded mean oscillation in the ball that has a finite radial limit at no point of the torus.

Let B_n denote the unit ball in $\mathbf{C}^n$ and let σ_n be Lebesgue measure on its boundary, ∂B_n, normalized so that $\sigma_n(\partial B_n) = 1$. If f is a holomorphic in B_n, we say that $f \in H^p(B_n)$, $0 < p < \infty$, if

$$\|f\|_p^p = \sup_{0<r<1} \int_{\partial B_n} |f(r\xi)|^p \, d\sigma_n(\xi) < \infty;$$

we say $f \in H^\infty(B_n)$ if $\|f\|_\infty = \sup_{\xi \in B_n} |f(\xi)| < \infty$. If $f \in H^2(B_n)$ we say that $f \in \mathrm{BMO}(B_n)$ if $\exists$ a constant C such that for all $F \in H^2(B_n)$ we have $|\int_{\partial B_n} F\bar{f}\, d\sigma_n| \leq C\|F\|_1$. Then $\mathrm{BMO}(B_n)$ serves as the dual of $H^1(B_n)$ and we have $H^\infty(B_n) \subseteq \mathrm{BMO}(B_n) \subseteq H^p(B_n)$, $0 < p < \infty$. For a more intrinsic description $\mathrm{BMO}(B_n)$, see [**1**].

Next we describe some function spaces in the open unit disc U in the complex plane. If μ is a positive measure on U then $A^p(d\mu)$ will denote the space of holomorphic functions in $L^p(d\mu)$, $0 < p \leq \infty$. When $d\mu(r, \theta) = (1 - r)^\alpha \, dr\, d\theta$, $\alpha > -1$, we use the notation $A^p(d\mu) = A^p_\alpha$. Finally we say that g is a Bloch function, $g \in \mathfrak{B}(U)$, if

$$\|g\|_{\mathfrak{B}} = \sup_{|z|<1} (1 - |z|)\,|g'(z)| < \infty.$$

We have a mapping π: $\mathbf{C}^n \to \mathbf{C}$ given by $\pi(z_1, \ldots, z_n) = n^{n/2}\prod_{j=1}^n z_j$. It is easily checked that $\pi(B_n) = U$, $\pi(\bar{B}_n) = \bar{U}$, and that $\pi^{-1}(\partial U) = T_n = \{(z_1, \ldots, z_n): |z_j| = n^{-1/2}, j = 1, \ldots, n\}$. In this paper it is shown that if $g \in A^p_{(n-3)/2}$ then $g \circ \pi \in H^p(B_n)$, and that if $g \in \mathfrak{B}(U)$ then $g \circ \pi \in \mathrm{BMO}(B_n)$.

The following remarks are intended to motivate the results of this paper. It is known, see [8], for instance, that if $f \in H^p(B_n)$ then f has radial limits almost everywhere $(d\sigma_n)$. If $f \in H^\infty(B_n)$, then more can be said; for example, f has radial limits almost everywhere with respect to arclength on the curve $\Gamma = \{(0,\ldots,0, e^{i\theta})\colon 0 \leq \theta \leq 2\pi\}$. Of course $\sigma_n(\Gamma) = 0$. A. Nagel and W. Rudin [5] have generalized this to certain other curves in ∂B_n. The following question arises: are there growth conditions less restrictive than boundedness that still imply the existence of radial limits on sets of zero Lebesgue measure? It is natural to look for the existence of radial limits on submanifolds of ∂B_n that are nowhere complex tangential. One such submanifold with the largest possible dimension is the torus T_n, described above. This torus is the distinguished boundary of the poly disc $D_n = \{(z_1,\ldots,z_n)\colon |z_j| < n^{-1/2}, j = 1,\ldots,n\} \subseteq B_n$. It is known that if f is holomorphic and of bounded characteristic in D_n, then f has radial limits almost everywhere on T_n, see [7]. It has already been noted that $\pi^{-1}(\partial U) = T_n$; so that if g manifests a certain property near ∂U, $g \circ \pi$ will manifest that same property near T_n. So the results of this paper show that a function in $H^p(B_n)$ can be expected to behave no better near T_n than a function in $A^p_{(n-3)/2}$ can be expected to behave near ∂U. A similar statement can be made about the spaces $\mathrm{BMO}(B_n)$ and $\mathfrak{B}(U)$. It is known [6] that there is a function $g \in \mathfrak{B}(U)$ that has a finite radial limit at no point of ∂U. It follows that the function $G = g \circ \pi$ is in $\mathrm{BMO}(B_n)$ but has a finite radial limit at no point of T_n, in particular the restriction of G to D_n cannot have bounded characteristic.

If we let $D = \{(z, z,\ldots,z)\colon |z| < n^{-1/2}\} \subseteq D_n \subseteq B_n$, then it follows from [8] and [4] that $H^p(B_n)$ and $H^p(D_n)$ have the *same* restriction to D. That is to say, even though the restriction of $H^p(B_n)$ to D_n contains functions of unbounded characteristic, $H^p(B_n)$ and $H^p(D_n)$ have the same restriction to the diagonal D.

1. The first result is a calculation upon which the results of this paper are based. We point out that the case $n = 2$ of Theorem 1 is computationally much simpler than the general case. Moreover, the proof shows that $w_2(r) = 2r(1 - r^2)^{-1/2}$. A formula for $w_n(r)$, $n \geq 3$, is not obvious.

THEOREM 1. *For each integer* $n \geq 2$, $\exists w_n\colon (0, 1) \to [0, \infty) \ni$

(i) $\int_0^1 w_n(r)\, dr < \infty$,

(ii) $0 < \lim_{r\to 1} w_n(r)(1 - r)^{(3-n)/2} < \infty$.

(iii) *If g is a continuous complex valued function defined on $\overline{U}$ then*

$$\int_{\partial B_n} g \circ \pi \, d\sigma_n = \int_0^{2\pi} \int_0^1 g(re^{i\theta}) w_n(r) \, dr \, d\theta.$$

Proof. The proof proceeds by induction. We assume the result is true for $n-1$, $n \geq 3$. If G is a continuous function defined on ∂B_n, then we have, **[8]**,

$$(1.1) \qquad \int_{\partial B_n} G d\sigma_n = \int_{B_{n-1}} \int_0^{2\pi} G\left(\xi, \left(1 - |\xi|^2\right)^{1/2} e^{i\theta}\right) \frac{d\theta}{2\pi} d\nu_{n-1}(\xi),$$

where ν_{n-1} denotes Lebesgue measure on $\mathbf{C}^{n-1}$, normalized so that $\nu_{n-1}(B_{n-1}) = 1$. Next we introduce polar coordinates, see **[8]**, then the right hand side of (1.1) becomes,

$$(1.2) \quad 2(n-1) \int_0^1 r^{2n-3} \int_{\partial B_{n-1}} \int_0^{2\pi} G\left((r\xi, 1 - r^2)^{1/2} e^{i\theta}\right) \frac{d\theta}{2\pi} d\sigma_{n-1}(\xi) \, dr.$$

Now we fix r and θ and look at the integral over ∂B_{n-1} in (1.2) in the case $G = g \circ \pi$; we obtain

$$(1.3) \qquad \int_{\partial B_{n-1}} g\left(a_n a_{n-1}^{-1} (1 - r^2)^{1/2} r^{n-1} e^{i\theta} a_{n-1} \prod_{j=1}^{n-1} z_j \right) d\sigma_{n-1}(z),$$

where $a_n = n^{n/2}$.

By the induction hypothesis, (1.3) is equal to

$$(1.4) \qquad \int_0^{2\pi} \int_0^1 g\left(a_n a_{n-1}^{-1} r^{n-1} (1 - r^2)^{1/2} e^{i\theta} \rho e^{i\psi}\right) w_{n-1}(\rho) \, d\rho \, d\psi.$$

Inserting (1.4) into (1.2) and changing the order of integration we arrive at,

$$(1.5) \quad \int_{\partial B_n} g \circ \pi \, d\sigma_n$$
$$= 2(n-1) \int_0^1 \int_0^1 \int_0^{2\pi} \int_0^{2\pi} g\left(a_n a_{n-1}^{-1} r^{n-1} (1 - r^2)^{1/2} \rho e^{i(\theta + \psi)}\right)$$
$$\cdot \frac{d\theta}{2\pi} d\psi \, w_{n-1}(\rho) \, d\rho \, r^{2n-3} \, dr.$$

Next replace $\theta + \psi$ by θ, integrate out ψ, and interchange the order of integration again and we have

$$(1.6)\quad \int_{\partial B_n} g \circ \pi \, d\sigma_n = 2(n-1)\int_0^{2\pi}\int_0^1 w_{n-1}(\rho)\int_0^1 g\left(a_n a_{n-1}^{-1} r^{n-1}(1-r^2)^{1/2}\rho e^{i\theta}\right) \cdot r^{2n-3}\, dr\, d\rho\, d\theta.$$

Let $h(r) = a_n a_{n-1}^{-1} r^{n-1}(1-r^2)^{1/2}$; from elementary calculus it follows that h is increasing on the interval $[0, b_n]$ and decreasing on the interval $[b_n, 1]$, where $b_n = [(n-1)/n]^{1/2}$. Moreover, $h(0) = h(1) = 0$ and $h(b_n) = 1$. We also see that h' vanishes to order $n-2$ at 0, to order $-1/2$ at 1, and to order 1 at b_n. We break the innermost integral in (1.6) into two pieces, the integral from 0 to b_n, and the integral from b_n to 1. In each of these we make the substitution $t = h(r)$. If λ denotes the inverse function of h (in either case) then each of the two integrals takes the form

$$(1.7)\qquad \int_0^1 g(t\rho e^{i\theta})\lambda(t)^{-(n-1)}[h'(\lambda(t))]^{-1}\, dt.$$

We are interested in the behaviour of the "weight" $w(t) = \lambda(t)^{-(n-1)}[h'(\lambda(t))]^{-1}$, when t is near 1. First note that $\lambda(1) = b_n$. Next we may calculate that $h'(\lambda(t))$ vanishes like $(1-t)^{1/2}$ when t approaches 1. Now if we substitute (1.7) back in for the inner integral in (1.6) we get,

$$(1.8)\qquad \int_{\partial B_n} g \circ \pi \, d\sigma_n = \int_0^{2\pi}\int_0^1 w_{n-1}(\rho)\int_0^1 g(t\rho e^{i\theta})w(t)\, dt\, d\rho\, d\theta,$$

where $w(t) = (c_1 + o(1))(1-t)^{-1/2}$ as $t \to 1$, $c_1 > 0$. (We have absorbed the constant $2(n-1)$ into w.) Finally, in the inner integral in (1.8) we make the substitution $r = t\rho$, and then interchange the order of integration to obtain

$$(1.9)\qquad \int_{\partial B_n} g \circ \pi \, d\sigma_n = \int_0^{2\pi}\int_0^1 g(re^{i\theta})w_n(r)\, dr\, \theta,$$

where

$$w_n(r) = \int_r^1 w\left(\frac{r}{\rho}\right) w_{n-1}(\rho)\frac{d\rho}{\rho}.$$

We now check that w_n has the right properties. First of all

$$\int_0^1 w_n(r)\, dr = \int_0^1 w(r)\, dr \int_0^1 w_{n-1}(r)\, dr < \infty,$$

so (i) is satisfied. Now we know that $w(r/\rho) = (c_1 + o(1))(1 - r/\rho)^{1/2}$, as $r/\rho \to 1$, and $w_{n-1}(\rho) = (c_2 + o(1))(1 - \rho)^{(n-4)/2}$, as $\rho \to 1$. It follows that

$$w_n(r) = \int_r^1 w\left(\frac{r}{\rho}\right) w_{n-1}(\rho) \frac{d\rho}{\rho}$$
$$= (c + o(1)) \int_r^1 (\rho - r)^{-1/2} (1 - \rho)^{(n-4)/2} \, d\rho \quad \text{as } r \to 0,$$

for some $c > 0$. In this integral we make the substitution

$$s = (\rho - r)/(1 - \rho)$$

we arrive at

$$w_n(r) = (c + o(1))(1 - r)^{(n-3)/2} \int_0^1 s^{-1/2}(1 - s)^{(n-4)/2} \, ds$$
$$= (c + o(1))(1 - r)^{(n-3)/2} \beta\left(\tfrac{1}{2}, (n-2)/2\right).$$

To complete the proof we should check that the theorem is true for $n = 2$. This is done by the same method as the induction step given above. Indeed, it is somewhat simpler and the details will be omitted.

Now if g is continuous on $\overline{U}$ we may apply Theorem 1 to the function $|g|^p$ and conclude that

$$\int_{\partial B_n} |Tg|^p \, d\sigma_n = \int_V |g|^p \, d\mu_n,$$

where $Tg = g \circ \pi$, and $d\mu_n(r, \theta) = w_n(r) \, dr \, d\theta$. It is now clear that T extends uniquely to be an isometry of $L^p(d\mu_n)$ into $L^p(d\sigma_n)$. If g is holomorphic, then it is obvious from Theorem 1 that $g \in L^p(d\mu_n)$ if and only if $g \in A^p_{(n-3)/2}$. Also if g is holomorphic, then so is Tg. We may conclude with

COROLLARY 1. *T is a bounded, linear, one-to-one map of $A^p_{(n-3)/2}$ into $H^p(B_n)$.*

2. In this section we show that if $g \in \mathfrak{B}(U)$ then $Tg = g \circ \pi \in \mathrm{BMO}(B_n)$. To do this it is sufficient to show that if $g \in \mathfrak{B}(U)$ $\exists$ constant $C \ni$ if F is a holomorphic polynomial in n variables then

$$\left| \int F \overline{g \circ \pi} \, d\sigma_n \right| \leq C \int |F| \, d\sigma_n.$$

To do this we proceed as follows. Since $F \in L^2(d\sigma_n)$ we have

$$\int F\overline{Tg}\, d\sigma_n = \int T^*F\bar{g}\, d\mu_n$$

where T^* is the adjoint of the isometry T: $L^2(d\mu_n) \to L^2(d\sigma_n)$. The proof is then accomplished in two steps. The first is to show that if F is a holomorphic polynomial in $\mathbf{C}^n$ then T^*F is a holomorphic polynomial in $\mathbf{C}$ and $\int |T^*F|\, d\mu_n \leq \int |F|\, d\sigma_n$. The second step is to show that if $g \in \mathfrak{B}(U)$ then $\exists$ constant $C \ni$ for any holomorphic polynomial h of one variable we have

$$\left|\int h\bar{g}\, d\mu_n\right| \leq C\int |h| d\mu_n.$$

The first step is quite easy. The second is slightly trickier than it may appear.

LEMMA 2.1. *If* $F(z) = \sum F_\alpha z^\alpha$ *is a holomorphic polynomial then* $(T^*F)(z) = \sum_{k\geq 0} F_{(k,\ldots,k)} n^{-nk/2} z^k$. *Moreover* $\int |T^*F|\, d\mu_n \leq \int |F|\, d\sigma_n$.

Proof. To prove the first part we show that $T^*z^\alpha = 0$ unless $\alpha_1 = \alpha_2 = \cdots = \alpha_n$, and then we show $T^*z_1^k \cdots z_n^k = n^{-nk/2}z^k$. Because of the rotational invariance of σ_n, the integral $I_\alpha = \int z^\alpha \overline{g(\pi(z))}\, d\sigma_n(z)$ is unchanged if z_k is replaced by $z_k e^{i\theta}$ and z_l by $z_l e^{-i\theta}$. Since $g \circ \pi$ is also unchanged by these substitutions we see that $I_\alpha = e^{i(\alpha_k - \alpha_l)\theta} I_\alpha$, for any θ. It follows that $I_\alpha = 0$ unless $\alpha_\alpha = \alpha_l$ for all k, $l = 1,\ldots,n$. If $\alpha = (k,\ldots,k)$ then

$$\begin{aligned}\int z^\alpha \overline{g(\pi(z))} d\sigma_n(z) &= \int (z_1 \cdots z_n)^k \overline{g(n^{n/2}z_1 \cdots z_k)} d\sigma_n(z) \\ &= n^{-nk/2}\int h(\pi(z))\, d\sigma_n(z),\end{aligned}$$

where $h(z) = z^k\overline{g(z)}$. By Theorem 1 (iii) we see that

$$\int z^\alpha \overline{g(\pi(z))} d\sigma_n(z) = n^{-nk/2}\int z^k \overline{g(z)} d\mu_n(z).$$

Since T: $L^p(d\mu_n) \to L^p(d\sigma_n)$, $1 \leq p < \infty$, is an isometry it follows that T^*: $L^q(d\sigma_n) \to L^q(d\mu_n)$, $1 < q \leq \infty$, has norm at most 1. So if F is a holomorphic polynomial in n variables

$$\int |T^*F|^q\, d\mu_n \leq \int |F|^q\, d\sigma_n$$

for each $q > 1$. Now we let $q \to 1$, by the bounded convergence theorem

$$\int |F|^q \, d\sigma_n \to \int |F| d\sigma_n \quad \text{and} \quad \int |T^*F|^q \, d\mu_n \to \int |T^*F| d\mu_n.$$

Next we want to show that if $g \in \mathfrak{B}(U)$ $\exists$ a constant $C \ni$ for every holomorphic polynomial h in one variable we have

$$\left| \int h\bar{g} \, d\mu_n \right| \leq c \int |h| d\mu_n.$$

In other words, the mapping $h \mapsto \int h\bar{g} \, d\mu_n$ is continuous on $A^1(d\mu_n)$, the closure in $L^1(d\mu_n)$ of the holomorphic polynomials. We have already observed that since $w_n(r)$ behaves like $(1-r)^{(n-3)/2}$ as $r \to 1$, $A^1(d\mu_n)$ and $A^1_{(n-3)/2}$ have the same elements. Now A^1_α has another name in the literature, it is called B^p where $p = 1/(2+\alpha)$. The dual of B^p (and hence of A^1_α) is known to be a certain space of Lipschitz function depending on p, see [3]. However, the duality is effected by an integral over the "boundary" of U rather than by an integral over U itself. We seem to be saying that the dual of B^p is $\mathfrak{B}(U)$, for all p, if we use the area pairing. This can be seen as follows: by results from [3] any two B^p spaces are isomorphic by means of fractional derivatives. In particular B^p is isomorphic to $B^{1/2}$. From [3] we know that the dual of $B^{1/2}$ is the Zygmund class Λ^*. By a theorem of Zygmund, [2] Λ^* is the set of indefinite integrals of $\mathfrak{B}(U)$. However, rather than following this tortuous path we prefer to give a simple direct proof of what we need.

LEMMA 2.2. *If $\alpha > -1$ and $g \in \mathfrak{B}(U)$ then $\exists$ a constant $C \ni$ for every holomorphic polynomial h we have*

$$\left| \int_0^{2\pi} \int_0^1 h(re^{i\theta}) \overline{g(re^{i\theta})} (1-r)^\alpha \, dr \, d\theta \right| \leq C \int |h(re^{i\theta})| (1-r)^\alpha \, dr.$$

Proof. If f is holomorphic in U, f_k will denote its kth Taylor coefficient. We then calculate that

$$\begin{aligned} \int_0^1 \int_0^{2\pi} h(re^{i\theta}) \overline{g(re^{i\theta})} d\theta (1-r)^\alpha \, dr &= 2\pi \sum h_k \bar{g}_k \int_0^1 r^{2k} (1-r)^\alpha \, dr \\ &= 2\pi \sum h_k \bar{g}_k \frac{\Gamma(2k+1)\Gamma(\alpha+1)}{\Gamma(2k+\alpha+2)}. \end{aligned}$$

Also we see that

$$\int_0^1 \int_0^{2\pi} h(re^{i\theta})\overline{re^{i\theta}g'(re^{i\theta})}\,d\theta(1-r)^{\alpha+1}\,dr$$

$$\begin{aligned}
&= 2\pi \sum h_k g_k k \int_0^1 r^{2k}(1-r)^{\alpha+1}\,dr \\
&= 2\pi \sum h_k \bar{g}_k k \frac{\Gamma(2k+1)\Gamma(\alpha+2)}{\Gamma(2k+\alpha+3)} \\
&= 2\pi \sum h_k g_k k(\alpha+1) \frac{\Gamma(2k+1)\Gamma(\alpha+1)}{(2k+\alpha+2)\Gamma(2k+\alpha+2)} \\
&= 2\pi(\alpha+1) \sum h_k \bar{g}_k \frac{\Gamma(2k+1)\Gamma(\alpha+1)}{\Gamma(2k+\alpha+2)} \cdot \frac{k}{2k+\alpha+2} \\
&= 2\pi(\alpha+1) \sum h_k \bar{g}_k \frac{\Gamma(2k+1)\Gamma(\alpha+1)}{\Gamma(2k+\alpha+2)} \left\{ \frac{1}{2} - \frac{\alpha/2+1}{2k+\alpha+2} \right\} \\
&= \frac{\alpha+1}{2} \int_0^1 \int_0^{2\pi} h(re^{i\theta})\overline{g(re^{i\theta})}\,d\theta(1-r)^{\alpha}\,dr \\
&\qquad -2\pi \frac{(\alpha+1)(\alpha+2)}{2} \sum \frac{h_k \bar{g}_k}{2k+\alpha+2} \int_0^1 r^{2k}(1-r)^{\alpha}\,dr.
\end{aligned}$$

In other words

$$\begin{aligned}
&\int_0^1 \int_0^{2\pi} h(re^{i\theta})\overline{g(re^{i\theta})}\,d\theta(1-r)^{\alpha}\,dr \\
&\qquad = \frac{2}{(\alpha+1)} \int_0^1 \int_0^{2\pi} h(re^{i\theta})\overline{re^{i\theta}g'(re^{i\theta})}\,d\theta(1-r)^{\alpha+1}\,dr \\
&\qquad\quad +2\pi(\alpha+2) \sum \frac{h_k \bar{g}_k}{2k+\alpha+2} \int_0^1 r^{2k}(1-r)^{\alpha}\,dr.
\end{aligned}$$

Using the fact that $g \in \mathfrak{B}(\mathfrak{U})$, the first term above has modulus at most

$$\frac{2}{\alpha+1} \|g\|_{\mathfrak{B}} \int |h|(1-r)^{\alpha}\,dr\,d\theta.$$

Using the fact that $|g_k| \leq C\|g\|_{\mathfrak{B}}$ for some constant C, the modulus of the second term is at most

$$\sum 2\pi(\alpha+2)C\|g\|_{\mathfrak{B}} \frac{|h_k|}{k+1} \int_0^1 r^{2k}(1-r)^{\alpha}\,dr.$$

The proof will be finished if we show that there is a constant C such that

$$\sum \frac{|h_k|}{k+1} \int_0^1 r^{2k}(1-r)^{\alpha}\,dr \leq C \int_0^1 \int_0^{2\pi} |h(re^{i\theta})|(1-r)^{\alpha}\,d\theta\,dr,$$

for any holomorphic polynomial h. To see this we apply the well known inequality of Hardy and Littlewood [2] on a circle of radius $r < 1$ to obtain

$$\sum \frac{|h_k| r^k}{k+1} \leq C \int_0^{2\pi} |h(re^{i\theta})| d\theta,$$

and hence

$$\sum \frac{|h_k| r^{2k}}{k+1} (1-r)^\alpha \leq C \int_0^{2\pi} |h(re^{i\theta})| d\theta (1-r)^\alpha.$$

Now we just integrate on r from 0 to 1.

Next we want to show that Lemma 2.2 remains true if the measure $(1-r)^{(n-3)/2}\, dr\, d\theta$ is replaced by the measure $d\mu_n$. Since $A^1_{(n-3)/2}$ and $A^1(d\mu_n)$ are the "same" it may seem obvious that they have the "same" dual. A little reflection shows that we need more information to draw this conclusion. It turns out that we do know enough about μ_n to get the result we want.

LEMMA 2.3. *If $n \geq 2$ and $g \in \mathfrak{B}(U)$ are given, $\exists$ a constant $C \ni$ for every holomorphic polynomial h we have*

$$\left| \int h \bar{g}\, d\mu_n \right| \leq C \int |h|\, d\mu_n.$$

Proof. The idea is to show that $\int h\bar{g}\, d\mu_n$ and $\int h\bar{g}(1-r)^{(n-3)/2}\, dr\, d\theta$ differ by a manageable amount and then to apply Lemma 2.2. As before we calculate that

$$\begin{aligned}
\int_0^1 \int_0^{2\pi} & h(re^{i\theta})\overline{g(re^{i\theta})}(1-r)^{(n-3)/2}\, d\theta\, dr \\
&= 2\pi \sum h_k \bar{g}_k \int_0^1 r^{2k}(1-r)^{(n-3)/2}\, dr \\
&= 2\pi \sum h_k \bar{g}_k \frac{\Gamma(2k+1)\Gamma((n-1)/2)}{\Gamma(2k+1+(n-1)/2)}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\int h\bar{g}\, d\mu_n &= \int_0^1 \int_0^{2\pi} h(re^{i\theta})\overline{g(re^{i\theta})} d\theta\, w_n(r)\, dr \\
&= 2\pi \sum h_k \bar{g}_k \int_0^1 r^{2k}\, w_n(r)\, dr.
\end{aligned}$$

To calculate $\int_0^1 r^{2k} w_n(r)\,dr$ we use Theorem 1, (iii);

$$\int_0^1 r^{2k} w_n(r)\,dr = \frac{1}{2}\int_0^{2\pi}\int_0^1 |re^{i\theta}|^{2k} w_n(r)\,dr\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\int_0^1 g(re^{i\theta}) w_n(r)\,dr\,d\theta, \quad \text{where } g(z) = |z|^{2k}.$$

So, by Theorem 1 (iii) we see that

$$\int_0^1 r^{2k} w_n(r)\,dr$$

$$= \frac{1}{2\pi}\int g\circ\pi\,d\sigma_n = \frac{1}{2\pi}\int |n^{n/2} z_1\cdots z_n|^{2k}\,d\sigma_n(z)$$

$$= \frac{n^{nk}}{2\pi}\int |z_1\cdots z_n|^{2k}\,d\sigma_n(z) = \frac{n^{nk}}{2\pi}(n-1)!\frac{(k!)^n}{(n-1+(nk))!}$$

$$= \frac{(n-1)!}{2\pi} n^{nk}\frac{\Gamma(k+1)^n}{\Gamma(nk+n)};$$

here we have used the formula found in **[8]**. So we conclude that

$$\int h\bar{g}\,d\mu_n = \sum h_k \bar{g}_k (n-1)! n^{nk}\frac{\Gamma(k+1)^n}{\Gamma(nk+n)}.$$

We will use Stirling's formula:

$$\Gamma(x+1) = (2\pi e)^{1/2}\left(\frac{x}{e}\right)^{x+1/2}\left[1 + O\left(\frac{1}{x}\right)\right], \quad \text{as } x\to\infty.$$

We have

$$n^{nk}\frac{\Gamma(k+1)^n}{\Gamma(nk+n)}$$

$$= n^{nk}\frac{\left[(2\pi e)^{1/2}(ke^{-1})^{k+1/2}\right]^n}{(2\pi e)^{1/2}((nk+n+1)e^{-1})^{nk+n-1/2}}\left[1+O\left(\frac{1}{k}\right)\right]$$

$$= C_n n^{nk}\frac{k^{nk+n/2}}{(nk)^{nk+n-1/2}(1+(n-1)/nk)^{nk+n-1/2}}\left[1+O\left(\frac{1}{k}\right)\right]$$

$$= C_n' \frac{1}{k^{(n-1)/2}}\,\frac{1}{(1+(n-1)/nk)^{nk+n-1/2}}\cdot\left[1+O\left(\frac{1}{k}\right)\right],$$

where C_n, C_n' depend only on n, not on k. Next note that

$$(1+(n-1)/nk)^{nk} = e^{n-1}[1+O(1/k)].$$

We may conclude that

$$n^{nk}\frac{\Gamma(k+1)^n}{\Gamma(nk+n)} = C_0\frac{1}{k^{(n-1)/2}}\left[1+O\left(\frac{1}{k}\right)\right],$$

where C_0 depends only on n, not on k. A similar calculation shows us that

$$\frac{\Gamma(2k+1)}{\Gamma(2k+1+(n-1)/2)} = C_1\frac{1}{k^{(n-1)/2}}\left[1+O\left(\frac{1}{k}\right)\right]$$

for some constant C_1. We may conclude that

$$\int h\bar{g}\,d\mu_n = C\int h\bar{g}(1-r)^{(n-3)/2}\,dr\,d\theta$$
$$+O\left[\sum\frac{h_k\bar{g}_k}{k^{(n+1)/2}}\right]$$

for some constant C. By Lemma 2.2 the first term above has its modulus at most a constant times

$$\int|h|(1-r)^{(n-3)/2}\,dr\,d\theta$$

which is in turn bounded by a constant times $\int|h|\,d\mu_n$. The second term is bounded by a constant times

$$\sum\frac{|h_k|}{k}\frac{1}{k^{(n-1)/2}}$$

which is again bounded by a constant times

$$\sum\frac{|h_k|}{k}\int_0^1 r^{2k}(1-r)^{(n-3)/2}\,dr,$$

by Stirling's formula. This last expression, as was seen at the end of the proof of Lemma 2.2 is at most a constant times $\int|h|(1-r)^{(n-3)/2}\,dr\,d\theta$, which is, finally, majorized by $\int|h|\,d\mu_n$. This completes the proof.

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Received September 22, 1981 and in revised form May 5, 1982.

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