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ON ORTHOGONAL COMPLETION OF REDUCED RINGS

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It was proved by the author earlier that every orthogonal extension of a reduced ring R is a subring of $Q(R)$, the maximal two sided ring of quotients of R and the orthogonal completion of R, if it exists, is unique upto an isomorphism. Here, in Theorem 2, we prove that the orthogonal completion of R, if it exists, is a ring of right quotients $Q_F(R)$ of R with respect to an idempotent filter F of dense right ideals of R. Furthermore, it is shown in Proposition 5 that $Q_F(R)$ is an orthogonal extension of R if and only if for every $q \in Q_F(R)$, there exists a maximal orthogonal subset $\{e_i\colon i \in I\}$ of idempotents of $Q(R)$ such that q maps (by left multiplication) the right R-submodule of $Q(R)$ generated by $q^{-1}R \cup \{e_i\colon i \in I\}$ into R. Also an orthogonal extension $Q_F(R)$ is an orthogonal completion of R if and only if for every R-submodule M_R of $Q(R)_R$ generated by a maximal orthogonal subset of idempotents of $Q(R)$ and for every $f \in \operatorname{Hom}_R(M, R)$ there exists a $q \in Q_F(R)$ such that $f(m) = qm$ for every $m \in M$ (Proposition 6). Thus we obtain a necessary and sufficient condition for a reduced ring to have an orthogonal completion without any reference to its idempotent which improves earlier known results derived by Burgess and Raphael. By examples we show that reduced rings without proper idempotents may also have an orthogonal completion.

Introduction. Abian [**2**] showed that the canonical order relation '$\leq$' of Boolean rings can be defined for *reduced rings R* (a ring with no nonzero nilpotent element) by writing $a \leq b$ if $ab = a^2$ and this order relation makes R into a partially ordered multiplicative semigroup. Reduced rings under this relation '$\leq$' were studied by Abian [**1**] and Chacron [**5**] to characterise the direct produce of integral domains, division rings and fields. Their studies involved the concepts of orthogonal completeness and orthogonal completion of reduced rings. These two concepts, on their own merit, were studied by Burgess, Raphael and Stephenson [**3**], [**4**], [**11**]. They proved that reduced rings which have enough idempotents (i-dense) or satisfy certain chain conditions have an orthogonal completion. In this paper we shall provide a necessary and sufficient condition for a reduced ring to have an orthogonal completion.

In what follows, *all rings referred to will have* 1, *the identity element, and R will always denote a reduced ring*. In a reduced ring R, every idempotent is central and for every subset X of R, right and left annihilators of X in R coincide. A subset X of R is called an *orthogonal subset* of R if for every $x, y \in X$, $x \neq y$ implies $xy = 0$. An element $a \in R$ is said to be an *upper bound* of an orthogonal subset X of R if $xa = x^2$ for every $x \in X$. An upper bound a of X is called a supremum of X in R if for every upper bound b of X in R, $a \leq b$. It is obvious from this definition that a supremum of an orthogonal subset of R, if it exists, is unique. We denote the supremum of an orthogonal subset X in R by $\sup_R X$. It can be easily proved that an upper bound a of an orthogonal subset X of R is the supremum of X in R if and only if $\operatorname{ann}_R X = \operatorname{ann}_R(a)$. Also, for every orthogonal subset X of R and for every $r \in R$, $\sup_R rX = r(\sup_R X)$ provided $\sup_R X$ exists (see Raphael and Stephenson [**11**], page 347).

A reduced ring R is said to be *orthogonally complete* if every orthogonal subset X of R has a supremum in R. A reduced ring $\hat{R} \supseteq R$ is an *orthogonal extension* of R if every element of $\hat{R}$ is the supremum of an orthogonal subset of R. An orthogonal extension $\hat{R}$ of R is said to be an *orthogonal completion* of R if every orthogonal subset of R has a supremum in $\hat{R}$. It follows easily from this that an orthogonal extension $\hat{R}$ of R is an orthogonal completion of R if and only if $\hat{R}$ is orthogonally complete. Obviously, every orthogonal extension of a reduced ring is a reduced ring.

A ring $S \supseteq R$ is said to be a ring of *right quotients* of R if for every $s \in S$, $s^{-1}R = \{r \in R\colon sr \in R\}$ is a dense right ideal of R and $s(s^{-1}R) \neq 0$. We denote the ring of right quotients of R with respect to the idempotent filter of all dense right ideals of R by $Q_r(R)$. Every ring of right quotients may be regarded as a subring of $Q_r(R)$ in the canonical way ([**8**], page 99).

Let $Q(R) = \{q \in Q_r(R)\colon Dq \subseteq R$ for some dense left ideal D of $R\}$. Then obviously, $R \subseteq Q(R)$. It is proved by Wong and Johnson [**14**] that $Q(R)$ is a subring of $Q_r(R)$ and it is unique (up to isomorphism over R) maximal two sided ring of quotients of R. Also for every reduced ring R, $Q(R)$ is reduced (see Steinberg [**12**], page 466). It is proved in [**10**] (page 483) that every orthogonal subset X of R has a supremum in $Q(R)$ and since $Q(Q(R)) = Q(R)$, $Q(R)$ is orthogonally complete.

For every non-zero element a of a reduced ring R there exists an idempotent $0 \neq e \in Q(R)$ such that $\operatorname{ann}_{Q(R)}(a) = \operatorname{ann}_{Q(R)}(e)$ and $ae = ea = a$. For a proof of this see Lambek [7], Theorem 6.6.

We denote the injective hull of a right R-module M by $I(M)$.

LEMMA 1. *Let R be a ring, S a proper subring of $Q_r(R)$ such that $R \subseteq S$ and D a right ideal of R. Then* $\operatorname{Hom}_R(R/D, I(Q_r(R)/S)) = 0$ *if and only* *if for every* $x \in R$ *and for every* $q \in Q_r(R)$, $(D : x) \subseteq (S : q)$ *implies* $q \in S$.

Proof. $\operatorname{Hom}_R(R/D, I(Q_r(R)/S)) = 0$

$\Leftrightarrow$ for every $x + D \in R/D$ and for every $0 \neq q + S \in Q_r(R)/S$ there exists an $r \in R$ such that $xr \in D$ and $qr \notin S$

$\Leftrightarrow$ for every $x \in R$ and for every $q \in Q_r(R)\backslash S$, there exists an $r \in R$ such that $r \in (D : x)$ and $r \notin (S : q)$

$\Leftrightarrow$ for every $x \in R$ and for every $q \in Q_r(R)\backslash S$, $(D : x) \not\subseteq (S : q)$

$\Leftrightarrow$ for every $x \in R$ and for every $q \in Q_r(R)$, $(D : x) \subseteq (S : q)$ implies $q \in S$.

THEOREM 2. *Let* R *be a reduced ring which admits an orthogonal completion* $\hat{R}$. *Then there exists an idempotent filter* $\mathbf{F}$ *of dense right ideals of* R *such that* $\hat{R} = Q_F(R)$.

Proof. If $\hat{R} = Q_r(R)$, then $\hat{R} = Q_F(R)$ where $\mathbf{F}$ is the idempotent filter of all dense right ideal of R. Hence assume that $\hat{R} \subset Q_r(R)$ and let $\mathbf{F}$ denote the collection of all those dense right ideals of R for which $\operatorname{Hom}_R(R/D, I(Q_r(R)/\hat{R})) = 0$. Then $\mathbf{F}$ is an idempotent filter of dense right ideals of R corresponding to the torsion theory cogenerated by $I(Q_r(R)/\hat{R})$. Now consider $Q_F(R)$, which is a subring of $Q_r(R)$. Let $q \in Q_F(R)$. Then there exists a $D \in \mathbf{F}$ such that $qD \subseteq R$. Hence $D = (D : 1) \subseteq (\hat{R} : q)$, which by Lemma 1 implies that $q \in \hat{R}$. Thus $Q_F(R) \subseteq \hat{R}$.

On the other hand, let $q \in \hat{R}$ and suppose $0 \neq f \in \operatorname{Hom}_R(R/q^{-1}R, I(Q_r(R)/\hat{R}))$. Let $0 \neq p + \hat{R} \in f(R/q^{-1}R) \cap Q_r(R)/\hat{R}$ and $a \in R$ be such that $f(a + q^{-1}R) = p + \hat{R}$. Then for every $r \in (q^{-1}R : a) = (R : qa)$, $f(a + q^{-1}R)r = 0$. Hence $p(R : qa) \subseteq \hat{R}$.

Since $q \in \hat{R}$ and $a \in R$, qa, $qa + 1 \in \hat{R}$. Also, $(R : qa) = (R : qa + 1)$. Hence there exist orthogonal subsets $X = \{x_i : i \in I\}$, $Y = \{y_j : j \in J\}$ of $(R : qa)$ such that $qa = \sup_{\hat{R}} X$ and $qa + 1 = \sup_{\hat{R}} X$. Since $p(R : qa) \subseteq \hat{R}$, pX, $pY \subseteq \hat{R}$. Hence for every $z \in X$ and $r \in R$,

$$\begin{aligned}(\sup_{\hat{R}} pX)(rz) &= \sup_{\hat{R}}[(pX)(rz)] && \text{(Cor. 1.2, [}\mathbf{11}\text{])}\\ &= \sup_{\hat{R}} p[X(rz)] && \text{(associativity)}\\ &= p(zrz) && \text{(orthogonality of } X).\end{aligned}$$

Also, since $qa = \sup_{\hat{R}} X$, $z(qa) = (qa)z = z^2$. Hence $[(qa)rz - zrz]^2 = 0$ and since $\hat{R}$ is a reduced ring, this implies that $(qa)rz = zrz$ for every $z \in X$ and $r \in R$. Now consider the dense ideal $D = \{\Sigma a_i x_i b_i + r\colon a_i, b_i \in R, x_i \in X \text{ and } r \in \operatorname{ann}_R X\}$. For every $\Sigma a_i x_i b_i + r \in D$

$$\begin{aligned}(\sup_{\hat{R}} pX)\left(\sum a_i x_i b_i + r\right) &= \sum (\sup_{\hat{R}} pX) a_i x_i b_i \qquad (\text{Prop. 1.1, } [\mathbf{11}])\\ &= \sum p(x_i a_i x_i b_i)\\ &= p\left(\sum (qa) a_i x_i b_i\right)\\ &= p(qa)\left(\sum a_i x_i b_i + r\right).\end{aligned}$$

Hence $(pqa - \sup_{\hat{R}} pX)D = 0$ proving that $\sup_{\hat{R}} pX = pqa$. Similarly, $p(qa + 1) = \sup_{\hat{R}} pY$. But this implies that $p \in \hat{R}$ which contradicts the fact that $p + \hat{R}$ is a nonzero element of $Q_r(R)/\hat{R}$. Hence $\operatorname{Hom}_R(R/q^{-1}R, I(Q_r(R)/\hat{R})) = 0$ proving that $q^{-1}R \in \mathbf{F}$. Therefore $q \in Q_F(R)$ and hence $\hat{R} = Q_F(R)$ as was required to be proved.

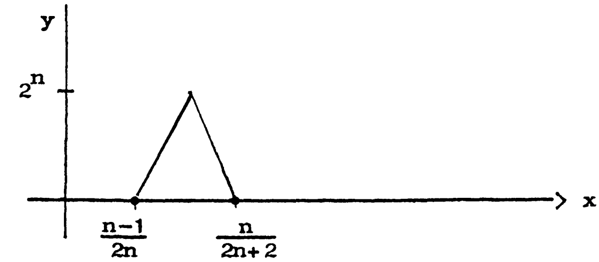
As an immediate consequence of this theorem we have the following result.

Proposition 3. *Let R be a reduced indecomposable ring which has an orthogonal completion R. Let $\mathbf{F}$ be an idempotent filter of dense right ideals of R such that $\hat{R} = Q_F(R)$. Then no dense right ideal $D \in \mathbf{F}$ can be expressed as a nontrivial direct sum of two right ideals.*

Proof. If $D = D_1 \oplus D_2$, then $e \in Q_F(R)$ which is defined by $e(d_1 + d_2) = d_1$ for every $d_1 \in D_1$, $d_2 \in D_2$ is an idempotent of $\hat{R} = Q_F(R)$. If $D_1 \neq 0 \neq D_2$, then e is a proper idempotent of $\hat{R}$. Since $\hat{R}$ is an orthogonal completion of R and $e \in \hat{R}$, there exists an orthogonal set $\{e_I\colon i \in R\} \subseteq R$ such that $e = \sup_{\hat{R}} e_i$. But then each e_i should be a proper idempotent of R which is a contradiction since R is indecomposable.

Corollary 4. *$C[0, 1]$, the ring of continuous real values function on the interval $[0, 1]$ has no orthogonal completion.*

Proof. Consider the orthogonal subset $\{f_n\colon n \in N\} \subseteq C[0, 1]$, where for a typical $n \in N$, f_n is given by the following.



Then $\sup f_n$ is not continuous at $1/2$. Hence $C[0, 1]$is not orthogonally complete. Now suppose $C[0, 1]$ has an orthogonal completion $\hat{R}$. Then by Theorem 2 there exists an idempotent filter $\mathbf{F}$ of dense ideals of $C[0, 1]$ such that $\hat{R} = Q_F(C[0, 1])$. Let $f \in Q(C[0, 1])$ be the supremum of $\{f_n: n \in N\}$ and $M = \{g \in C[0, 1]: g(1/2) = 0\}$. The $f^{-1}(C[0, 1]) = \{g \in C[0, 1]: fg \in C[0, 1]\} \subseteq M$. Hence $M \in \mathbf{F}$. Since $C[0, 1]$ has no proper idempotents and M can be expressed as a proper direct sum of ideals of $C[0, 1]$, by Proposition 3 this is a contradiction. Hence $C[0, 1]$ has no orthogonal completion.

Burgess and Raphael [**3**] proved this result using the properties of continuous real valued functions over a closed interval.

Theorem 2 provides us a necessary condition for a reduced ring $\hat{R}$ to be an orthogonal completion of R. But it fails to provide a sufficient condition for a reduced ring R to have an orthogonal completion because for many subrings S of $Q_r(R)$ there exist idempotent filters $\mathbf{F}$ such that $S = Q_F(R)$. The following proposition characterises the nature of idempotent filters which give rise to an orthogonal extension of R.

In what follows, we shall write $\sup X$ for $\sup_{Q(R)} X$. Also, we would like to remind our reader that for an orthogonal subset X of R, Burgess and Raphael ([**3**], Lemma 11) proved that $\sup_R X = \sup X$ whenever $\sup_R X$ exists.

PROPOSITION 5. *Let R be a reduced ring and $\mathbf{F}$ an idempotent filter of dense right ideals of R. Then $Q_F(R)$ is an orthogonal extension of R if and only if for every $q \in Q_F(R)$, there exists a maximal orthogonal subset $\{e_i: i \in I\}$ of idempotents of $Q(R)$ such that q maps (by left multiplication) the right R-submodule of $Q(R)$ generated by $q^{-1}R \cup \{e_i: i \in I\}$ into R.*

Proof. Suppose $Q_F(R)$ is an orthogonal extension of R. Then $Q_F(R) \subseteq Q(R)$. Let $a \in Q_F(R)$. Then there exists an orthogonal subset $\{a_i: i \in I'\}$ of R such that $a = \sup\{a_i: i \in I'\}$. Let $E = \{e_i: i \in I'\}$ be the orthogonal subset of central idempotents of $Q(R)$ such that a_i, $e_i = a_i$ and $\operatorname{ann}_{Q(R)}(a_i) = \operatorname{ann}_{Q(R)}(e_i)$ for every $i \in I'$ (see [7], Theorem 6.6).

Embed E in a maximal orthogonal subset $\{e_i\colon i \in I\}$ of idempotents of $Q(R)$. Then, since

$$\operatorname{ann}_{Q(R)}\{a_i\colon i \in I'\} = \operatorname{ann}_{Q(R)}(a) = \operatorname{ann}_{Q(R)}\{e_i\colon i \in I'\},$$

$ae_i = 0$ for every $i \in I \backslash I'$. Also, since a is the supremum of $\{a_i\colon i \in I'\}$, $a_i = a_i e_i \leq ae_i$ for every $i \in I'$. But, since

$$\operatorname{ann}_{Q(R)}(e_i) \subseteq \operatorname{ann}_{Q(R)}(ae_i) \subseteq \operatorname{ann}_{Q(R)}(a_i),$$

we have $a_i = ae_i$, proving that $ae_i \in R$ for every $i \in I$.

Now if we let M be the right R-submodule of $Q(R)$ generated by $q^{-1}R \cup \{e_i\colon i \in I\}$ then obviously the left multiplication by q determines a homomorphism from M into R.

Conversely, suppose for every $q \in Q_F(R)$ there exists a maximal orthogonal subset $\{e_i\colon i \in I\}$ of idempotents of $Q(R)$ such that q maps (by left multiplication) the right R-submodule of $Q(R)$ generated by $q^{-1}R \cup \{e_i\colon i \in I\}$ into R. Consider the ideal $D = \{\Sigma e_i R\} \cap R$. Since $\{e_i\colon i \in I\}$ is a maximal orthogonal subset of $Q(R)$, $\operatorname{ann}_R D = 0$. Hence D is a dense right and left ideal of R. Since e_i's are central in $Q(R)$, $Dq \subseteq R$ and therefore $q \in Q(R)$. It follows from this that $Q_F(R) \subseteq Q(R)$.

Now, since $\{e_i\colon i \in I\}$ is a maximal orthogonal subset of $Q(R)$, q is the supremum of the orthogonal subset $\{qe_i\colon i \in I\}$ of R. Hence $Q_F(R)$ is an orthogonal extension of R. This completes the proof.

The following proposition gives us a necessary and sufficient condition for an orthogonal extension to be an orthogonal completion.

PROPOSITION 6. *Let* **F** *be an idempotent filter of dense right ideals of* R *such that* $Q_F(R)$ *is an orthogonal extension of* R. *Then* $Q_F(R)$ *is an orthogonal completion of* R *if and only if for every* R*-submodule* M_R *of* $Q(R)_R$ *generated by a maximal orthogonal subset of idempotents of* $Q(R)$ *and for every* $f \in \operatorname{Hom}_R(M, R)$ *there exists a* $q \in Q_F(R)$ *such that* $f(m) = qm$ *for every* $m \in M$.

Proof. Since $Q_F(R)$ is an orthogonal extension of R. $Q_F(R)$ is a subring of $Q(R)$. Suppose $Q_F(R)$ is an orthogonal completion of R, M_R is an R-submodule of $Q(R)$ generated by a maximal orthogonal subset $\{e_i\colon i \in I\}$ of idempotents of $Q(R)$ and $f \in \operatorname{Hom}_R(M, R)$. Then (as in the proof of Proposition 5) the ideal $D = (\Sigma_i e_i R) \cap R = M \cap R$ of R is a dense right and left ideal of R. Since $f|D \in \mathrm{Hom}_R(D_R, R_R)$, we can find a $q \in Q_r(R)$ such that $f(d) = qd$ for every $d \in D$. Let $i \in I$. Then since D is a dense right ideal of R and $e_i \in Q_r(R)$, $e_i^{-1}D$ is *a* dense right ideals of R. Now, $qe_i - f(e_i) \in Q_r(R)$ and for every $r \in e_i^{-1}D$,

$$\begin{aligned}(qe_i - f(e_i))r &= qe_ir - f(e_i)r \\ &= f(e_ir) - f(e_i)r \\ &= f(e_i)r - f(e_i)r \\ &= 0.\end{aligned}$$

Since $e_i^{-1}D$ is a dense right ideal of R, it follows from this that $qe_i = f(e_i)$ for every $i \in I$. Hence $f(m) = qm$ for every $m \in M$. Further, for every $i \in I$ and $d \in D$,

$$\begin{aligned}(qe_i - e_iq)d &= q(e_id) - e_i(qd) = f(e_id) - e_if(d) \\ &= q(de_i) - (qd)e_i = 0.\end{aligned}$$

(Here we used the fact that e_i's are central in $Q(R)$ and that $qd \in R$.) Since D is a dense right ideal of R, it follows from this that $qe_i = e_iq$ for every $i \in I$. Therefore, $Dq = ((\Sigma_i e_iR) \cap R)q \subseteq R$. Thus $q \in Q(R)$ and hence $\sup\{qe_i \colon i \in I\} = q \cdot \sup\{e_i \colon i \in I\} = q \cdot 1 = q$. Since $\{qe_i \colon i \in I\} \subseteq R \subseteq Q_F(R)$ and $Q_F(R)$ is orthogonally complete, this implies that $q \in Q_F(R)$, as was required.

Now we prove the converse. Let $\{a_i \colon i \in I'\} \subseteq R$ be an orthogonal subset and $\{e_i \colon i \in I'\}$ be the orthogonal subset of idempotents of $Q(R)$ such that $a_ie_i = e_ia_i = a_i$ and $\mathrm{ann}_{Q(R)}(a_i) = \mathrm{ann}_{Q(R)}(e_i)$ for every $i \in I'$. Let $\{e_i \colon i \in I\}$ be a maximal orthogonal subset of idempotents of $Q(R)$ containing $\{e_i \colon i \in I'\}$ and $M = \Sigma_{i \in I} e_iR$. Define $f \colon M \to R$ by

$$\begin{aligned}f(e_i) &= a_i \quad \text{if } i \in I' \\ &= 0 \quad \text{otherwise}\end{aligned}$$

and extend f by linearity over sums. Then $f \in \mathrm{Hom}_R(M, R)$. Hence by our assumption there exists an $a \in Q_F(R)$ such that $f(m) = am$ for every $m \in M$. Now

$$aa_i = a(a_ie_i) = a(e_ia_i) = f(e_i)a_i = a_i^2$$

and hence $a_i \leq a$ for every $i \in I'$. Also, if $x \in \mathrm{ann}_{Q(R)}\,\{a_i \colon i \in I'\}$ then, since

$$(ae_i)x = f(e_i)x,$$

$(ae_i)x = 0$ for every $i \in I$. Hence,

$$ax = a.1.x = a(\sup\{e_i \colon i \in I\})x = \sup\{(ae_i)x \colon i \in I\} = 0.$$

Thus $\operatorname{ann}_{Q(R)}\{a_i \colon i \in I'\} \subseteq \operatorname{ann}_{Q(R)}(a)$. Since a is an upper bound of $\{a_i \colon i \in I'\}$, $\operatorname{ann}_{Q(R)}(a) \subseteq \operatorname{ann}_{Q(R)}\{a_i \colon i \in I'\}$. Hence a is the supremum of $\{a_i \colon i \in I'\}$. Since, $Q_F(R) \subseteq Q(R)$ and $a \in Q_F(R)$, $a = \sup_{Q_F(R)}\{a_i \colon i \in I'\}$. Thus we see that every orthogonal subset of R has a supremum in $Q_F(R)$. Since $Q_F(R)$ is an orthogonal extension of R, it follows that $Q_F(R)$ is the orthogonal completion of R. This completes the proof.

Combining Propositions 5 and 6 we get the following result.

THEOREM 7. *A reduced ring R has an orthogonal completion if and only if it has an idempotent filter* **F** *of dense right ideals of R such that* (i) *for every $q \in Q_F(R)$ there exists a maximal orthogonal subset $\{e_i \colon i \in I\}$ of idempotents of $Q(R)$ such that q maps (by left multiplication) the right R-submodule M of $Q(R)$ generated by $q^{-1}R \cup \{e_i \colon i \in I\}$ into R and*

(ii) *for every R-submodule M of $Q(R)$ generated by maximal orthogonal subset of idempotents of $Q(R)$ and for every $f \in \operatorname{Hom}_R(M, R)$, there exists a $q \in Q_F(R)$ such that $f(m) = qm$ for every $m \in M$.*

Thus a reduced ring R has an orthogonal completion if and only if it has an idempotent filter **F** of dense right ideals of R such that $Q_F(R)$ consists of exactly those elements of $Q(R)$ for which there exists a maximal orthogonal subset $\{e_i \colon i \in I\}$ of idempotents of $Q(R)$ such that $qe_i \in R$ for every $i \in I$. Also, it follows from this that a reduced ring R is orthogonally complete if and only if it contains all those elements of $Q(R)$ for which there exists a maximal orthogonal subset $\{e_i \colon i \in I\}$ of idempotents of $Q(R)$ such that $qe_i \in R$ for every $i \in I$.

As an application of Theorem 7, we prove the following result which was established in **[4]** for commutative rings.

COROLLARY 8. *Every reduced i-dense ring has an orthogonal completion.*

Proof. Let **F** be the idempotent filter of all those dense right deals of R which contain a maximal orthogonal subset of idempotents of R and consider $Q_F(R)$. Since all members of **F** contain a maximal orthogonal subset of idempotents of R, $Q_F(R)$ is an orthogonal extension of R. Now let M_R be any R-submodule of $Q(R)_R$ generated by a maximal orthogonal subset of idempotents of $Q(R)$ and $f \in \mathrm{Hom}_R(M, R)$. Then, since each idempotent of $Q(R)$ is the supremum of an orthogonal subset of idempotents of R, it follows that M_R contains a maximal orthogonal subset of idempotents of R and hence a member D of $\mathbf{F}$. Therefore, there exists a $q \in Q_F(R)$ such that $f(d) = qd$ for every $d \in D$.

Now let m be an arbitrary element of M. Then, since $M \subseteq Q(R)$, $m^{-1}D = \{r \in R\colon\ mr \in D\}$ is a dense right ideal of R and for every $d' \in m^{-1}D$, $md' \in D$ and hence

$$(f(m) - qm)d' = f(m)d' - (qm)d' = f(md') - q(md') = 0.$$

Thus $f(m) = qm$ for every $m \in M$ and hence by Theorem 7 $Q_F(R)$ is the orthogonal completion of R. This completes the proof.

It follows from this result that every reduced Baer ring has an orthogonal completion (see [**5**], Theorems 18).

EXAMPLE 9. For every $n \geq 1$, let $R_n = k[x, y, z]$ be a polynomial ring in commuting indeterminates over a field k such that $yz = 0$. Let R be the subring of $\prod_{n=1}^{\infty} R_i$ generated by $\bigoplus_{n=1}^{\infty} R_n$ and $1 \in \prod_{n=1}^{\infty} R_n$. Let $D = \bigoplus_{n=1}^{\infty} R_n$ and $\mathbf{F}$ be the idempotent filter of ideals of R which contain D. Since D contains a maximal orthogonal subset of idempotents of R and hence of $Q(R)$, it follows from Theorem 7 that $Q_F(R)$ is an orthogonal extension of R. Also, since every R-submodule M of $Q(R)$ generated by a maximal orthogonal subset $\{f_i\colon\ i \in I\}$ of idempotents of $Q(R)$ contains $\{e_n\colon\ n \in N\}$ (where e_n is that element of $R \subseteq \prod_{i=1}^{\infty} R_i$ whose nth coordinate is 1 and all other coordinates are 0), it follows that $D \subseteq M$. Hence if $g \in \mathrm{Hom}_R(M, R)$ then $g|D \in \mathrm{Hom}_R(D, R)$. Let $q \in Q_F(R)$ be such that $g(d) = qd$ for every $d \in D$.

Now, let m be an arbitrary element of M. Then as in the proof of Corollary 8, it can be proved that $g(m) = qm$ for every $m \in M$. Hence by Theorem 7, $Q_F(R)$ is the orthogonal completion of R. It can be easily verified that $Q_F(R) = \prod_{i=1}^{\infty} R_i$. It is also interesting to observe that R is not i-dense.

Theorem 7 produces conclusive results when applied to reduced rings without proper idempotents as well. This is done in the next example.

EXAMPLE 10. Let S denote the ring $\prod_{n=1}^{\infty} k_n[x]$ where $k_n = \mathbf{Z}/(2)$ for every $n \in N$ and let $M = (\mathbf{x}^2 + \mathbf{x})S$ where $\mathbf{x}$ is the element of S all of whose co-ordinates are x. Let R be the subring of S generated by $M \cup \{\mathbf{1}\} \cup \{a_n : n \in \mathbf{N}\} \cup \{b_n : n \in \mathbf{N}\}$ where

$$a_1 = (x+1, x+1, 0, 0, 0, 0, 0, \ldots\ldots\ldots\ldots),$$
$$a_2 = (0, 0, x+1, x+1, 0, 0, \ldots\ldots\ldots\ldots),$$
$$a_3 = (0, 0, 0, 0, x+1, x+1, 0, 0, \ldots\ldots\ldots\ldots),$$
$$\ldots\ldots\ldots\ldots$$
$$b_1 = (x, 0, 0, 0, 0, 0, 0, \ldots\ldots\ldots\ldots),$$
$$b_2 = (0, x, x, 0, 0, 0, 0, \ldots\ldots\ldots\ldots),$$
$$b_3 = (0, 0, 0, x, x, 0, 0, \ldots\ldots\ldots\ldots),$$
$$\ldots\ldots\ldots\ldots$$

Every power of a_n (resp. b_n) can be expressed as $a_n + m$ (resp. $b_n + m$) where $m \in M$. Also, $a_n b_m \in M$, $a_n M \subseteq M$ and $b_n M \subseteq M$. Hence every element of R can be expressed in the form

$$\sum \alpha_i a_i + \sum \beta_j b_j + m + \mathbf{n}$$

where $\alpha_i, \beta_j \in \mathbf{Z}/(2)$ for every $i, j, m \in M$ and $\mathbf{n} = \mathbf{0}$ or $\mathbf{1}$. It can be easily verified that R has no proper idempotent.

Let D be the ideal of R generated by $\{a_n : n \in \mathbf{N}\}$, $\{b_n : n \in \mathbf{N}\}$ and let $\mathbf{F}$ be the idempotent filter of all those ideals of R which contain D. Since $\operatorname{ann}_R D = 0$, $\mathbf{F}$ is an idempotent filter of dense ideals of R. Let f: $D \to R$ be an R homomorphism. Then since (taking $a_0 = 0$)

$$a_i = (b_i + a_i + b_{i+1})a_i,$$
$$b_i = (a_{i-1} + b_i + a_i)b_i,$$

it follows that

$$f(a_i) = [f(b_i) + f(a_i) + f(b_{i+1})]a_i$$

and

$$f(b_i) = [f(a_{i-1}) + f(b_i) + f(a_i)]b_i.$$

Let $q = \sup_i {}_{Q(R)} f(a_i) + \sup_j {}_{Q(R)} f(b_j)$. Then, since every element of D is of the form of

$$d = \sum \alpha_i a_i + \sum \beta_j b_j + m$$

where $\alpha_i, \beta_j \in \mathbf{Z}/(2)$ and $m = \Sigma a_k r_k$ (or $\Sigma b_k s_k$) is a finite sum with r_k's (or s_k's) in R. Hence it follows that $f(d) = qd$ for every $d \in D$. We show that either $q \in R$ or there exists a proper maximal orthogonal subset $E = \{e_i: i \in I\}$ of idempotents of $Q\max(R)$ such that $qE \subseteq R$.

Since $f(b_i) + f(a_i) + f(b_{i+1}) \in R$, it is of the form $\Sigma\alpha_i a_i + \Sigma\beta_j b_j + m + \mathbf{n}$ and it follows from this that $f(a_i) = \alpha_1' a_i + m_i$ where $m_i \in M$ is such that $m_i a_j = 0$ for every $i \neq j$ and $\alpha_i' \in \mathbf{Z}/(2)$. Hence, by Lemma 5 of **[10]**,

$$\begin{aligned}\sup_i f(a_i) &= \sup_i(\alpha_i' a_i + m_i) = \sup_i \alpha_i' a_i + \sup_i m_i \\ &= \sup \alpha_i' a_i + m \quad \text{where } m \in M.\end{aligned}$$

Similarly,

$$\begin{aligned}\sup_j f(b_j) &= \sup_j(\beta_j' b_j + m_j') = \sup_j \beta_j' b_j + \sup_j m_j' \\ &= \sup_j \beta_j' b_j + m' \quad \text{where } m' \in M.\end{aligned}$$

Hence,

$$q = \sup_i \alpha_i' a_i + \sup \beta_j' b_j + (m + m')$$

where $\alpha_i', \beta_j' \in \mathbf{Z}/(2)$ and $m + m' \in M$. Thus $q = r + m''$ where $r = \sup_i \alpha_i' a_i + \sup_j \beta_j' b_j$ and $m'' = m + m'$.

It follows from these discussions that $Q_F(R)$ is the subring of S obtained by adjoining elements of the form of $\sup \alpha_i a_i + \sup \beta_j b_j$ to R where $\alpha_i, \beta_j \in \mathbf{Z}/(2)$ for all i, j. Now using Theorem 7, it can be verified that $Q_F(R)$ is the orthogonal completion of R.

This example shows that the existence of proper idempotents in R is not a necessary condition for R to have an orthogonal completion.

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