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ON THE VECTOR FIELDS ON AN ALGEBRAIC HOMOGENEOUS SPACE

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We construct a holomorphic vector field V with isolated zeros on an algebraic homogeneous space $X = G/P$ and show that the Koszul complex defined by V gives much information concerning the cohomology groups of X. Our results give useful examples to the studies of J. B. Carrell and D. Lieberman.

1. Koszul complex. Let X be a compact Kähler manifold of dimension n. We assume the manifold X admits a holomorphic vector field V whose zero set Z is simple isolated and nonempty. The following complex of sheaves is said to be the Koszul complex defined by V:

$$0 \to \Omega^n \xrightarrow{\partial} \Omega^{n-1} \xrightarrow{\partial} \cdots \to \Omega^1 \xrightarrow{\partial} \Omega^0 = \mathcal{O}_X \to 0, \tag{1.1}$$

where the differential ∂ is the contraction map $i(V)$. The structure sheaf of Z is $\mathcal{O}_Z = \mathcal{O}_X/i(V)\Omega^1$. To make the differentials of degree $+1$, we substitute $K^p = \Omega^{-p}$:

$$0 \to K^{-n} \xrightarrow{\partial} K^{-n+1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} K^0 \to 0. \tag{1.2}$$

For any locally free sheaf $\mathcal{F}$, we denote by $K(\mathcal{F})$ the complex obtained by tensoring $\mathcal{F}$ with (1.2) over $\mathcal{O}_X$. Let $\mathcal{H}^*(\mathcal{F})$ be the cohomology sheaves of the complex $K(\mathcal{F})$. Then, from the assumptions, it follows that $\mathcal{H}^q(\mathcal{F}) = 0$ for $-n \leq q < 0$ and $\mathcal{H}^0(\mathcal{F}) = \mathcal{F} \otimes \mathcal{O}_Z$, whose support is contained in Z. We abbreviate $\mathcal{F}_Z = \mathcal{F} \otimes \mathcal{O}_Z$. The hypercohomology $\mathbf{H}^*(X, K(\mathcal{F}))$ can be calculated by using the double Čech complex $\check{C}^*(\mathcal{U}, K(\mathcal{F}))$ in the usual manner. See [**3**]. Corresponding to the natural two filtrations in $\check{C}^*(\mathcal{U}, K(\mathcal{F}))$, we get the following spectral sequences which converge to $\mathbf{H}^{p+q}(X, K(\mathcal{F}))$:

$$'E_1^{p,q} = H^q(X, K^p(\mathcal{F})), \tag{1.3}$$

$$''E_2^{p,q} = H^p(X, \mathcal{H}^q(\mathcal{F})). \tag{1.4}$$

From the above remark, it follows that $\mathbf{H}^r(X, K(\mathcal{F})) = 0$ for $r \neq 0$ and $\mathbf{H}^0(X, K(\mathcal{F})) = H^0(Z, \mathcal{F}_Z)$. Note that the space $H^0(Z, \mathcal{O}_Z)$, i.e., in case $\mathcal{F} = \mathcal{O}_X$, can be interpreted as the ring of complex-valued functions on Z.

Let $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ be locally free sheaves and ϕ: $\mathcal{F}_1 \times \mathcal{F}_2 \to \mathcal{F}_3$ a bilinear map. Then by using the exterior product in K^*, we obtain a bilinear map

$$(1.5) \qquad \phi\colon \mathbf{H}^p(X, K(\mathcal{F}_1)) \times \mathbf{H}^q(X, K(\mathcal{F}_2)) \to \mathbf{H}^{p+q}(X, K(\mathcal{F}_3)).$$

Further if we denote by $F\mathbf{H}^p(K(\mathcal{F}_i))$ the filtration on $\mathbf{H}^p(X, K(\mathcal{F}_i))$ induced from the $'E_1$-terms (1.3), then the map keeps the filtrations

$$(1.6) \qquad \phi\colon F_r\mathbf{H}^p(K(\mathcal{F}_1)) \times F_s\mathbf{H}^q(K(\mathcal{F}_2)) \to F_{r+s}\mathbf{H}^{p+q}(K(\mathcal{F}_3))$$

for $p, q, r, s \in \mathbf{Z}$. In particular if we take $\mathcal{F}_i = \mathcal{O}_X$, $1 \le i \le 3$, and ϕ: $\mathcal{O}_X \times \mathcal{O}_X \to \mathcal{O}_X$ the multiplication, we can introduce a natural ring structure in $\mathbf{H}^0(X, K)$ which is compatible with the wedge product pairing of the groups $'E_1^{-p,q} = H^q(X, \Omega^p)$. Further we have the following known results. See [**2**], [**3**].

LEMMA 1. *Suppose the manifold X and the vector field V are as above. Then*

(1) *In case* $\mathcal{F} = \mathcal{O}_X$, *all the differentials of* (1.3) *vanish.*

(2) *Therefore comparing* (1.3) *and* (1.4), *we have*

$$(1.7) \qquad H^p(X, \Omega^q) = 0 \quad \text{for } p \neq q.$$

(3) *The space* $H^0(Z, \mathcal{O}_Z)$ *has the canonical filtration induced from the filtered hypercohomology ring* $\mathbf{H}^0(X, K)$ *such that*:

$$(1.8) \qquad H^0(Z, \mathcal{O}_Z) = F_{-n} \supseteq F_{-n+1} \supseteq \cdots \supseteq F_0 \supseteq \{0\},$$

$$(1.9) \qquad F_p \cdot F_q \subseteq F_{p+q},$$

$$(1.10) \qquad F_{-p}/F_{-p+1} \cong H^p(X, \Omega^p),$$

$$(1.11) \qquad H^*(X, \mathbf{C}) \cong \operatorname{gr} H^0(Z, \mathcal{O}_Z) = \bigoplus_{p=0}^{n} F_{-p}/F_{-p+1}.$$

2. *V*-equivariant vector bundles. The following definition and results are in [**3**].

DEFINITION. We say that a vector bundle $\mathcal{E}$ on X is V-equivariant if the derivation V: $\mathcal{O}_X \to \mathcal{O}_X$ can be lifted to $\mathcal{E}$, i.e., there exists a $\mathbf{C}$-linear map $\tilde{V}$: $\mathcal{E} \to \mathcal{E}$ such that

$$(2.1) \qquad \tilde{V}(f \cdot s) = V(f) \cdot s + f \cdot \tilde{V}(s)$$

where f is a local section of $\mathcal{O}_X$ and s that of $\mathcal{E}$.

Let $\{f_{ij}\}$ be a set of transition matrices of $\mathfrak{E}$. Then the set $\{df_{ij} \cdot f_{ij}^{-1}\}$ defines the Atiyah-Chern class $c(\mathfrak{E})$ of $\mathfrak{E}$ in $H^1(X, Hom(\mathfrak{E}, \mathfrak{E}) \otimes \Omega^1)$. And the class $i(V)c(\mathfrak{E})$ in $H^1(X, Hom(\mathfrak{E}, \mathfrak{E}))$ is the obstruction for $\mathfrak{E}$ to be V-equivariant. See [3]. If we put $\mathfrak{F} = Hom(\mathfrak{E}, \mathfrak{E})$, the cohomology groups $H^1(X, Hom(\mathfrak{E}, \mathfrak{E}) \otimes \Omega^1)$ and $H^1(X, Hom(\mathfrak{E}, \mathfrak{E}))$ can be interpreted as the $'E_1^{-1,1}$ and $'E_1^{0,1}$-terms, respectively, of the spectral sequence (1.3). Therefore each V-equivariant vector bundle $\mathfrak{E}$ defines the hypercohomology class $\tilde{c}(\mathfrak{E})$ lying in $F_{-1}\mathbf{H}^0(X, K(Hom(\mathfrak{E}, \mathfrak{E})))$. Here the class $\tilde{c}(\mathfrak{E})$ is well defined only up to $F_0\mathbf{H}^0(X, K(Hom(\mathfrak{E}, \mathfrak{E})))$ and is called the hyper-Chern class of $\mathfrak{E}$. We denote by σ_d: $Hom(\mathfrak{E}, \mathfrak{E}) \to \mathcal{O}_X$, $0 \leq d \leq r =$ rank $\mathfrak{E}$, the vector bundle homomorphisms defined by the rule:

$$(2.2) \qquad \det(tI + \mathfrak{A}) = \sum_{d=0}^{r} \sigma_d(\mathfrak{A})t^{r-d}, \qquad \mathfrak{A} \in Hom(\mathfrak{E}, \mathfrak{E}).$$

The mapping σ_d is usually called the dth elementary function and is a polynomial map of degree d. We denote by e_d: $F_{-d}\mathbf{H}^0(X, K) \cong F_{-d} \to H^d(X, \Omega^d)$ the mapping which induces the canonical isomorphism $F_{-d}/F_{-d+1} \cong H^d(X, \Omega^d)$.

LEMMA 2. *The map σ_d determines the classes $\sigma_d(c(\mathfrak{E}))$ and $\sigma_d(\tilde{c}(\mathfrak{E}))$ which belong to $H^d(X, \Omega^d)$ and $F_{-d}\mathbf{H}^0(X, K)$, respectively. We have*:

(1) $(-1)^d\sigma_d(c(\mathfrak{E}))$ *is the dth Chern class of $\mathfrak{E}$ and coincides with* $(-1)^d e_d(\sigma_d(\tilde{c}(\mathfrak{E})))$.

(2) *Let* $\tilde{V}_Z \in H^0(Z, Hom(\mathfrak{E}, \mathfrak{E})_Z) \cong \mathbf{H}^0(X, K(Hom(\mathfrak{E}, \mathfrak{E})))$ *denote the restriction of $\tilde{V}$ to Z. Then $(-1)^d\sigma_d(\tilde{V}_Z)$ belongs to $H^0(Z, \mathcal{O}_Z)$ and is equal to* $(-1)^d\sigma_d(\tilde{c}(\mathfrak{E}))$.

3. Semisimple Lie algebras. Let $\mathfrak{g}$ be a complex semisimple Lie algebra. We choose a compact form $\mathfrak{k}$ and define a $*$-operation on $\mathfrak{g}$ with respect to $\mathfrak{k}$. Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$. If we put $\mathfrak{h} = \mathfrak{b} \cap \mathfrak{b}^*$ then $\mathfrak{h}$ becomes a Cartan subalgebra of $\mathfrak{g}$. Let $\Delta \subset \mathfrak{h}_{\mathfrak{R}}^*$ be the root system of $\mathfrak{h}$ in $\mathfrak{g}$. The set Δ is divided into two classes, the positive roots Δ_+ and negative roots Δ_- with respect to $\mathfrak{b}$. We denote by Π the set of simple roots corresponding to Δ_+. Then any root $\phi \in \Delta$ can be written as $\phi = \sum_{\alpha \in \Pi} n_\alpha(\phi)\alpha$ where $n_\alpha(\phi)$ are nonnegative or nonpositive integers according to $\phi \in \Delta_+$ or Δ_-. The algebra $\mathfrak{g}$ has the root space decomposition

$$(3.1) \qquad \mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Delta_+} \mathfrak{g}_\alpha + \sum_{\beta \in \Delta_-} \mathfrak{g}_\beta,$$

where

$$\mathfrak{b} = \mathfrak{h} + \sum_{\alpha \in \Delta_+} \mathfrak{g}_\alpha.$$

For any $\alpha \in \Delta$, $\dim \mathfrak{g}_\alpha = 1$, and from the definition of $\mathfrak{g}_\alpha$ it follows that

$$\text{(3.2)} \qquad \mathrm{ad}(H)(X) = [H, X] = \alpha(H)X, \qquad X \in \mathfrak{g}_\alpha, H \in \mathfrak{h}.$$

Let $\mathfrak{p}$ be a parabolic Lie subalgebra of $\mathfrak{g}$ which contains $\mathfrak{b}$. Then there exists a decomposition of $\mathfrak{g}$ corresponding to $\mathfrak{p}$.

LEMMA 3. *Let* $\mathfrak{g}$, $\mathfrak{p}$ *be as above. We put* $\mathfrak{n} = \{Z \in \mathfrak{g} \mid (Z, Y) = 0$ *for all* $Y \in \mathfrak{p}\}$ *where* $(\,,)$ *is the Killing form of* $\mathfrak{g}$. *Then* $\mathfrak{n}$ *is the maximal nilpotent ideal of* $\mathfrak{p}$ *and also the set of all nilpotent elements in the radical of* $\mathfrak{p}$. *Further if we define* $\mathfrak{g}_1 = \mathfrak{p} \cap \mathfrak{p}^*$, *then we have the decomposition*

$$\text{(3.3)} \qquad \mathfrak{g} = \mathfrak{n}^* + \mathfrak{g}_1 + \mathfrak{n}, \qquad \mathfrak{p} = \mathfrak{g}_1 + \mathfrak{n}.$$

Moreover $\mathfrak{g}_1$ *lies in the normalizers of both* $\mathfrak{n}$ *and* $\mathfrak{n}^*$.

For any subspace $\mathfrak{a}$ which is invariant by the adjoint action of $\mathfrak{h}$, we define the set $\Delta(\mathfrak{a}) \subseteq \Delta$ as follows:

$$\text{(3.4)} \qquad \Delta(\mathfrak{a}) = \{0 \neq \alpha \in \mathfrak{h}^*_{\mathfrak{R}} \mid [H, X] = \alpha(H)X \text{ for some } 0 \neq X \in \mathfrak{a} \text{ and any } H \in \mathfrak{h}\}.$$

The subalgebras $\mathfrak{g}_1$, $\mathfrak{n}$ and $\mathfrak{n}^*$ are invariant and we have:

$$\text{(3.5)} \qquad \Delta(\mathfrak{g}_1) = \{\phi \in \Delta \mid n_\alpha(\phi) = 0 \text{ for all } \alpha \in \Pi \cap \Delta(\mathfrak{p})\},$$

$$\text{(3.6)} \qquad \Delta(\mathfrak{n}) = \{\phi \in \Delta_+ \mid n_\alpha(\phi) > 0 \text{ for all } \alpha \in \Pi \cap \Delta(\mathfrak{p})\},$$

$$\text{(3.7)} \qquad \Delta(\mathfrak{n}^*) = -\Delta(\mathfrak{n}).$$

Let G be a simply-connected complex semisimple Lie group whose Lie algebra is $\mathfrak{g}$. Let B, T and P be the Borel subgroup of G with Lie algebra $\mathfrak{b}$, the Cartan subgroup with Lie algebra $\mathfrak{h}$, and the parabolic subgroup with Lie algebra $\mathfrak{p}$, respectively. The homogeneous space $X = G/P$ becomes compact. Further, the space X can be embedded into a certain projective space by using the representation theory of G. Hence we call the space X an algebraic homogeneous space. Let G_1, N and N^* be the Lie subgroups of G corresponding to $\mathfrak{g}_1$, $\mathfrak{n}$ and $\mathfrak{n}^*$, respectively. Then the group P is the semidirect product of G_1 and N, and, further, $P \cap N^* = \{I\}$. See [7].

Let $N(T)$ be the normalizer of T in G. We call the group $W = N(T)/T$ the Weyl group of G with respect to T. We put $W_1 = N(T) \cap P/T \subset W$

and $W^1 = W/W_1$. The group $N(T)$ acts on T, $\mathfrak{h}$ and Δ as follows:

$$\mathfrak{w} \cdot \exp H \cdot \mathfrak{w}^{-1} = \exp(\mathrm{Ad}(\mathfrak{w})H), \tag{3.8}$$

$$(\mathrm{Ad}(\mathfrak{w})^*\alpha)(H) = \alpha\big(\mathrm{Ad}(\mathfrak{w})^{-1}H\big), \tag{3.9}$$

for $\mathfrak{w} \in N(T)$, $H \in \mathfrak{h}$, $\alpha \in \Delta$. But if $\mathfrak{w} \in T$, the actions of $\mathfrak{w}$ are all trivial. Hence we can regard as the group W acts on T, $\mathfrak{h}$ and Δ. For simplicity, we use the same letter $\mathfrak{w}$ for $\mathfrak{w}$, $\mathrm{Ar}(\mathfrak{w})$ and $\mathrm{Ad}(\mathfrak{w})^*$.

4. Main results. We first prove the following proposition.

PROPOSITION 1. *If we act the maximal torus T on $X = G/P$ then the set $W^1 = W/W_1 = N(T)/N(T) \cap P$ is naturally realized as the set of all T fixed points in X.*

Proof. An element $\bar{g} \in X$ is fixed by the action of T if and only if $g^{-1}Tg \subset P$ where g is a representative of $\bar{g}$ in G. Since the group $g^{-1}Tg$ is also a maximal torus of G contained in P, there exists $p \in P$ such that $g^{-1}Tg = pTg^{-1}$. This means $gp \in N(T)$. Hence $\bar{g}$ defines a coset $\widetilde{gp}$ in W^1. If two fixed points $\bar{g}$ and $\bar{g}'$ define the same coset in W^1 then $gp = g'p'p''$ for some $p, p' \in P$ and $p'' \in N(T) \cap P$. So $\bar{g} = \bar{g}'$ in X. If we take an element $\mathfrak{w} \in N(T)$ then the coset corresponding to $\bar{\mathfrak{w}}$ is $\tilde{\mathfrak{w}} \in W^1$. Hence the mapping is onto.

Let us consider the following diagram:

$$\begin{array}{ccccccc} & & G & & G/P & & G/P \\ & & \cup & & \cup & & \cup \\ \mathfrak{n}^* & \overset{\phi}{\to} & N^* & \overset{\psi}{\to} & \bar{N}^* & \overset{\mathfrak{w}}{\to} & \mathfrak{w}\bar{N}^* \\ \Cup & & \Cup & & \Cup & & \Cup \\ Z & \to & \exp Z & \to & \overline{\exp Z} & \to & \overline{\mathfrak{w} \exp Z}. \end{array} \tag{4.1}$$

We write an element Z of $\mathfrak{n}^*$ as $Z = \sum_{\alpha \in \Delta(\mathfrak{n}^*)} z_\alpha X_\alpha$ with respect to the basis $X_\alpha \in \mathfrak{g}_\alpha$, $\alpha \in \Delta(\mathfrak{n}^*)$, of $\mathfrak{n}^*$. Since the Lie algebra $\mathfrak{n}^*$ is nilpotent, we have $\log(\exp Z) = Z$ and hence the map ϕ is one-to-one and onto. Since $N^* \cap P = \{I\}$, the mapping ψ is also one-to-one. The left multiplication of $\mathfrak{w}$ is clearly one-to-one. Hence we can take the pair $(\mathfrak{w}\bar{N}^*, \phi^{-1} \circ \psi^{-1} \circ \mathfrak{w}^{-1})$ as a coordinate neighborhood near $\mathfrak{w} \in W^1$ and then the functions $\{z_\alpha(\mathfrak{w}\bar{n}^*)\}_{\alpha \in \Delta(\mathfrak{n}^*)}$ become the local coordinates.

THEOREM 1. *The quotient set* $W^1 = W/W_1$ *can be canonically embedded into* $X = G/P$ *as the set of all* T*-fixed points, and the pair* $(\mathfrak{w}\bar{N}^*, \phi^{-1} \circ \psi^{-1} \circ \mathfrak{w}^{-1})$ *is a coordinate neighborhood near* $\mathfrak{w} \in W^1$. *The sets* $\mathfrak{w}\bar{N}^*$, $\mathfrak{w} \in W^1$, *are all* T *invariant Zariski open sets. In fact if we multiply* $\exp H \in T$ *on* $\mathfrak{w}\bar{N}^*$, *the local coordinate* $\{z_\alpha(\mathfrak{w}\bar{n}^*)\}_{\alpha\in\Delta(\mathfrak{n}^*)}$ *changes to* $\{e^{(\mathfrak{w}\alpha)(H)} \cdot z_\alpha(\mathfrak{w}\bar{n}^*)\}_{\alpha\in\Delta(\mathfrak{n}^*)}$. *Further, the space* X *is covered with the family of the open sets* $\mathfrak{w}\bar{N}^*$, i. e., $X = \bigcup_{\mathfrak{w}\in W^1} \mathfrak{w}\bar{N}^*$.

Proof. The first sentence has been proved. Let $\mathfrak{w}_0$ be the element of W whose length is maximal among all. Then since $\mathfrak{w}_0^{-1}N\mathfrak{w}_0 = N^*$, $\mathfrak{w}_0\bar{N}^* = N\mathfrak{w}_0P/P$. Namely the set $\mathfrak{w}_0\bar{N}^*$ is the Bruhat cell of maximal dimension and is a Zariski open set. So $\mathfrak{w}\bar{N}^* = \mathfrak{w}\mathfrak{w}_0^{-1}\mathfrak{w}_0\bar{N}^*$, $\mathfrak{w} \in W^1$, are all Zariski open sets. Since, for $\exp Z \in N^*$,

$$\begin{aligned}\exp H \cdot \mathfrak{w}\exp Z \cdot P &= \mathfrak{w}\mathfrak{w}^{-1}\exp H\mathfrak{w} \cdot \exp Z \cdot \mathfrak{w}^{-1}\exp(-H)\mathfrak{w} \cdot P \\ &= \mathfrak{w} \cdot \exp(\mathfrak{w}^{-1}(H)) \cdot \exp Z \cdot \exp(-\mathfrak{w}^{-1}(H)) \cdot P \\ &= \mathfrak{w} \cdot \exp\big(\mathrm{Ad}(\exp(\mathfrak{w}^{-1}(H)))Z\big) \cdot P \\ &= \mathfrak{w} \cdot \exp\big(\mathrm{Exp}(\mathrm{ad}(\mathfrak{w}^{-1}(H)))Z\big) \cdot P\end{aligned}$$

and

$$\mathrm{Exp}\big(\mathrm{ad}(\mathfrak{w}^{-1}(H))\big) \cdot Z \in \mathfrak{n}^*,$$

then

$$(\phi^{-1} \circ \psi^{-1} \circ \mathfrak{w}^{-1})(\exp H \cdot \mathfrak{w}\,\overline{\exp Z}) = \mathrm{Exp}\big(\mathrm{ad}(\mathfrak{w}^{-1}(H))\big) \cdot Z.$$

If we write $Z = \sum_{\alpha\in\Delta(\mathfrak{n}^*)} z_\alpha X_\alpha$, we have

$$\begin{aligned}\mathrm{ad}(\mathfrak{w}^{-1}(H)) \cdot Z &= \Big[\mathfrak{w}^{-1}(H), \sum_{\alpha\in\Delta(\mathfrak{n}^*)} z_\alpha X_\alpha\Big] \\ &= \sum_{\alpha\in\Delta(\mathfrak{n}^*)} \alpha(\mathfrak{w}^{-1}(H))z_\alpha X_\alpha = \sum_{\alpha\in\Delta(\mathfrak{n}^*)} (\mathfrak{w}\alpha)(H)z_\alpha X_\alpha\end{aligned}$$

and, hence,

$$\mathrm{Exp}\big(\mathrm{ad}(\mathfrak{w}^{-1}(H))\big) \cdot Z = \sum_{\alpha\in\Delta(n^*)} e^{(\mathfrak{w}\alpha)(H)} z_\alpha X_\alpha.$$

To prove $X = \bigcup_{\mathfrak{w}\in W^1} \mathfrak{w}\bar{N}^*$, we need the following fact. See [**6**].

Fact. Let Y be a compact Kähler manifold which satisfies $H^1(Y, \mathbf{C}) = 0$. Then if a complex connected solvable Lie group S acts holomorphically on Y, it always has a fixed point inside any analytic subvariety that S leaves invariant.

The space X satisfies above assumptions and we can take T as S. Then since $\mathfrak{w}\bar{N}^*$ is a T invariant Zariski open set, the complement $X' = X - \bigcup_{\mathfrak{w} \in W^1} \mathfrak{w}\bar{N}^*$ becomes a T invariant subvariety. Hence if X' is not empty, it must have a T fixed point. But this is a contradiction. This completes the proof.

Since the Lie group G acts on $X = G/P$ from the left side, the space X has many global holomorphic vector fields. For an element $H \in \mathfrak{h}$, let us define a holomorphic vector field V_H on X by the rule

$$(4.2) \qquad (V_H f)(\bar{g}) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\{f(\exp(\varepsilon H)\bar{g}) - f(\bar{g})\},$$

where $\bar{g} \in X$ and f is a local function near $\bar{g}$. Then the above theorem implies that the vector field V_H is expressible on $\mathfrak{w}\bar{N}^*$ in the explicit form

$$(4.3) \qquad V_H = \sum_{\alpha \in \Delta(\mathfrak{n}^*)} (\mathfrak{w}\alpha)(H) z_\alpha \frac{\partial}{\partial z_\alpha}.$$

If H belongs to the Weyl chambers then $0 \neq (\mathfrak{w}\alpha)(H) \in \mathbf{R}$ for all $\mathfrak{w} \in W^1$, $\alpha \in \Delta(\mathfrak{n}^*)$. Hence the set of all vanishing points of V_H agrees with W^1 and V_H vanishes in the first order there.

Let us quote the following fact from C. Kosniowsky [5].

Fact. Let M be a compact complex manifold of dimension n and A a holomorphic vector field with simple isolated zeros $\{\zeta_1, \ldots, \zeta_k\}$. Let us consider the Lie derivative L_A: $T^*_\zeta(M) \to T^*_\zeta(M)$ at $\zeta \in \{\zeta_1, \ldots, \zeta_k\}$ and denote by $\{\theta_1(\zeta), \ldots, \theta_n(\zeta)\}$ its eigenvalues. Then we have

$$\chi_p = \sum_q (-1)^q h^{p,q} = (-1)^p \cdot {}^\#\{\zeta_i \mid \operatorname{Re}\theta_j(\zeta_i) > 0 \text{ for exactly } p \text{ indices } j,\ 1 \le j \le n\},$$

where $h^{p,q} = \dim H^q(X, \Omega^p)$.

Theorem 2 is well known.

THEOREM 2. *Let $X = G/P$. Then the numbers $h^{p,q}$ are determined as follows*:

(1) $h^{p,q} = 0$ *for* $p \neq q$,

(2) $h^{p,p} = \{\mathfrak{w} \in W^1 \mid (\mathfrak{w}\alpha)(H) > 0$ *for exactly p weights α*, $\alpha \in \Delta(\mathfrak{n}^*)\}$.

Proof. (1) has been shown in Lemma 1. By using (4.3) we can easily calculate the eigenvalues of the Lie derivative L_{V_H} at the zero point $\mathfrak{w} \in W^1$. In fact they are the values $\{2(\mathfrak{w}\alpha)(H)\}_{\alpha\in\Delta(\mathfrak{n}^*)}$. After noting $\chi_p = (-1)^p \cdot h^{p,p}$, we complete the proof.

THEOREM 3. *Let $X = G/P$. Let $\mathfrak{S}$ be a homogeneous vector bundle which is induced from a representation $\phi\colon P \to GL(V)$. Then*:

(1) *The vector bundle $\mathfrak{S}$ is V_H-equivariant.*

(2) *The representative $(-1)^d\sigma_d(\tilde{V}_{H,Z})$ of the dth Chern class, $0 \le d \le r = \operatorname{rank} \mathfrak{S}$, of $\mathfrak{S}$ in $H^0(Z, \mathcal{O}_Z)$ takes the value $\sigma_d(d\phi(\mathfrak{w}^{-1}(H)))$ at $\mathfrak{w} \in W^1$. Here we denote the differential of ϕ by $d\phi\colon \mathfrak{p} \to \mathfrak{gl}(V)$.*

REMARK. For the line bundle case, i.e., $r = 1$, see E. Akyilidiz [**1**].

Proof. The vector bundle $\mathfrak{S}$ is obtained by dividing $G \times V$ by the equivalence relation $(g, v) \sim (gp, \phi^{-1}(p)v)$ for $g \in G$, $p \in P$, $v \in V$. Therefore a local section v of $\mathfrak{S}$ can be interpreted as the V-valued function on some open set U of G which satisfies $v(g) = \phi(p)v(gp)$ for $g, gp \in U$, $p \in P$. Similarly a local function f on X can be considered as the function satisfying $f(g) = f(gp)$. For these $v(g)$ we define

$$(4.4)\qquad (\tilde{V}_H v)(g) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\{v(\exp(\varepsilon H)g) - v(g)\};$$

then

$$\begin{aligned} &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\{\phi(p)v(\exp(\varepsilon H)gp) - \phi(p)v(gp)\} \\ &= \phi(p)\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\{v(\exp(\varepsilon H)gp) - v(gp)\} \\ &= \phi(p)(\tilde{V}_H v)(gp). \end{aligned}$$

Hence $(\tilde{V}_H v)(g)$ is also a local section of $\mathfrak{S}$. On the other hand, let f be a local function; then

$$\begin{aligned} (4.5)\quad (\tilde{V}_H(fv))(g) &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\{f(\exp(\varepsilon H)g)v(\exp(\varepsilon H)g) - f(g)v(g)\} \\ &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\{(f(\exp(\varepsilon H)g) - f(g))v(\exp(\varepsilon H)g)\} \\ &\quad + \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\{f(g)(v(\exp(\varepsilon H)g) - v(g))\} \\ &= (V_H f)(g)v(g) + f(g)(\tilde{V}_H v)(g). \end{aligned}$$

This means $\tilde{V}_H$ is a lifting of V_H to $\mathfrak{S}$. Hence $\mathfrak{S}$ is V_H-equivariant. Let $v(g)$ be a local section of $\mathfrak{S}$ which takes a constant vector v along the set $\mathfrak{w}N^*$. Then

$$(4.6)\quad (\tilde{V}_H v)(\mathfrak{w}\exp Z) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\{v(\exp(\varepsilon H)\mathfrak{w}\exp Z) - v(\mathfrak{w}\exp Z)\}$$

$$= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\{\phi(\mathfrak{w}^{-1}\exp(-\varepsilon H)\mathfrak{w})$$

$$\cdot v(\mathfrak{w}\mathfrak{w}^{-1}\exp(\varepsilon H)\mathfrak{w}\ \exp Z\mathfrak{w}^{-1}\exp(-\varepsilon H)\mathfrak{w})$$

$$-v(\mathfrak{w}\exp Z)\}$$

$$= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\{\phi(\mathfrak{w}^{-1}\exp(-\varepsilon H)\mathfrak{w})v - v\}$$

$$= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\{\mathrm{Exp}(-\varepsilon d\phi(\mathfrak{w}^{-1}(H)))v - v\}$$

$$= -d\phi(\mathfrak{w}^{-1}(H))v(\mathfrak{w}\exp Z).$$

Therefore if we choose a basis of local sections of $\mathfrak{S}$ on $\mathfrak{w}\bar{N}^*$ from these sections, we can write $\tilde{V}_{H,Z} = -d\phi(\mathfrak{w}^{-1}(H))$ by using matrix notation. So we have

$$(4.7)\qquad \det(tI - \tilde{V}_{H,Z}) = \det(tI - (-d\phi)(\mathfrak{w}^{-1}(H)))$$

$$= \sum_{d=0}^{r}(-1)^d\sigma_d(-d\phi(\mathfrak{w}^{-1}(H)))t^{r-d}$$

$$= \sum_{d=0}^{r}\sigma_d(d\phi(\mathfrak{w}^{-1}(H)))t^{r-d}.$$

The proof of Theorem 3 is completed.

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