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ON F-SPACES AND F'-SPACES

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Two problems concerning F-spaces and F'-spaces are investigated. The first problem is to characterize those F-spaces whose product with every P-space is an F-space. A new necessary condition is obtained which is in fact a characterization of those F-spaces whose product with any P-space with only one non-isolated point is an F-space. As a corollary an example of a locally compact F-space and a P-space whose product is not an F-space is obtained. The second problem is to verify a conjecture of Comfort, Hindman and Negrepontis. It is shown that each weakly Lindelöf F'-space is an F-space. Also, each zero-dimensional weakly Lindelöf F'-space is strongly zero-dimensional.

0. Introduction. All spaces considered are completely regular and βX is the Čech-Stone compactification of X. A space X is: an *F-space* if disjoint cozero subsets of X are contained in disjoint zero sets; a *P-space* if each zero set is closed and open; an *F'-space* if disjoint cozero subsets have disjoint closures; and *basically disconnected* (BD) if each cozero set has clopen closure. In 1960, Curtis [**Cu**] showed that if a product of spaces is an F-space then it can be expressed as a product of a P-space with an F-space. The interesting question is then to characterize those F-spaces Y so that $X \times Y$ is an F-space for each P-space X. In [**CHN**], the analogous question for F'-spaces and BD spaces is solved. In [**N**], Negrepontis showed that Y be compact is sufficient and Hindman [**H**] weakened this considerably. A space Y is *weakly Lindelöf* if each open cover of Y has a countable subcollection whose union i sdense. Hindman's condition is a weakening of weakly Lindelöf.

The only necessary condition on an F-space Y to arise is from the characterization for F'-spaces. Clearly if a space is an F-space then it must also be an F'-space. In [**H**] it is shown that this condition is not sufficient. We obtain a new necessary condition on Y in order that $X \times Y$ is an F-space for each P-space X. This condition is actually a characterization of those F-spaces whose product with each P-space with a unique non-isolated point is an F-space. As a consequence of this we are able to construct an example of a locally compact F-space Y and a P-space X such that $X \times Y$ is not an F-space. In addition this provides an example of a locally compact F'-space which is not an F-space, answering a question related to the author by M. Henriksen.

In the second section it is shown that a weakly Lindelöf F'-space is an F-space and if it is zero dimensional then it is strongly zero dimensional. This verifies a conjecture in [**CHN**]. We provide an example which shows that no reasonable weakening of weakly Lindelöf can replace it in the first result.

We list some well known facts about F-spaces and F'-spaces which we shall use without mention. We refer the reader to the Gillman and Jerison text [**GJ**] for undefined notions and terminology.

0.1 THEOREM. (1) *If X is an F-space then βX is an F-space.*

(2) *Each C^*-embedded subspace of an F-space (F'-space) is an F-space (F'-space).*

(3) *An open subset of an F'-space is an F'-space.*

(4) *A space is an F'-space iff it is locally an F'-space.*

(5) *Each cozero subset of an F-space is C^*-embedded.*

(6) *X is an F-space iff disjoint cozero subsets of X are completely separated.* □

1. On products of F-spaces. In this section a new necessary condition is found for an F-space Y to have the property that $X \times Y$ is an F-space for each P-space X. It comes in the form of a characterization of those F-spaces Y so that $X \times Y$ is an F-space for each P-space with at most one non-isolated point. For convenience, let us let $P1$-spaces be the class of P-spaces with a unique non-isolated point. For a $P1$-space, X, let p_X be the unique non-isolated point and $\mathcal{N}_X$ be the neighborhood filter of p_X. A few preliminary results are needed.

1.1 DEFINITION. [**CHN**] a space Y is CLWL (countably locally weakly Lindelöf) if for each point $y \in Y$ and countable collection of open covers $\mathcal{U}_n$ of Y there is a neighborhood U of y so that each $\mathcal{U}_n$ has a countable subcollection whose union is dense in U. □

We shall only need the following consequence of a space being CLWL.

1.2 THEOREM. [**CHN**] *Let $f \in C^*(X \times Y)$ for a P-space X and a CLWL space Y, then for each $(x, y) \in X \times Y$ there are neighborhoods U of x and V of y so that $f|_{U \times \{y'\}}$ is constant for each $y' \in V$.* □

1.3 THEOREM. [**CHN**] *An F'-space Y is CLWL iff for each P-space X (or each $P1$-space X), $X \times Y$ is an F'-space.* □

Notice that if Y is an F-space with the property that $X \times Y$ is an F-space for each $P1$-space X then, by 1.3, Y must be CLWL. The following definition will be convenient.

1.4 DEFINITION. Let X be a P1-space and Y any space. For an indexed collection $\{C_i: i \in X\backslash\{p_X\}\}$ of subsets of Y the

X-$\lim_Y \{C_i: i \in X\backslash\{p_X\}\} = \bigcap \{\mathrm{cl}_Y \bigcup (C_i: i \in U): U \cup \{p_X\} \in \mathfrak{N}_X\}$.

Call the space Y, *X-paracompact*, if whenever X-$\lim_Y\{C_i: i \in X\backslash\{p_X\}\} = \varnothing$ for cozero subsets, C_i, of Y there are zero sets $Z_i \supset C_i$ with

$$X\text{-}\lim_Y \{Z_i: i \in X\backslash\{p_X\}\} = \varnothing. \qquad \square$$

Observe that a BD space is X-paracompact for any $P1$-space X. For a function f with domain $X \times Y$ and $x \in X$, let f_x be the function on Y defined by $f_x(y) = f(x, y)$ for $y \in Y$. We shall require the following lemma.

1.5 LEMMA. *Let X be a $P1$-space and let $\{S_x: x \in X\backslash\{p_X\}\}$ be subsets of Y so that X-$\lim_Y\{S_x: x \in X\backslash\{p_X\}\} = \varnothing$. If h is a bounded real-valued function on $X \times Y$ with $h_x \in C^*(Y)$ for $x \in X$ and for $x \in X\backslash\{p_X\}$, $\{y \in Y: h_x(y) \neq h_{p_X}(y)\} \subset S_x$, then $h \in C^*(X \times Y)$.*

Proof. It suffices to show continuity of h at a point (p_X, y), for $y \in Y$ arbitrary. Choose $U \in \mathfrak{N}_X$ so that $y \notin \mathrm{cl}_Y \bigcup \{S_x: x \in U\backslash\{p_X\}\}$ and let V be a neighborhood of y so that $V \cap S_x = \varnothing$ for $x \in U\backslash\{p_X\}$. It follows that h is continuous on $U \times V$ since for $x \in U$, $h|_{\{x\}\times V} = h_{x|V} = h_{p_X|V}$. $\square$

1.6 THEOREM. *If X is a $P1$-space and Y is a CLWL, X-paracompact F-space then $X \times Y$ is an F-space.*

Proof. Let A_0 and A_1 be disjoint cozero subsets of $X \times Y$. We shall show that A_0 and A_1 are completely separated in $X \times Y$. Since $\{p_X\} \times Y$ is an F-space we can choose disjoint zero sets of Y, W_0 and W_1, so that $A_i \cap \{p_X\} \times Y \subseteq \{p_X\} \times W_i$ for $i = 0, 1$. We may also choose $g_0, g_1 \in C^*(X \times Y)$ so that the cozero set of g_i is $A_i\backslash(X \times W_i)$ for $i = 0, 1$. For $x \in X\backslash\{p_X\}$ choose, C_x, a cozero subset of Y so that

$$\{x\} \times C_x = [A_0\backslash(X \times W_0) \cup A_1\backslash(X \times W_1)] \cap (\{x\} \times Y).$$

We show that X-$\lim_Y\{C_x: x \in X\backslash\{p_X\}\} = \varnothing$. Indeed, let $y \in Y$ be arbitrary. Since Y is CLWL there are neighborhoods U of p_X and V of y

so that g_0 and g_1 are constant on $U \times \{y'\}$ for each $y' \in V$ by 1.2. Now $g_0((p_X, y')) = g_1((p_X, y')) = 0$ for all $y' \in Y$, so $(U \times V) \cap \bigcup\{\{x\} \times C_x : x \in X\setminus\{p_X\}\} = \varnothing$. Hence $V \cap \bigcup\{C_x: x \in U\setminus\{p_X\}\} = \varnothing$ and y is not an X-$\lim_Y\{C_x: x \in X\setminus\{p_X\}\}$. By the X-paracompactness of Y we choose $Z_x \supset C_x$ so that X-$\lim_Y\{Z_x: x \in X\setminus\{p_X\}\} = \varnothing$.

We shall define $h \in C^*(X \times Y)$ to witness that A_0 and A_1 are completely separated by defining $h_x \in C^*(Y)$ for $x \in X$. Let $h_{p_X} \in C^*(Y)$ be such that $h_{p_X}(W_0) = 0$ and $h_{p_X}(W_1) = 1$. For $x \in X$ and $i = 0, 1$ let $A_i(x) \supset Y$ be such that $A_i \cap \{x\} \times Y = \{x\} \times A_i(x)$. Recall that, for $x \in X$ and $i = 0, 1$, $A_i(x)\setminus W_i \subset Z_x$. The cozero set $Y\setminus Z_x \cup A_0(x) \cup A_1(x)$ is C^*-embedded in Y so we may choose $h_x \in C^*(Y)$ so that $h_{x|Y\setminus Z_x} = h_{p_X|Y\setminus Z_x}$, $h_x(A_0(x)) = 0$, and $h_x(A_1(x)) = 1$. h_x is well defined since, if $y \in A_i(x)\setminus Z_x$ then $y \in W_i$, so $h_{p_X}(y) = i$. By Lemma 1.5, $h \in C^*(X \times Y)$ and h is as required. □

If X is a $P1$-space then let $\hat{X}$ be the quotient space of $X \times \{0, 1\}$ obtained by identifying $(p_X, 0)$ and $(p_X, 1)$ to the single point $p_{\hat{X}}$. Note that $\hat{X}$ is a $P1$-space. In general, $\hat{X}$ is not homeomorphic to X. The only examples that I know where $\hat{X}$ is not homeomorphic to X are constructed from countably complete ultrafilters on discrete spaces of measurable cardinaltiy. We introduce the above idea because we cannot prove the converse of 1.6 but we can come very close.

1.7 THEOREM. *If X is a $P1$-space and Y is not X-paracomapct then $\hat{X} \times Y$ is not an F-space.*

Proof. Since Y is not X-paracompact, choose C_x, $x \in X\setminus\{p_X\}$, cozero subsets of Y so that X-$\lim_Y\{C_x: x \in X\setminus\{p_X\}\} = \varnothing$ but if $C_x \subset Z_x$, $x \in X$, Z_x a zero set then X-$\lim_Y\{Z_x: x \in X\setminus\{p_X\}\} \neq \varnothing$. Define $A_0 = \bigcup\{(x, 0) \times C_x: x \in X\setminus\{p_X\}\}$ and $A_1 = \bigcup\{(x, 1) \times C_x: x \in X\setminus\{p_X\}\}$. We show that A_0 and A_1 are disjoint cozero subsets of $\hat{X} \times Y$. For each $x \in X\setminus\{p_X\}$, choose $g_x \in C^*(Y)$ so that $1 \geq g_x(y) \geq 0$ for $y \in Y$ and C_x is the cozero set of g_x. Define $h \in C^*(\hat{X} \times Y)$ so that for $x \in X\setminus\{p_X\}$, $h_{(x,0)} = g_x$, $h_{(x,1)} = -g_x$ and $h_{p_X}(y) = 0$ for $y \in Y$. Now, since C_x is the cozero set of both $h_{(x,0)}$ and $h_{(x,1)}$ for $x \in X\setminus\{p_X\}$, one easily sees that h satisfies the conditions of 1.5 and so $h \in C^*(\hat{X} \times Y)$. Observe that A_0 is the preimage of the positive reals under h and A_1 is the preimage of the negative reals. Hence A_0 and A_1 are disjoint cozero subsets of $\hat{X} \times Y$.

Suppose that B_0 and B_1 are disjoint zero subsets of $\hat{X} \times Y$ containing A_0 and A_1 respectively. For each $x \in X\setminus\{p_X\}$ choose a zero set Z_x of Y

so that, for $i = 0, 1$, $\{(x, i)\} \times C_x \subset \{(x, i)\} \times Z_x \subset \{(x, i)\} \times Y \cap B_i$. Now, by our assumption on $\{C_x\colon x \in X\backslash\{p_X\}\}$, there is a $y \in X\text{-lim}_Y\{Z_x\colon x \in X\backslash\{p_X\}\}$. However $(p_{\hat{X}}, y) \in \bar{B}_0 \cap \bar{B}_1$, for if U is a neighborhood of $p_{\hat{X}}$ and V is a neighborhood of y then $V \cap \bigcup\{Z_x\colon (x,0) \in U\} \neq \varnothing$ and $V \cap \bigcup\{Z_x\colon (x,1) \in U\} \neq \varnothing$. Hence $U \times V \cap B_0 \neq \varnothing$ and $U \times V \cap B_1 \neq \varnothing$. □

1.8 Corollary. *For a space Y, $X \times Y$ is an F-space for each $P1$-space X iff Y is a CLWL F-space which is X-paracompact for each $P1$-space X.* □

The condition in 1.7 has proven very useful for constructing counter examples to conjectures concerining F-spaces. A very similar construction (which was originally identical) is used in **[D]** to show that Fine and Gillman's well known result, that open subspaces of F-spaces of weight c are F-spaces, is equivalent to CH. Another example is an answer to a question related by M. Henriksen; there is a locally compact F'-space which is not an F-space. With the same example, more can be shown. In **[CHN]**, it is asked if the condition that an F-space, Y, be CLWL is sufficient to ensure that $X \times Y$ is an F-space for each P-space X. Hindman **[H]**, showed that this is not the case but our example shows that not even local compactness is sufficient.

1.9 Example. There is a locally compact F-space Y and a P-space X such that $X \times Y$ is not an F-space. The simple idea behind the construction of this example is to construct an F-space Y which is not X-paracompact for some $P1$-space X. To do so we merely 'remove' the X-limits of a collection of cozero sets while leaving some X-limits of zero set containing them. The simplest example of a $P1$-space is $X = \omega_1 \cup \{p\}$ where neighborhoods of p are cocountable and points of ω_1 are isolated. To construct an 'appropriate' F-space Y we start with $S = \omega_1 \times \omega^*$, where again ω_1 has the discrete topology. Let $C \subset \omega^*$ be any cozero set whose closure is not a zero set. For $\alpha \in \omega_1$, let $C_\alpha = \{\alpha\} \times C$. Let K be the X-limits of $\{C_\alpha\colon \alpha \in \omega_1\}$ in βS; i.e. $K = \bigcap_{\gamma \in \omega_1} \text{cl}_{\beta S} \bigcup_{\gamma < \alpha} C_\alpha$. Let us show that $Y = \beta S \backslash K$ is not X-paracompact. Indeed, if Z_α, $\alpha \in \omega_1$, is a zero set of Y containing C_α then we may choose $y_\alpha \in \{\alpha\} \times \omega^* \cap Z_\alpha \backslash \overline{C_\alpha}$ for $\alpha \in \omega_1$. In S, the sets $\{y_\alpha\colon \alpha \in \omega_1\}$ and $\bigcup_{\alpha \in \omega_1} C_\alpha$ are completely separated. It follows that there is an X-limit of $\{Z_\alpha\colon \alpha \in \omega_1\}$ (even of $\{\{y_\alpha\}\colon \alpha \in \omega_1\}$) in βS which is not in K. Since X is homeomorphic to $\hat{X}$ and Y is not X-paracompact, $X \times Y$ is not an F-space by 1.7. However S is an F-space and therefore Y is a locally compact F-space.

1.10 Example. There is a locally compact F'-space U which is not an F-space and which is an open subset of a compact F-space. Let X, S, K and Y be as in 1.9. As mentioned in the Introduction the product of a P-space with a compact F-space is an F-space [**N**]. Therefore $X \times \beta S$ and $T = \beta(X \times \beta S)$ are F-spaces. Let $U = T \setminus (\{p\} \times K)$; U is obviously locally compact as it is an open subset of T and so is an F'-space. However, we shall show that $X \times Y$ is C^*-embedded in U from which it follows that U is not an F-space. It suffices to show that $X \times Y$ is C^*-embedded in $X \times Y \cup \{u\}$ for an arbitrary $u \in U$ (6H of [**GJ**]). Let C be a clopen neighborhood of $u \in U \setminus (X \times Y)$ such that $C \cap (\{p\} \times \beta S) = \varnothing$ (since $u \notin \{p\} \times \beta S$). Therefore there is an $\alpha \in \omega_1$ such that $C \subset \mathrm{cl}_T(\alpha \times \beta S)$. Now, Y is C^*-embedded in βS so $\alpha \times Y$ is C^*-embedded in $\alpha \times \beta S$ and therefore in T. It follows that $X \times Y$ is C^*-embedded in $X \times Y \cup C$. □

2. Weakly Lindelöf F'-spaces. A Lindelöf space is normal so, evidently, a Lindelöf F'-space is an F-space. A space is ccc if each cellular family of open subsets is countable. It is easy to show that a ccc F'-space is also an F-space. A natural generalization of both the properties Lindelöf and ccc is weakly Lindelöf. It is very natural to conjecture that a weakly Lindelöf F'-space is an F-space [**CHN**]. We verify this conjecture. Let us first consider the special case of a zero dimensional space.

2.1 Theorem. *Each zero dimensional weakly Lindelöf F'-space X is a strongly zero dimensional F-space.* (*A space X is strongly zero dimensional if βX is zero dimensional.*)

Proof. It suffices to show that disjoint cozero subsets C_0, C_1 of X are separated by clopen subsets of X. Since X is an F'-space, $\overline{C_0} \cap \overline{C_1} = \varnothing$. Cover X with clopen subsets which meet at most one of C_0 and C_1. Since X is weakly Lindelöf we may choose a countable subcollection, $\mathcal{C}$, whose union is dense in X. Let $\{A_n : n \in \omega\}$ be all those elements of $\mathcal{C}$ which do not intersect C_1 and let $\{B_n : n \in \omega\} = \mathcal{C} \setminus \{A_n : n \in \omega\}$. For each $n \in \omega$, let $A'_n = A_n \setminus \bigcup_{k<n} B_k$ and $B'_n = B_n \setminus \bigcup_{k \le n} A_k$. Clearly, for any $n, k \in \omega$, $A'_k \cap B'_n = \varnothing$ and $\bigcup_{n\in\omega} A'_n \cup B'_n$ is dense in X. Therefore $U = \mathrm{cl}_X \bigcup_{n\in\omega} A'_n$ is disjoint from $V = \mathrm{cl}_X \bigcup_{n\in\omega} B'_n$ because X is an F'-space and both are clopen. Also $U \cap C_1 = \varnothing$ and $C_0 \cap V = \varnothing$, again because X is an F'-space. Therefore C_0 and C_1 are separated by clopen sets. □

2.2 Theorem. *Each weakly Lindelöf F'-space is an F-space.*

Proof. Let C_0 and C_1 be disjoint cozero subsets of a weakly Lindelöf F'-space X. We inductively constuct cozero sets $A(i/2^n)$ and $B(i/2^n)$ for $n \in \omega$ and $0 \leq i \leq 2^n$. Let $A(0) = C_0$, $A(1) = X$, $B(0) = X$ and $B(1) = C_1$. Suppose that $n \in \omega$ and for $0 \leq i \leq 2^n$ we have contructed cozero subsets $A(i/2^n)$ and $B(i/2^n)$ so that (i) for $0 \leq i \leq 2^n$, $A(i/2^n) \cup B(i/2^n)$ is dense in X, and (ii) for $0 \leq i < 2^n$, $A(i/2^n) \cap B((i+1)/2^n) = \varnothing$. Let $m = n+1$ and let i be an even integer with $0 \leq i < 2^m$. We construct $A((i+1)/2^m)$ and $B((i+1)/2^m)$. By inductive assumption, $A(i/2^m) \cap B((i+2)/2^m) = \varnothing$ and therefore have disjoint closures since X is an F'-space. So we can cover X with cozero sets which intersect at most one of $A(i/2^m)$ and $B((i+2)/2^m)$. Since X is weakly Lindelöf we may choose a countable subcollection $\mathcal{U}$ whose union is dense in X. Let $A((i+1)/2^m)$ be the union of those elements of $\mathcal{U}$ which do not intersect $B((i+2)/2^m)$ and let $B((i+1)/2^m)$ be the union of those elements of $\mathcal{U}$ which do not intersect $B((i+2)/2^m)$ and hence do not intersect $A(i/2^m)$. Since $\bigcup\mathcal{U}$ is dense, so is $A((i+1)/2^m) \cup B((i+1)/2^m)$ and the induction hypotheses are satisfied. Let us note that (i) and (ii) ensure that for $n \in \omega$ and $0 \leq i < 2^n$, $\overline{A(i/2^n)} \subseteq \operatorname{int}\overline{A((i+1)/2^n)}$. Indeed, $\overline{A(i/2^n)} \cap \overline{B((i+1)/2^n)} = \varnothing$ so $\overline{A(i/2^n)} \subset X\setminus\overline{B((i+1)/2^n)}$ and $X\setminus\overline{B((i+1)/2^n)} \subset \overline{A((i+1)/2^n)}$ since $\overline{A((i+1)/2^n)} \cup \overline{B((i+1)/2^n)} = X$. Define the real-valued function f on X by $f(x) = \inf\{i/2^n\colon x \in \overline{A(i/2^n)}\}$. To check the continuity of f at $x \in X$, let $\varepsilon > 0$ be arbitrary. Choose $n \in \omega$ large enough so that there is an i, $0 \leq i \leq 2^n$ with $f(x) - \varepsilon \leq (i-1)/2^n < f(x) \leq i/2^n < (i+1)/2^n \leq f(x) + \varepsilon$. Since $f(x) \leq i/2^n$, $x \in \overline{A\big((2i+1)/2^{n+1}\big)}$. If $i = 2^n$, let $A((i+1)/2^n) = X$ and if $i = 0$ let $A((i-1)/2^n) = \varnothing$. Then

$$U = \operatorname{int}\overline{A((i+1)/2^n)}\setminus\overline{A((i-1)/2^n)}$$

is an open neighborhood of x since $\overline{A\big((2i+1)/2^{n+1}\big)} \subset \operatorname{int}\overline{A((i+1)/2^n)}$ and $x \notin \overline{A((i-1)/2^n)}$ because $f(x) > (i-1)/2^n$. Clearly, for $y \in U$, $f(x) - \varepsilon \leq f(y) \leq f(x) + \varepsilon$ hence f is continuous. Since $C_0 = A(0)$, $f(C_0) = 0$ and $f(C_1) = 1$ because $\overline{A(i/2^n)} \cap C_1 = \varnothing$ for each $i/2^n < 1$. Therefore C_0 and C_1 are completely separated showing that X is an F-space. □

It is not difficult to show that a weakly Lindelöf subspace of an F-space is an F-space and is C^*-embedded. The key to this fact is that a cozero subset of a weakly Lindelöf space is weakly Lindelöf and the following result which is an easy consequence of a result in **[CHN]**.

2.3 THEOREM. *Suppose X is an F-space, A, B are weakly Lindelöf subspaces of X with $\bar{A} \cap B = A \cap \bar{B} = \varnothing$; then A and B are completely separated in X.* □

A more general property than weakly Lindelöf has been found which can take its place in 2.3. A space X has the *P-cover* property if there are no closed P-sets of βX contained in $\beta X \setminus X$ [**DF**]. ($P \subset X$ is a P-set if each G_δ containing P is a neighborhood of P.)

2.4 THEOREM [**DF**] *Suppose A, B are sets with the P-cover property contained in an F-space X with $\bar{A} \cap B = A \cap \bar{B} = \varnothing$, then A and B are completely separated in X.* □

One can show that if an F-space X has a subspace A which is itself an F-space with the P-cover property then A is C^*-embedded in X [**DF**]. Now, 2.2 would follow from 2.3 if each F'-space can be embedded in an F-space, the following example shows that this is not the case. I believe that this is the first example given of an F'-space which cannot be embedded in an F-space. One might suspect that the result 2.4 suggests that weakly Lindelöf could be weakened 'towards' P-cover property in 2.1. For instance, call a space X WL^2 if there is a weakly Lindelöf cozero set $C \subset X$ such that any cozero subset of X disjoint from C is also weakly Lindelöf. The P-cover property is a much weaker property than WL^2. A space X is *almost compact* if given disjoint zero sets at least one is compact. Another natural generalization of weakly Lindelöf is almost weakly Lindelöf, AWL. A space X is AWL if given disjoint cozero subsets of X at least one is weakly Lindelöf. The following very interesting example seems convincing that no reasonable weakening of weakly Lindelöf can take place in 2.2.

2.5 EXAMPLE. There is an almost compact, AWL, WL^2 F'-space which is not an F-space (which obviously cannot be embedded in an F-space). Let X be the P-space whose underlying set is $\omega_2 + 1$ and possessing the G_δ-topology obtained from the order topology. Van Douwen [**vD**] has shown that each P-space can be embedded in an extremally disconnected space (a space in which open sets have open closures). In particular, X can be C^*-embedded in $E(2^{\omega_2})$ as a nowhere dense set, where $E(2^{\omega_2})$ is the compact ccc extremally disconnected space which can be mapped onto 2^{ω_2} by a perfect irreducible map (see [**D**] or [**DvM**] for a proof and [**W**] for a survey on $E(X)$). The space X^2 is a P-space and we assert that

$\beta X^2\backslash\{(\omega_2, \omega_2)\}$ is almost compact (this is proved the same way as $8L$ in **[GJ]**). Let E_0 and E_1 be two copies of $E(2^{\omega_2})$ containing respectively X_0 and X_1, copies of X. Let S be the quotient space of $E_0 \cup E_1 \cup X_0 \times X_1$ obtained by identifying $X_0 \subset E_0$ with $X_0 \times \{\omega_2\}$ and $X_1 \subset E_1$ with $\{\omega_2\} \times X_1$. Since X_0 and X_1 are C^*-embedded in E_0 and E_1 respectively, we may think of βS as $E_0 \cup E_1 \cup \beta(X_0 \times X_1)$ with the obvious identifications. Our example, advertised above, is the space $Y = \beta S\backslash\{(\omega_2, \omega_2)\}$ where (ω_2, ω_2) is a point in each of E_0, E_1 and $\beta(X_0 \times X_1)$. One should think of $Y \cap S$ as a plank, $X_0 \times X_1$, with the corner, (ω_2, ω_2), deleted and copies of $E(2^{\omega_2})\backslash\{\omega_2\}$, E_0 and E_1, glued to the 'top' edge and 'right' edge respectively. To see that Y is almost compact, i.e. Y is C^*-embedded in βS, simply observe that $E_0\backslash\{(\omega_2, \omega_2)\}$ is C^*-embedded in E_0, similarly for E_1, and $\beta(X_0 \times X_1)\backslash\{(\omega_2, \omega_2)\}$ is C^*-embedded in $\beta(X_0 \times X_1)$. Now, if C is a cozero subset of βS with $(\omega_2, \omega_2) \notin C$ then $C \cap Y$ is Lindelöf. Therefore Y is AWL. It is well known that $E(2^{\omega_2})$ is ccc so we may choose C_0 and C_1, Lindelöf dense cozero subsets of $E_0\backslash\{(\omega_2, \omega_2)\}$, and $E_1\backslash\{(\omega_2, \omega_2)\}$ respectively. Let $C = C_0 \cup C_1$, which is a Lindelöf cozero subset of Y. Suppose that U is a cozero subset of Y which is disjoint from C. Clearly, U is then a subset of $\beta(X_0 \times X_1)$ which is disjoint from $X_0 \times \{\omega_2\} \cup \{\omega_2\} \times X_1$. It follows that U is Lindelöf and hence that Y is WL^2. Now we show that Y is an F'-space. Each of the subspaces of βS, $E_0 \cup \beta(X_0 \times X_1)$ and $E_1 \cup \beta(X_0 \times X_1)$ are obtained by identifying a compact subset, βX_i, of an F-space, E_i, with a compact P-set $\beta X_0 \times \{\omega_2\}$ or $\{\omega_2\} \times \beta X_1$ of $\beta(X_0 \times X_1)$. From this fact it follows that both the above subspaces of βS are F-spaces (see **[DF]**). So Y is an F'-space since each point of $Y = \beta S\backslash\{(\omega_2, \omega_2)\}$ has a neighborhood contained in one of the above F-spaces. Finally, Y is not an F-space because βS is not an F'-space. Indeed, the point (ω_2, ω_2) is in the closures of the disjoint cozero subsets C_0 and C_1. □

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