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ON UNITS OF PURE QUARTIC NUMBER FIELDS

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Let $K = Q(\sqrt[4]{D^4 \pm d})$ be a pure quartic number field, where D and d are natural numbers such that d divides D^3 and d is fourth power free. Then $\varepsilon = \pm(\sqrt[4]{D^4 \pm d} + D)/(\sqrt[4]{D^4 \pm d} - D)$ is a unit of K whose relative norm to the quadratic subfield of K is 1. We consider the condition for ε to be a member of a system of fundamental units of K.

1. Introduction. There have been many investigations concerning units of pure extensions of the rational number field of degree $n > 2$ generated by $\sqrt[n]{D^n \pm d}$, where D and d are natural numbers satisfying certain conditions ([**2**], [**4**], [**7**], [**9**], etc.). In general, suppose d divides D^{n-1} or, if n is a power of a prime number p, d divides pD^{n-1}. Then the numbers

$$\varepsilon_k = \frac{\omega^k - D^k}{(\omega - D)^k}, \qquad \omega = \sqrt[n]{D^n \pm d},$$

where k runs over all the divisors of n except 1, are units and, moreover, independent in the real algebraic number field generated by ω [**1**], [**2**], [**4**]. (The proof of independence of the ε_k's given by Halter-Koch and Stender [**4**] is incomplete. But the proof can be corrected by a slight modification.) When $n = 3$, 4 or 6, the number of such divisors is equal to the rank of the unit group of the field $Q(\omega)$, where Q denotes the rational number field. In this paper we shall treat these units in the case $n = 4$.

The following result is established by Stender [**8**], [**9**]:

Let D and d be two natural numbers such that $d \mid 2D^3$, and put $A = D^4 \pm d$ and $\omega = \sqrt[4]{A}$. Suppose that d is fourth power free and A/d or $2A/d$ is square free, according as $d \mid D^3$ or $d \mid 2D^3$. Then

$$\varepsilon_2 = \pm\frac{\omega + D}{\omega - D} \quad \text{and} \quad \varepsilon_4 = \begin{cases} \dfrac{d}{(\omega - D)^4} & \text{if } d \text{ is not a square,} \\ \dfrac{\sqrt{d}}{(\omega - D)^2} & \text{if } d \text{ is a square and } d \neq 1, \\ \pm\dfrac{1}{\omega - D} & \text{if } d = 1 \end{cases}$$

form a system of fundamental units of $Q(\omega)$, except for the three cases $\omega^k = 8 = 2^4 - 8$, $\omega^4 = 12 = 2^4 - 4$ and $\omega^4 = 20 = 2^4 + 4$.

In this paper we shall remove the above assumption on A/d, and study the properties of ε_2.

2. Known facts. First, we state a few known facts on units of a pure quartic number field. Let A be a natural number which is fourth power free; then we can write $A = fg^2h^3$ with natural numbers f, g, h such that fgh is square free. We suppose $fh \neq 1$. Then the pure quartic number field $K = Q(\sqrt[4]{A})$ generated by $\sqrt[4]{A}$ contains a unique quadratic subfield, namely $Q(\sqrt{fh})$. Any integer α of K is of the form

$$\alpha = \frac{1}{k}\left(x_0 + x_1\sqrt[4]{fg^2h^3} + x_2\sqrt{fh} + x_3\sqrt[4]{f^3g^2h}\right)$$

with rational integers x_0, x_1, x_2, x_3 and $k = 1, 2$ or 4, and its conjugate relative to $Q(\sqrt{fh})$ is

$$\alpha' = \frac{1}{k}\left(x_0 - x_1\sqrt[4]{fg^2h^3} + x_2\sqrt{fh} - x_3\sqrt[4]{f^3g^2h}\right).$$

Now let $\varepsilon_0 > 1$ be the smallest unit of K such that $\varepsilon_0\varepsilon_0' = 1$, and $\varepsilon^* > 0$ the fundamental unit of $Q(\sqrt{fh})$.

LEMMA 1 ([**5**], [**6**]). *ε_0 and ε^* or ε_0 and $\sqrt{\varepsilon^*\varepsilon_0}$ form a system of fundamental units of K; the former case occurs if and only if neither ε^* nor $-\varepsilon^*$ is the norm of a unit of K to $Q(\sqrt{fh})$.*

In any case, ε_0 appears as a member of a system of fundamental units of K. The following result will aid in determining ε_0:

LEMMA 2 ([**6**]). *Let A_1 and A_2 be two positive rational integers such that $Q(\sqrt[4]{A_1A_2^3}) = Q(\sqrt[4]{A})$. Then the indeterminate equation*

$$A_1x^4 - A_2y^4 = \pm C \quad \text{with } C = 1, 2, 4$$

has at most one positive integer solution. If (a, b) is a positive integer solution of this equation, then $\pm(a\sqrt[4]{A_1} + b\sqrt[4]{A_2})/(a\sqrt[4]{A_1} - b\sqrt[4]{A_2})$ is a unit of $Q(\sqrt[4]{A})$ whose relative norm to $Q(\sqrt{A})$ is 1, and furthermore is equal to ε_0 or ε_0^2 with the only two exceptions $x^4 - 5y^4 = 1$ and $4x^4 - 3y^4 = 1$.

3. Theorems. From now on, we take A so that $K = Q(\sqrt[4]{A}) = Q(\sqrt[4]{D^4 \pm d})$, and suppose that $d \mid D^3$ and d is fourth power free. Then there is a natural number u satisfying

$$u^4 A = D^4 \pm d.$$

We write $d = d_1 d_2^2 d_3^3$ with natural numbers d_1, d_2, d_3 such that $d_1 d_2 d_3$ is square free. It is easy to see that $d_1 \mid f$, $d_2 \mid g$, $d_3 \mid h$.

Now we write $\sqrt[4]{A} = \omega$ and put

$$\varepsilon_2 = \pm \frac{u\omega + D}{u\omega - D},$$

which is a unit of K. In the special case where $u = 1$ and A/d is square free, i.e. $g = d_2$, $h = d_3$, as already mentioned in the introduction, Stender's result **[8]**, **[9]** states that ε_2 is contained in a system of fundamental units of K with the exception of three cases. Moreover **[3]**,

$$\varepsilon^* = \begin{cases} \dfrac{d}{(\omega^2 - D^2)^2}, & d_1 d_3 \neq 1, \\[2ex] \pm \dfrac{\sqrt{d}}{\omega^2 - D^2}, & d = d_2^2, \end{cases}$$

and $\pm \varepsilon^*$ are the norms of no unit of K to $Q(\sqrt{fh})$.

Since

$$Q\left(\sqrt[4]{D^4 \pm d}\right) = Q\left(\sqrt[4]{D'^4 \mp d'}\right),$$

where $D' = u(f/d_1)(g/d_2)(h/d_3)$, $d' = (f/d_1)^3 (g/d_2)^2 (h/d_3)$, $d' \mid D'^3$ and d' is fourth power free, we treat below exclusively the plus case, i.e. $u^4 A = D^4 + d$. We write simply $\varepsilon_2 = \varepsilon$:

$$\varepsilon = \frac{u\omega + D}{u\omega - D} = \frac{1}{d}\,(2D^4 + d + 2D^3 u\omega + 2D^2 u^2 \omega^2 + 2Du^3 \omega^3).$$

Obviously $\varepsilon\varepsilon' = 1$. We then consider whether there exists a unit η of K such that $\eta\eta' = 1$ and $\varepsilon = \eta^2$.

Let

$$\eta = \frac{1}{k}\left(x_0 + x_1 \sqrt[4]{fg^2h^3} + x_2 \sqrt{fh} + x_3 \sqrt[4]{f^3 g^2 h}\right)$$

be a unit of K with $\eta\eta' = 1$. Then

$$(1) \qquad x_0^2 + x_2^2 fh - 2x_1 x_3 fgh = k^2,$$

$$x_1^2gh + x_3^2fg - 2x_0x_2 = 0, \tag{2}$$

and

$$\eta^2 = \frac{1}{k^2}\Big(x_0^2 + x_1^2fh + 2x_1x_3fgh + 2(x_0x_1 + x_2x_3f)\sqrt[4]{fg^2h^3}$$
$$+ \left(x_1^2gh + x_3^2fg + 2x_0x_2\right)\sqrt{fh} + 2(x_0x_3 + x_1x_2h)\sqrt[4]{f^3g^2h}\Big).$$

Hence (1) and (2) imply that $\varepsilon = \eta^2$ if and only if

$$\frac{D^4}{d} = \frac{2}{k^2}x_1x_3fgh = \frac{1}{k^2}\left(x_0^2 + x_2^2fh\right) - 1, \tag{3}$$

$$\frac{D^3}{d}u = \frac{1}{k^2}(x_0x_1 + x_2x_3f), \tag{4}$$

$$\frac{D^2}{d}u^2gh = \frac{2}{k^2}x_0x_2 = \frac{1}{k^2}\left(x_1^2gh + x_3^2fg\right), \tag{5}$$

$$\frac{D}{d}u^3gh^2 = \frac{1}{k^2}(x_0x_3 + x_1x_2h). \tag{6}$$

From (3)–(6) we have

$$2x_0x_2(x_0x_1 + x_2x_3f)h = 2x_1x_3fgh(x_0x_3 + x_1x_2h).$$

It easily follows from this, together with (2), that

$$(x_0x_1 - x_2x_3f)(x_1^2h - x_3^2f) = 0,$$

from which, as $fh \neq 1$ is not a square,

$$x_0x_1 = x_2x_3f. \tag{7}$$

REMARK 1. It is easily shwon that in the above situation the following facts hold:

$$k = 1 \quad \text{if } 4 \mid d,\ 2 \mid fh \text{ or } 2 \nmid fgh,$$
$$k = 2 \quad \text{if } 2 \nmid d \text{ and } 2 \mid g.$$

We prove here the following:

THEOREM 1. *Notations being as above, suppose that* $u^4A = D^4 + d$ *and* $A \neq 5^3, 2^23^3$. *Then* $\varepsilon = \varepsilon_0$ *or* ε_0^2, *and moreover* $\varepsilon = \varepsilon_0^2$ *if and only if* $A = d$ *or* $4d$ *and either* $2(u^2 + \sqrt{d/A})$ *or* $2(u^2 - \sqrt{d/A})$ *is a square.*

Proof. It follows from Lemma 2 that $\varepsilon = \varepsilon_0$ or ε_0^2. Suppose that $\varepsilon = \eta^2$ with $\eta\eta' = 1$ as above. Then from (3) we have

$$u^4 A = u^4 f g^2 h^3 = D^4 + d = \frac{2}{k^2} x_1 x_3 fghd + d.$$

This implies $A = d$ or $4d$ because $d_1 \mid f$, $d_2 \mid g$, $d_3 \mid h$, and if $2 \mid fh$, $k = 1$ by Remark 1. Furthermore, from (3)–(7) we obtain

$$x_1 = \frac{k^2 D^3 u}{2d}\frac{1}{x_0}, \quad x_2 = \frac{k^2 D^2 u^2 gh}{2d}\frac{1}{x_0}, \quad x_3 = \frac{D}{ufgh} x_0,$$

and

$$\frac{D}{d} u^3 gh^2 = \frac{1}{k^2}\left(\frac{D}{ufgh} x_0^2 + \frac{k^4 D^5 u^3 gh}{4d^2}\frac{1}{x_0^2}\right).$$

From the last equation we have

$$\begin{aligned} 0 &= x_0^4 - \frac{k^2 u^4 A}{d} x_0^2 + \frac{k^4 u^4 D^4 A}{4d^2} \\ &= \begin{cases} \left(x_0^2 - \dfrac{k^2 u^2 (u^2+1)}{2}\right)\left(x^2 - \dfrac{k^2 u^2 (u^2-1)}{2}\right), & A = d, \\ \left(x_0^2 - k^2 u^2 (2u^2+1)\right)\left(x_0^2 - k^2 u^2 (2u^2-1)\right), & A = 4d. \end{cases} \end{aligned}$$

Since x_0 is a rational integer, $(u^2 \pm 1)/2$ or $2u^2 \pm 1$ must be a square, according as $A = d$ or $A = 4d$. Conversely, if these conditions are satisfied, then

$$x_0 = kuv, \quad x_1 = \frac{kD^3}{2d}\frac{1}{v}, \quad x_2 = \frac{kD^2 ugh}{2d}\frac{1}{v}, \quad x_3 = \frac{kD}{fgh} v,$$

where

$$v = \begin{cases} \sqrt{\dfrac{u^2 \pm 1}{2}}, & A = d, \\ \sqrt{2u^2 \pm 1}, & A = 4d, \end{cases}$$

satisfy conditions (1)–(6). Thus the theorem follows.

REMARK 2. In the above theorem, $u \neq 1$ if $A = d$. Since the fundamental unit of the real quadratic number field $Q(\sqrt{2})$ is $1 + \sqrt{2}$, the natural numbers u such that $(u^2 \pm 1)/2$ is a square are given by $u + \sqrt{u^2 + 1} = (1 + \sqrt{2})^{2l+1}$ or $u + \sqrt{u^2 - 1} = (1 + \sqrt{2})^{2l}$ for some $l \geq 1$.

Moreover, the natural numbers u such that $2u^2 \pm 1$ is a square are given by $\sqrt{2u^2+1} + u\sqrt{2} = (1+\sqrt{2})^{2l}$ or $\sqrt{2u^2-1} + u\sqrt{2} = (1+\sqrt{2})^{2l-1}$ for some $l \geq 1$.

In the minus case we have the following:

THEOREM 2. *Suppose $u^4A = D^4 - d$ and $A \neq 5, 2^23$. Then $\varepsilon = \varepsilon_0$ or ε_0^2, and $\varepsilon = \varepsilon_0^2$ if and only if $d = 1$ or 4 and either $2(D^2 + \sqrt{d})$ or $2(D^2 - \sqrt{d})$ is a square.*

Proof. Immediate from Theorem 1 and the remark at the beginning of this section.

REMARK 3. In the above theorem, $D \neq 1$ if $d = 1$. The natural numbers D such that $D^2/2 \pm 1$ is a square are given by $\sqrt{2(D^2+2)} + D\sqrt{2} = 2(1+\sqrt{2})^{2l}$ or $\sqrt{2(D^2-2)} + D\sqrt{2} = 2(1+\sqrt{2})^{2l-1}$ for some $l \geq 1$.

COROLLARY ([**9**]). *If $A = D^4 \pm d$ and A/d is square free, there exists no unit η of K such that $\eta\eta' = 1$ and $\varepsilon = \eta^2$, with the single exception of $A = 12 = 2^4 - 4$.*

Proof. By Theorem 1 such a unit cannot exist in the plus case, and hence Theorem 2 shows that $d = 1$ or 4. Then, from the assumption, we have $A = 4f = D^4 - d = 12$, namely $D = 2$, $d = 4$, which gives the only exception stated above.

REMARK 4. Stender [**10**] has obtained some sufficient conditions for $\varepsilon = \varepsilon_0$, which can also be deduced from Theorems 1 and 2.

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