PACIFIC JOURNAL OF MATHEMATICS
Vol. 110, No. 1, 1984

HOMOTOPICALLY TRIVIAL TOPOSES

M. V. MIELKE

We give a number of equivalent conditions for a topos to be homotopically trivial and then relate these conditions to the logic of the topos. This is accomplished by constructing a family of intervals that can detect complemented, regular subobjects of the terminals. It follows that these conditions generally are weaker than the Stone condition but are equivalent to it if they hold locally. As a consequence we obtain an extension of Johnstone's list of conditions equivalent to DeMorgan's law. Thus, for example, the fact that there is no nontrivial homotopy theory in the category of sets is equivalent to the fact, among others, that maximal ideals in commutative rings are prime. Moreover, any topos has a 'best approximation' by a locally homotopically trivial topos.

1. Homotopy in a topos. The notion of (singular) homotopy in a topos is the notion of homotopy based, as in the topological case, on an interval, where by an *interval* I in a topos E is meant an internally linearly ordered object of E with disjoint minimum m: $1 \to I$ and maximum M: $1 \to I$ elements, i.e., $m \cap M = 0$. More precisely, for an interval I in E, an (ordered) pair of maps f, g: $A \to B$ in E is said to be *directly I-homotopic* (abbreviated DI-homotopic) if there is a map h: $A \times I \to B$ such that $f = h(\mathrm{id} \times m)$ and $g = h(\mathrm{id} \times M)$: $A \simeq A \times 1 \to A \times I \to B$, and to be *I-homotopic* if there is a finite sequence $\{j_k\}$, $k = 1,\dots,n+1$, of maps $A \to B$ with $j_1 = f$, $j_{n+1} = g$ and j_k DI-homotopic to j_{k+1} or vice versa, for $k = 1,\dots,n$. E is said to be *$(D)I$-homotopically trivial* if every pair of parallel maps in E are $(D)I$-homotopic. It is readily seen that E is both DI- and I-homotopically trivial for any interval I that is *trivial* in the sense that $I \simeq I_1 \amalg I_2$ and m, M factor through I_1, I_2 respectively. In fact we have:

1.1 PROPOSITION. *For an interval I in a topos E, the following conditions are equivalent:*

(1) *I is trivial.*

(2) *E is DI-homotopically trivial.*

(3) *E is I-homotopically trivial.*

We postpone the proof that (3) $\Rightarrow$ (1) until 2.2. Note that if I is trivial then DI-homotopy is both symmetric and transitive but the converse need not hold. For example the "topological topos" of [5] contains the standard unit internal $I = [0, 1]$ which is not trivial but DI-homotopy is symmetric and transitive. However, the converse does hold for intervals that are *irreducible* in the sense that $\neg m = M$ and $\neg M = m$. We have:

1.2 PROPOSITION. *For an irreducible interval I in a topos E, the following conditions are equivalent*:

(1) *I is trivial.*

(2) *DI-homotopy is symmetric.*

(3) *DI-homotopy is transitive.*

We give the proof in 3.2 and 4.1. It is clear, from 1.1, that E is homotopically trivial (i.e., all intervals in E are trivial) then D-homotopy (i.e., DI-homotopy, for all intervals I in E) is both symmetric and transitive, in fact:

1.3 PROPOSITION. *For a topos E, the following conditions are equivalent*:

(1) *E is homotopically trivial.*

(2) *D-homotopy is symmetric.*

(3) *D-homotopy is transitive.*

The proof depends on, and immediately follows, 4.3.

So far we have considered conditions equivalent to homotopy triviality. We now turn to a somewhat weaker condition. Recall that a Heyting algebra H is called a Stone lattice if the regular (i.e., the $\neg\neg$-closed) elements of H have complements, or, equivalently, if the equation $\neg x \vee \neg\neg x = t$ holds for all $x \in H$. (See [3] or problem 3, p. 162, [2].)

1.4 PROPOSITION. *For the following conditions on a topos E*: (1) *E is homotopically trivial*; (2) *the Heyting algebra of subobjects of* 1 *is a Stone lattice*; (3) *all regular subobjects in E have complements, the following implications hold*: (a) (1) *implies* (2), (b) (3) *implies* (1), *and* (c) *if E satisfies SG (subobjects of* 1 *generate E*), (2) *implies* (3).

The proof depends on, and follows, 5.1.

We can now state and prove the main result which is a strengthened form (without the SG condition) of 1.4 obtained by passing to local notions. A topos E is said to be *locally* homotopically trivial if the topos E/X is homotopically trivial for all $X \in E$.

1.5 THEOREM. *For a topos E, the following conditions are equivalent.*

(1) *E is locally homotopically trivial.*

(2) *Regular subobjects in E have complements.*

(3) *E is a Stone topos (i.e., the subobject classifier Ω is an internal Stone lattice).*

(4) *DeMorgan's law $(\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q)$ holds in E.*

Proof. Condition (1) implies, for $X \in E$, that E/X is homotopically trivial and thus, by 1.4(a), that the regular subobjects of 1 in E/X have complements. Since subobjects of 1 in E/X correspond bijectively to subobjects of X in E and since this correspondence respects regularity and complements, it readily follows that (1) implies (2). Conversely, if (2) holds then regular subobjects of 1 in E/X have complements, for all $X \in E$. Since regular subobjects of $(f\colon Y \to X) \in E/X$ correspond to regular subobjects of 1 in $((E/X)/f) \simeq E/Y$, it follows that regular subobjects of E/X have complements and consequently, by 1.4(b), (2) implies (1). The equivalence of (2), (3), and (4) is essentially Theorem 3 of Frink [**1**]. Also see [**3**].

There are many equivalent forms of 1.5(3) (see [**3**]) and of 1.5(4) (see [**4**]) and thus of 1.5(1). We list a few:

1.6 COROLLARY. *For a topos E, the following conditions are equivalent:*

(1) *E is locally homotopically trivial.*

(2) *In commutative, unitary rings in E, maximal ideals are prime.*

(3) *If E has a natural number object, the Dedeking reals are conditionally order-complete.*

(4) (*If* $E = \mathrm{Shv}(X)$) *X is extremally disconnected.*

(5) (*If* $E = (\mathrm{Set})^{C^{\mathrm{op}}}$) *for every pair f, g of maps of C with common codomain there are maps x, y such that $fx = gy$.*

There are also many conditions which imply 2.5(1).

1.7 COROLLARY. *Each of the following conditions on a topos E implies that it is locally homotopically trivial.*

(1) *E satisfies the axiom of choice.*

(2) *E is Boolean* ($\Omega \simeq [2]$).

(3) *E is linear (i.e., Ω is an interval in E).*

Proof. By 5.23 [**2**], (1) implies (2), clearly (2) implies (3), and (3) implies 1.5(3) by Theorem 3 [**3**].

Finally, in view of 1.4(c), we have:

1.8 COROLLARY. *If E satisfies SG then the conditions of* 1.3, 1.5, *and* 1.6 *are equivalent and all are implied by each condition of* 1.7.

1.9. REMARK. By 1.5 constructions and results involving DeMorgan toposes can equally well be interpreted in terms of locally homotopically trivial toposes. For example, the results of **[6]** may be viewed as showing that any topos has a 'best approximation' by a locally homotopically trivial topos, while those of [7] characterize the locally homotopically trivial toposes, among the Set-toposes satisfying SG, as the projective toposes.

We end this section with an example of a topos that is homotopically trivial but not locally homotopically trivial. The topos $E = (\mathrm{Set})^{A^{\mathrm{op}}}$ of A-sets, where A is the monoid on two generators a, b with $a^2 = \mathrm{id}$ is homotopically trivial. To see this, note that an interval in E consists of an interval (I, m, M) in Set together with two order and endpoint preserving maps a, b: $I \to I$ with $a^2 = \mathrm{id}$. However, if $a(x) \leq x$ then $x = a^2(x) \leq a(x)$, thus, by comparability and antisymmetry, $a = \mathrm{id}$. From this it readily follows that $I = I_1 \amalg I_2$, where $I_1 = \{x \mid b^n(x) = m$, for some integer $n \geq 0\}$ and $I_2 = I - I_1$, renders I trivial in E. We now lift the interval π: $2 \times [3] \to 2$ in Set/2, where $2 = \{0, 1\}$, $[3]$ is the ordinal $\{0 < 1 < 2\}$ and π is the projection on the first factor, to a nontrivial interval p: $I \to X$ in E/X, where X is 2 with A-action given by $a(i) = 1 - i$, $b(i) = 0$, $i \in 2$, I is $2 \times [3]$ with A-action given by $a(i, j) = (1 - i, j)$, $b(i, j) = (0,0)$ or $(0,2)$ as $j \leq i$ or $j > i$, $i \in 2$, $j \in [3]$, and $p = \pi$. It is easily seen that p is an interval in E/X for which there is no separation of $m = \{(0,0), (1,0)\}$ and $M = \{(0,2), (1,2)\}$ since $b(1,1) \in m$ while $ba(1,1) \in M$.

2. Separation. In this section we complete the proof of 1.1. We begin with some basic properties of the separation relation s on the set of subobjects of an object of E (for subobjects A, B of X, $A \mathrm{s} B$ iff A and B can be separated, i.e., $X \simeq X_1 \amalg X_2$ and $A \leq X_1$, $B \leq X_2$).

2.1 LEMMA. *Let* s *denote the separation relation on the set of subobjects of $X \in E$.*

(1) s *is symmetric.*

(2) $0 \mathrm{s} A$ *for all* $A \leq X$.

(3) *If* $A \leq A_1$ *and* $B \leq B_1$, *then* $A \mathrm{s} B$ *if* $A_1 \mathrm{s} B_1$.

(4) *A s $(B_1 \cup B_2)$ iff A s B_1 and A s B_2.*

(5) *If $A = \bigcup_{i=1}^{n} A_i$, $B = \bigcup_{j=1}^{m} B_j$ then A s B if, for each pair i, j, there is a pair A_i^j, B_j^i such that $A_i \leq A_i^j$, $B_j \leq B_j^i$, and A_i^j s B_j^i.*

Proof. (1), (2) and (3) are trivial and (3) implies the only if part of (4). For the converse suppose $A \leq C_0$, $B_1 \leq C_1$, $A \leq C_0'$, $B_2 \leq C_1'$ and $C_0 \amalg C_1 \simeq X \simeq C_0' \amalg C_1'$. Then

$$X = (C_0 \cup C_1) \cap (C_0' \cup C_1') = C_0'' \cup C_1'',$$

where $C_0'' = C_0 \cap C_0'$ and

$$\begin{aligned} C_1'' &= (C_0 \cap C_1') \cup (C_1 \cap C_0') \cup (C_1 \cap C_1') \\ &= C_1 \cup (C_0 \cap C_1') = (C_1 \cap C_0') \cup C_1'. \end{aligned}$$

Clearly $A \leq C_0''$, $B_1 \cup B_2 \leq C_1''$, and $C_0'' \cap C_1'' = 0$, i.e., A s $(B_1 \cup B_2)$. Multiple applications of (1), (3) and (4) give (5).

Before completing the proof of 1.1, note that if $i^p \colon 1 \to 1 \amalg 1 = [2]$, $p = 0, 1$, are the canonical injections then, for any interval I and map $h\colon I \to [2]$, we have, as subobjects of I,

$$(hm)^*(i^p) = m^*h^*(i^p) = m \cap h^*(i^p) \leq h^*(i^p)$$

and

$$(hM)^*(i^p) \leq h^*(i^p),$$

where, in general, f^* denotes the pullback along f. Since coproducts are universal in a topos, $h^*(i^p) \amalg h^*(i^{1-p}) \simeq I$ and consequently (i) $(hm)^*(i^p)$ s $(hM)^*(i^{1-p})$, and (ii) $(hm)^*(i^p)$ s $(hm)^*(i^{1-p})$, $p = 0, 1$.

2.2 *Proof that* (3) *implies* (1) *in* 1.1. If E is homotopically trivial then the maps $i^p\colon 1 \to [2]$, $p = 0, 1$ are I-homotopic and thus there is a sequence $\{j_k\}$, $k = 1, \ldots, n+1$, of maps $1 \to [2]$, with $j_1 = i^0$, $j_{n+1} = i^1$, and a sequence of homotopies h_k rendering j_k DI-homotopic to j_{k+1} or vice versa, $k = 1, \ldots, n$. Since

$$I = \bigcap_{k=1}^{n} \left(h_k^*(i^0) \cup h_k^*(i^1)\right) = \bigcup_{(p_1, \ldots, p_n)} \left(h_1^*(i^{p_1}) \cap \cdots \cap h_n^*(i^{p_n})\right),$$

$$m = m \cap I = \bigcup_{(p_1, \ldots, p_n)} m_1^{p_1} \cap \cdots \cap m_n^{p_n}$$

and

$$M = \bigcup_{(q_1, \ldots, q_n)} M_1^{q_1} \cap \cdots \cap M_n^{q_n},$$

where $p_i, q_i \in \{0, 1\}, i = 1,\ldots,n$, and

$$m_k^{p_k} = m \cap h_k^*(i^{p_k}) = (h_k m)^*(i^{p_k}) \quad \text{and} \quad M_k^{q_k} = (h_k M)^*(i^{q_k}).$$

By 2.1(5), $m \mathrel{s} M$ if for each pair $(p_1,\ldots,p_n; q_1,\ldots,q_n)$ of tuples there are subscripts i, j with $m_i^{p_i} \mathrel{s} M_j^{q_j}$ (we call a pair of tuples with this property a *separated* pair). We shall show that all pairs are separated by inductively (on k) analyzing the list of pairs that, possibly, are not separated. Now, by (i), $m_1^p \mathrel{s} M_1^{1-p}$, $p = 0, 1$, and thus every pair with $q_1 = 1 - p_1$, $p_1 = 0, 1$, is separated. Since $j_1 = i^0$, $h_1 m = i^0$ or $h_1 M = i^0$ and thus $m_1^1 = 0$ or $M_1^1 = 0$. By 2.1(2), then, $m_1^1 \mathrel{s} M_1^1$ and every pair with $p_1 = q_1 = 1$ is separated. Thus, after considering j_1 and h_1, we see that the only possible pairs not separated satisfy $p_1 = q_1 = 0$. Assume, now, that after considering $j_1,\ldots,j_k$ the only possible pairs not separated satisfy $p_1 = q_1 = 0$, $i = 1,\ldots,k$. Again, from (i), we have that every pair with $q_{k+1} = 1 - p_{k+1}$, $p_{k+1} = 0, 1$, is separated. Thus the only possible pairs not separated satisfy $p_i = q_i = 0$, $i = 1,\ldots,k$, $p_{k+1} = q_{k+1}$. However, since one of the following equations must hold — $h_k m = h_{k+1} m$, $h_k m = h_{k+1} M$, $h_{k+1} m = h_k M$, $h_{k+1} M = h_k M$ — it follows from (i) and (ii) that one of the following cases holds: $(m_k^0 = m_{k+1}^0) \mathrel{s} M_{k+1}^1$, $(m_k^0 = M_{k+1}^0) \mathrel{s} M_{k+1}^1$, $(m_{k+1}^1 = M_k^1) \mathrel{s} M_k^0$, $m_{k+1}^1 \mathrel{s} (M_{k+1}^0 = M_k^0)$ and since each of these cases separates pairs satisfying $p_k = q_k = 0$, $p_{k+1} = q_{k+1} = 1$, the only possible pairs not separated satisfy $p_i = q_i = 0$, $i = 1,\ldots,k+1$. By induction, then, the only possible pair not separated is $p_i = q_i = 0$, $i = 1,\ldots,n$. But by 2.1(2), $m_n^0 \mathrel{s} M_n^0$ since $j_{n+1} = i^1$ and, thus, either $h_n M = i^1$ or $h_n M = i^1$ and, consequently, either $m_n^0 = 0$ or $M_n^0 = 0$. Thus all pairs are separated and (3) implies (1).

3. Quotient objects. In this section we note some basic properties of certain quotient objects and prove part of 1.2.

For any object I in E, if $C \leq I$ then $R = (C \times C) \cup \Delta$ (Δ is the diagonal of I) is readily checked to be an equivalence relation on I, and, as such (equivalence relations are effective in a topos, p. 27 [**2**]) the corresponding coequalizer $R \rightrightarrows I \xrightarrow{q} I/C$ has the property that for any pair $f, g\colon A \to I$, $qf = qg$ iff $(f, g)\colon A \to I \times I$ factors through R iff there is an epi $\beta = \beta_1 + \beta_2\colon B_1 \amalg B_2 \to A$ such that $f\beta_1$ and $g\beta_1$ each factors through C and $f\beta_2 = g\beta_2$. Moreover, it is not difficult to show that the q-images $q(A)$, $q(B)$ of disjoint subobjects A, B of I are disjoint if $C \leq A$; in fact $B \simeq q(B)$, $q^*q(B) \simeq B$, and $q(C)$ is a subobject of 1.

3.1 LEMMA. *If C is a subobject of an object I in a topos E then the image $q(C)$ of C under the coequalizer $q\colon I \to I/C$ satisfies*

(a) $\neg q(C) = q(\neg C) \simeq \neg C$.

(b) $q^*(q(C)) \simeq C$.

(c) *The objects* $q(C)$ *and* $|C|$ *are isomorphic, where* $|C|$ *is the support of* C, *i.e., the image of* $C \to 1$.

(d) *If, in addition,* $A \cap C = 0$, *for* $A \leq I$, *then* $\neg q(A) = q(\neg A)$.

Proof. It is easily seen that $\neg q(B) \leq q(\neg B)$ for any $B \leq I$ and for any epi q with domain I. But, as noted above, $\neg C \simeq q(\neg C)$, $q(C) \cap q(\neg C) = 0 = q(A) \cap q(\neg A)$ since $C \cap \neg C = 0 = A \cap \neg A$ and, if $A \cap C = 0$, $C \leq \neg A$. Thus $q(\neg C) \leq \neg q(C)$, $q(\neg A) \leq \neg q(A)$ and (a) and (d) follow. (b) Since the commutative square $(C \overset{i}{\to} I \to I/C) = (C \to q(C) \to I/C)$ is readily seen to be a pushout with i mono, it is, by 1.28 [**2**], also a pullback. (c) follows from the above observation that $q(C)$ is a subobject of 1.

We can now give:

3.2 *Proof that* (1) $\Leftrightarrow$ (3) *in* 1.2. Let q: $I \amalg I \to J$ be the coequalizer determined, as above, by the subobject $m \amalg M$: $C = 1 \amalg 1 \to I \amalg I$ of $I \amalg I$. If i^p: $I \to I \amalg I$, $p = 0, 1$, are the canonical injections then the maps qi^p: $I \to J$ render $m_p = qi^p m$ DI-homotopic to $M_p = qi^p M$, $p = 0, 1$. Since $M_0 = m_1$, condition 1.2(3) gives a map h: $I \to J$ with $hm = m_0$ and $hM = M_1$. But then, in view of 3.1(b),

$$m \cap h^*(q(C)) = (hm)^*(q(C)) = (qi^0m)^*(q(C)) = m^*(i^0)^*q^*(q(C))$$
$$= m^*(i^0)^*(C) = m^*(M) = m \cap M = 0.$$

Hence $h^*(q(C)) \leq \neg m$ and similarly $h^*(q(C)) \leq \neg M$. Thus if I is irreducible then $h^*(q(C)) = 0$ and consequently h factors through $\neg q(C)$. Thus, in view of 3.1 (a), h induces a map

$$I \to \neg q(C) = \neg C = \neg M \amalg \neg m = m \amalg M \simeq 1 \amalg 1$$

that separates m and M since $hm = m_0$ and $hM = M_1$. Since (1) $\Rightarrow$ (3) is trivial, (1) $\Leftrightarrow$ (3).

4. Quotient intervals. In this section we make a detailed study of intervals, culminating in results which lead to proofs of 1.2 and 1.3. We begin with a precise description of an interval: an interval in a topos E consists of an object I of E together with a linear order $L \leq I \times I$ on I (L is reflexive, transitive, antisymmetric ($L \cap L^{-1} = \Delta$, where L^{-1} is the opposite relation to L and Δ is the diagonal of I) and comparable ($L \cup L^{-1} = I \times I$)) together with a minimum m: $1 \to I$ ($m \times \mathrm{id}$: $1 \times I \to I \times I$ factors through L) and a maximum M: $1 \to I$ ($\mathrm{id} \times M$ factors through L) element with $m \cap M = 0$. In terms of generalized elements (p. 157 [**2**]) a relation L on I is linear iff the relation $\leq$ induced by L on each set $E(A, I)$, $A \in E$ (for $f, g\colon A \to I$, $f \leq g$ iff $(f, g)\colon A \to I \times I$ factors through L) is reflexive, transitive, antisymmetric and for each pair $f, g\colon A \to I$, there is an epi $\alpha = \alpha_1 + \alpha_2\colon A_1 \amalg A_2 \to A$ such that $f\alpha_1 \leq g\alpha_1$ and $g\alpha_2 \leq f\alpha_2$.

We can now give:

4.1 *Proof that* (1) $\Leftrightarrow$ (2) *in* 1.2. The nontrivial part is (2) $\Rightarrow$ (1). If DI-homotopy is symmetric then, since the identity id: $I \to I$ renders m DI-homotopic to M, there is a map $h\colon I \to I$ with $hm = M$ and $hM = m$. By comparability there is an epi $\alpha = \alpha_1 + \alpha_2\colon A_1 \amalg A_2 \to I$ with $\alpha_1 \leq h\alpha_1$ and $h\alpha_2 \leq \alpha_2$. Hence the images $\alpha_i A_i \leq I$ satisfy $\alpha_1 A_1 \cup \alpha_2 A_2 = I$ and, by antisymmetry, $\alpha_1 A_1 \cap \alpha_2 A_2 \leq H$, where $j\colon H \to I$ is the equalizer of h and id. But then

$$H \cap M = j^*(M) = (hj)^*(M) = j^*h^*(M) = H \cap h^*(M).$$

Hence

$$\begin{aligned} M \cap H \leq M \cap h^*(M) &= M^*h^*(M) = (hM)^*(M) \\ &= m^*(M) = m \cap M = 0. \end{aligned}$$

Similarly, $m \cap H = 0$ and consequently $H \leq \neg M \cap \neg m$. Thus if I is irreducible then

$$\alpha_1 A_1 \cap \alpha_2 A_2 \leq H \leq m \cap M = 0$$

and, since $m \leq hM$ ($hM \leq M$), m (M) factors through $\alpha_1 A_1$ ($\alpha_2 A_2$) and m s M i.e., I is trivial.

If $C \leq I$, for I an interval in E, then the image L_C of L under $q \times q\colon I \times I \to (I/C) \times (I/C)$ is the relation on $J = I/C$ that is characterized, in terms of generalized elements $f, g\colon A \to J$, by $f \leq_C g$ iff there are an epi $\varepsilon\colon A_1 \to A$ and maps $f_1, g_1\colon A_1 \to I$ such that $qf_1 = f\varepsilon$, $qg_1 = g\varepsilon$ and $f_1 \leq g_1$. Further, it is not hard to see that L_C is reflexive, that $L_C \cup L_C^{-1} = J \times J$ (but is, in general, neither transitive nor antisymmetric) and that qm (qM) is the minimum (maximum) element of J (but they need not be disjoint). Thus $(I/C, L_C, qm, qM)$ is generally not an interval but under appropriate conditions on C it is. C is said to be *convex* in I if, for each triple $f, g, h\colon A \to I$, if $f \leq g \leq h$ and f, h each factors through C then so does g. Denoting the image of $X \to 1$ by $|X|$ we have:

4.2 Lemma. *If C is a convex subobject of an interval I in a topos E for which $|m \cap C| \cap |M \cap C| = 0$ then I/C is an interval in E.*

Proof. It suffices to prove that $\leq_C$ is transitive and antisymmetric and $qm \cap qM = 0$. To show that $f \leq_C g$ and $g \leq_C h$ imply $f \leq_C h$, it is sufficient to construct an epi ε': $A' \to A$ and maps f', h': $A' \to I$ such that $f' \leq h'$, $qf' = f\varepsilon'$ and $qh' = h\varepsilon'$. To this end note that if $f \leq_C g$ and $g \leq_C h$ then (using a common pullback if necessary) there are an epi ε: $A_1 \to A$ and maps f_1, g_1, g_1', h_1: $A_1 \to I$ such that $f_1 \leq g_1$, $g_1' \leq h_1$, $qf_1 = f\varepsilon$, $qg_1 = g\varepsilon = qg_1'$, and $qh_1 = h\varepsilon$. Further, by comparability, there is an epi $\alpha = \alpha_1 + \alpha_2$: $A_{11} \amalg A_{12} \to A_1$ with $f_1\alpha_1 \leq h_1\alpha_1$ and $h_1\alpha_2 \leq f_1\alpha_2$ and, since $qg_1\alpha_2 = qg_1'\alpha_2$ there is an epi $\beta = \beta_1 + \beta_2$: $B_1 \amalg B_2 \to A_{12}$ for which $g_1\alpha_2\beta_1$ and $g_1'\alpha_2\beta_1$ each factors through C and $g_1\alpha_2\beta_2 = g_1'\alpha_2\beta_2$. Hence it follows from $g_1'\alpha_2\beta_i \leq h_1\alpha_2\beta_i \leq f_1\alpha_2\beta_i \leq g_1\alpha_2\beta_i$ and the convexity of C, for $i = 1$, and the antisymmetry of $\leq$, for $i = 2$, that

$$qg_1'\alpha_2\beta_1 = qh_1\alpha_2\beta_1 = qf_1\alpha_2\beta_1 \quad \text{and} \quad h_1\alpha_2\beta_2 = f_1\alpha_2\beta_2$$

respectively. Clearly the maps $f' = f_1\alpha_1 + g_1'\alpha_2\beta_1 + f_1\alpha_2\beta_2$ ($h' = h_1\alpha_1 + g_1'\alpha_2\beta_1 + h_1\alpha_2\beta_2$): $A' = A_{11} \amalg B_1 \amalg B_2 \to I$ satisfy $f' \leq h'$ and $qf' = f\varepsilon'$, $qh' = h\varepsilon'$ for the epi $\varepsilon' = \varepsilon\alpha(\mathrm{id} \amalg \beta)$: $A' \to A_{11} \amalg A_{12} \to A_1 \to A$. Thus transitivity is proved. To show antisymmetry note that $f \leq_C g$ and $g \leq_C f$ imply there are an epi ε: $A_1 \to A$ and maps f_1, g_1, g_1', f_1': $A_1 \to I$ such that $f\varepsilon = qf_1 = qf_1'$, $q\varepsilon = qg_1 = gq_1'$ and $f_1 \leq g_1$, $g_1' \leq f_1'$. By comparability there is an epi $\alpha = \alpha_1 + \alpha_2$: $A_{11} \amalg A_{12} \to I$ so that $f_1\alpha_1 \leq g_1'\alpha_1$ and $g_1'\alpha_2 \leq f_1\alpha_2$. Since $qf_1\alpha_1 = qf_1'\alpha_1$ there is an epi $\beta = \beta_1 + \beta_2$: $B_1 \amalg B_2 \to A_{11}$ so that $f_1\alpha_1\beta_1$ and $f_1'\alpha_1\beta_1$ each factors through C and $f_1\alpha_1\beta_2 = f_1'\alpha_1\beta_2$. It then follows from $f_1\alpha_1\beta_i \leq g_1'\alpha_1\beta_i \leq f_1'\alpha_1\beta_i$ and the convexity of C, for $i = 1$, and the antisymmetry of $\leq$, for $i = 2$, that

$$qf_1\alpha_1\beta_1 = qg_1'\alpha_1\beta_1 \quad \text{and} \quad qf_1\alpha_1\beta_2 = qg_1'\alpha_1\beta_2.$$

Since β is epi we have that $qf_1\alpha_1 = qg_1'\alpha_1$, i.e., $f\varepsilon\alpha_1 = g\varepsilon\alpha_1$. A similar argument shows that $f\varepsilon\alpha_2 = g\varepsilon\alpha_2$ and thus, since both α and ε are epi, $f = g$ and antisymmetry is proved. Finally note that, since 1 is terminal, the inclusion j: $qm \cap qM \to 1$ is just the equalizer of qm and qM and consequently there is an epi $\alpha = \alpha_1 + \alpha_2$: $A_1 \amalg \alpha_2 \to qm \cap qM$ such that $mj\alpha_1$ and $Mj\alpha_1$ each factors through C and $mj\alpha_2 = Mj\alpha_2$. The first (resp. 2nd) condition implies that $\alpha_1 A_1$ (resp. $\alpha_2 A_2$) is a subobject of $|m \cap C| \cap |M \cap C| = 0$ (resp. of $m \cap M = 0$) and thus, since α is epi, $qm \cap qM = 0$. This proves the lemma.

The proof of 1.3 is based on:

4.3 LEMMA. *For each interval I in a topos E there is an irreducible interval J in E together with an endpoint preserving* epi $I \to J$.

Proof. If C is either the subobject $\neg m$ or $\neg M$ in any interval I, then clearly C is convex and $|m \cap C| \cap |M \cap C| = 0$. Thus there is an endpoint preserving epi q: $I \to I/C$ to the interval, by 4.2, I/C. Now the interval $I_1 = (I/(\neg m),\ m_1 = q(m),\ M_1 = q(M))$ satisfies $\neg m_1 = M_1$ since, by 3.1(d) with $A = m$, $\neg q(m) = q(\neg m)$, and, by 3.1(c) and the fact that $M \leq \neg m$, $q(\neg m) = q(M)$. Similarly, the interval $I_2 = (I_1/(\neg M_1),$ $m_2 = q(m_1)$, $M_2 = q(M_1))$ satisfies $\neg M_2 = m_2$ and $\neg m_2 = M_2$; the latter equality since, by 3.1(c) and $m_1 \leq \neg M_1$,

$$\neg m_2 = \neg q(m_1) = \neg q(\neg M_1),$$

and by 3.1(a) and $\neg\neg M_1 = \neg\neg\neg m_1 = \neg m_1 = M_1$,

$$\neg q(\neg M_1) = q(\neg\neg M_1) = M_2.$$

Thus I_2 is an irreducible interval and the composition $I \to I_1 \to I_2 = J$ is the desired epi.

4.4 *Proof of* 1.3. To see that (2), (3) each imply (1) note that each of these conditions implies, by 1.2, that all irreducible intervals in E are trivial. The result now follows from 4.3 since an interval I is trivial iff there is an endpoint (i.e., m, M) preserving epi $I \to [2]$, where $[2] = 1 \amalg 1$ is the (essentially unique) irreducible, trivial interval (m, M are the canonical injections, 1 is the terminal object of E).

5. Intervals from subobjects. In this section we construct a special class of intervals that can be used to detect regular subobjects of 1 with complements. The main result, and the basis of the proof of 1.4 is:

5.1 LEMMA. *For each subobject U of* 1 *in a topos E there is an interval $I(U)$ in E such that*:

(1) *$I(U)$ is irreducible iff U is regular, and*

(2) *$I(U)$ is trivial iff $\neg\neg U$ has a complement. Thus all regular subobjects of* 1 *have complements if all irreducible intervals in E are trivial.*

Proof. Clearly the functor $S \mapsto \amalg_S 1$ from the category of finite sets to the topos E maps linearly ordered sets to linearly ordered objects of E. In particular, the ordinal $\{0, 1, 2\}$ defines an interval $[3] = 1 \amalg 1 \amalg 1$ in E, in which $m = i^0$, $M = i^2$, where i^p: $1 \to [3]$, $p = 0, 1, 2$, are the canonical injections, and, in terms of generalized elements f, g: $A \to [3]$, $f \leq g$ iff, as subobjects of A, $g^*(m) \leq f^*(m)$ and $f^*(M) \leq g^*(M)$. For a subobject $U \to 1$, the obvious injection $C = U \amalg 1 \amalg \neg U \to [3]$ defines C as a convex subobject. To see this note that a map $f: A \to [3]$ factors through C iff $|f^*(m)| \leq U$ and $|f^*(M)| \leq \neg U$. Thus, if $f, g, h: A \to [3]$ with $f \leq g \leq h$ and f, g each factors through C then $g^*(m) \leq f^*(M)$, $|f^*(m)| \leq U$, $g^*(M) \leq h^*(M)$, and $|h^*(M)| \leq \neg U$; consequently $|g^*(m)| \leq U$, $|g^*(M)| \leq \neg U$ and g factors through C. Further, it is trivial to see that $|m \cap C| = U$ and $|M \cap C| = \neg U$. Hence, by 4.2, $I(U) = [3]/C$ is an interval in E with $m = q(i^0)$ and $M = q(i^2)$ for $q: [3] \to I(U)$ the defining epi. A direct calculation shows that, as subobjects of [3], $q^*(m) = 1 \amalg U \amalg 0$ and $q^*(M) = 0 \amalg \neg U \amalg 1$. Thus $\neg q^*(m) = q^*(M)$ and, since q is epi,

$$\neg m = qq^*(\neg m) = q(\neg q^*(m)) = qq^*(M) = M.$$

Further, $\neg\neg U = U$ iff $\neg q^*(M) = q^*(m)$ iff $\neg M = m$. This shows (1). For (2) note that if $\neg\neg U \amalg \neg U = 1$ then

$$\begin{aligned} C = ((C_1 &= U \amalg \neg\neg U \amalg 0) \amalg (C_2 = 0 \amalg \neg U \amalg \neg U)) \\ &\to ((I_1 = 1 \amalg \neg\neg U \amalg 0) \amalg (I_2 = 0 \amalg \neg U \amalg 1)) = [3] \end{aligned}$$

and $I(U) \simeq I_1/C_1 \amalg I_2/C_2$ renders $I(U)$ trivial.

5.2 *Proof of* 1.4. Part (a) is a direct consequence of 5.1. For (b) note that since the endpoints of an irreducible interval are obviously regular, they are mutually complementary and thus all irreducible intervals in E are trivial. The result now follows from 4.3 as in the proof of 1.3. For (c) it is sufficient to show that, for any regular subobject A of X in E, the obvious mono $A \amalg \neg A \to X$ is epi (and thus an isomorphism) which, under the SG assumption, can be accomplished by showing that each partial map $1 \to X$ factors through it. However, the pullback of A along any such partial map $U \to X$ defines a regular subobject U_1 of U, which satisfies, as a subobject of 1, $\neg\neg U_1 \amalg \neg U_1 = 1$ and consequently $(\neg\neg U_1 \cap U) \amalg (\neg U_1 \cap U) = U$. But since U_1 is regular in U, $U_1 = \neg\neg U_1 \cap U$, and since U_1 is the pullback of A, $(\neg U_1 \cap U)$ is the pullback of $\neg A$ and thus

$$U = U_1 \amalg (\neg U_1 \cap U) \to A \amalg \neg A \to X$$

gives the desired factorization.

We conclude with the observation that if $G \to 1$ is the generic subobject (i.e., classified by the generic element $1 \to \Omega^*$, p. 39 **[2]**) in E/Ω then the interval $I(G) = (I \to \Omega)$ in E/Ω is universal for the intervals $I(U)$ in the sense that $I(U) \simeq f^*(I)$, where $f: 1 \to \Omega$ is the classifying map of U. Moreover, the restriction of $I(G)$ to $\Omega_{\neg\neg}$ is the universal interval for the irreducible $I(U)$'s.

References

[1] O. Frink, *Pseudo-complements in semi-lattices*, Duke Math J., **29** (1962), 505–514.
[2] P. T. Johnstone, *Topos Theory*, L.M.S. Math Monograph No. 10 (1977) Academic Press.
[3] _____, *Conditions related to DeMorgan's Law*, *Applications of Sheaves*, Springer Lecture Notes in Math., **753** (1979), 479–491.
[4] _____, *Another condition equivalent to DeMorgan's Law*, Communications in Algebra, **7** (12) (1979), 1309–1312.
[5] _____, *On a topological topos*, Proc. London Math. Soc., (3) **38** (1979), 237–271.
[6] _____, *The Gleason cover of a topos*, *I*, J. Pure Appl. Algebra, **19** (1980), 171–192.
[7] _____, *The Gleason cover of a topos*, *II*, J. Pure Appl. Algebra, **22** (1981), 229–247.

Received December 9, 1981 and in revised form June 15, 1982.

UNIVERSITY OF MIAMI
CORAL GABLES, FL 33124