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ATOROIDAL, IRREDUCIBLE 3-MANIFOLDS AND 3-FOLD BRANCHED COVERINGS OF S^3

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Suppose M is a closed orientable 3-manifold. Then H. Hilden et al. proved that M is a 3-fold branched covering of S^3 branched over a fibered knot. In this paper we prove that, if M is irreducible and atoroidal, then M is either a 3-fold branched covering of S^3 branched over a simple, fibered knot, or a 2-fold branched covering of a closed orientable 3-manifold whose Heegaard genus is at most one.

Hilden [4], Hirsch [5] and Montesinos [11] proved independently that a closed, connected and orientable 3-manifold M is a 3-fold irregular branched covering of S^3 branched over a knot K. Further, it is known that K may be chosen to be a fibered knot. We do not know a reference for this refinement, which we need for our main theorem, so we give in §1 a sketch of the proof, shown to us by Hilden. Our main result is:

THEOREM. *Let M be a closed, connected and orientable 3-manifold. Suppose M is atoroidal and irreducible. Then at least one of the following holds.*

(i) *M is a 3-fold (cyclic or irregular) branched covering of S^3 branched over a simple, fibered knot.*

(ii) *There exist a closed, connected and orientable 3-manifold N whose Heegaard genus is at most one and a simple link L in N such that M is a 2-fold branched covering of N branched over L.*

Here M *atoroidal* means M contains no embedded incompressible torus. As is well known, classifying closed orientable 3-manifolds essentially reduces to the case of atoroidal irreducible 3-manifolds, by the Unique Prime Decomposition Theorem [9] and the Torus Decomposition Theorem [6], [7].

Recently Thurston announced that, if an atoroidal and irreducible 3-manifold M is a regular (in particular cyclic) branched covering of a closed, orientable 3-manifold, then M has a *geometric structure* (i.e. M admits a complete riemannian metric in which any two points have isometric neighborhoods). By this result and our Theorem, if M is a closed, orientable 3-manifold which is atoroidal and irreducible, then M

has a geometric structure or is a 3-fold irregular branched covering of S^3 branched over a simple, fibered knot.

By similar methods (see [**15**] for details) one can prove:

Suppose that Σ *is a homotopy* 3-*sphere, not necessarily irreducible. Then* Σ *is a* 3-*fold irregular branched covering of* S^3 *branched over a simple, fibered knot* K.

If the branch set K is a torus knot, then Σ is a graph manifold. By Montesinos [**10**, p. 249, Lemma 1], Σ is homeomorphic to S^3. Hence any homotopy 3-sphere is homeomorphic to S^3 or a 3-fold irregular branched covering of S^3 branched over a fibered, hyperbolic knot.

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1. Preliminaries. In this paper we work in the piecewise linear category and every 3-manifold is orientable.

Let L be a link in S^3 and ω: $\pi_1(S^3 - L) \to \Theta_3$ a transitive representation, where Θ_3 is the symmetric permutation group of 3-symbols. We say that ω is *simple* if it represents each meridian by a transposition in Θ_3. Then we denote by $M(L, \omega)$ the 3-fold irregular branched covering of S^3 (branched over L) which is determined by ω. We consider a regular projection of L. Let B be a 3-ball in S^3 as shown in Figure 1(a). In Figure 1(a), α, β and γ are three different transpositions in Θ_3 such that $\omega(x_\alpha) = \alpha$, $\omega(x_\beta) = \beta$, $\omega(x_\gamma) = \gamma$, where x_α (resp. x_β, x_γ) is the Wirtinger generator associated to an overpass x'_α (resp. x'_β, x'_γ).

We change the pair (L, ω) to (L', ω') as shown in Figure 1(b). By Montesinos [**11**], $M(L, \omega)$ is homeomorphic to $M(L', \omega')$. We say that (L', ω') is obtained by doing a *double-Montesinos move on* (L, ω) *in* B. We note that the number of components of L is equal to that of L'.

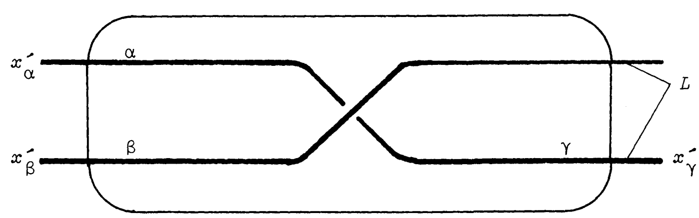

FIGURE 1(a)

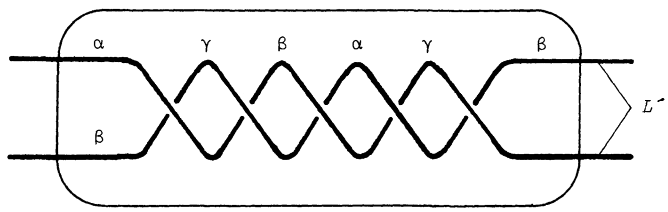

FIGURE 1(b)

Now we give a sketch of the proof of the following theorem of Hilden.

THEOREM (*Hilden*). *Every closed, connected* 3*-manifold* M *is a* 3*-fold irregular branched covering of* S^3 *branched over a fibered knot.*

Sketch of proof. By Hilden [**4**], Hirsch [**5**] or Montesinos [**11**], M is a 3-fold irregular branched covering of S^3 branched over a knot K. Let ω be the representation associated to the branched covering. By Alexander's Theorem, K is represented by a closed braid (see [**1**, p. 42, Theorem 2.1]). By doing some double-Montesinos moves on (K, ω), we obtain a new pair (K', ω') such that K' is represented by a closed positive braid (i.e. each crossing of the representation is positive). Figure 2 indicates the result. By Stallings [**16**, Theorem 2], K' is a fibered knot. □

Let F be a 2-manifold embedded in a 3-manifold M. Then a 2-disk D embedded in M is called a *compressing disk for F in M* if $F \cap D = \partial D$ and ∂D is not contractible in F. If F has a compressing disk in M, then we say that F is *compressible in M*, otherwise *incompressible in M*.

Let X be a submanifold of a manifold Y. Then we denote by $N(X, Y)$ a regular neighborhood of X in Y.

Let K be a knot in S^3. Then $E(K) = S^3 - \text{int } N(K, S^3)$ is called the *exterior of K in S^3*. We say that K is *simple* if $E(K)$ contains no incompressible torus which is not isotopic to $\partial E(K)$ in $E(K)$.

Let V be an unknotted solid torus in S^3 and K a knot in S^3 which is contained in V and such that ∂V is incompressible in $V - K$ and K is not isotopic in V to a core c of V. Let $f\colon V \to S^3$ be an embedding such that $f(c)$ is knotted in S^3 and $f(l)$ is homologous to zero in $S^3 - \text{int } f(V)$, where l is a meridian of the solid torus $S^3 - \text{int } V$. We set $T = f(\partial V)$. Then T is an incompressible torus in $E(f(K))$ which is not isotopic to $\partial E(f(K))$. We say that $f(c)$ is the *companion of $f(K)$ for T*, $f(K)$ is the *satellite of $f(c)$ for T* and K is the *preimage of $f(K)$ for T*. By Myers [**12**, Proposition 9.11], if $f(K)$ is a fibered knot, then $f(c)$ and K are also fibered knots and $g(f(c)), g(K) < g(f(K))$, where $g(K)$ denotes the genus of K.

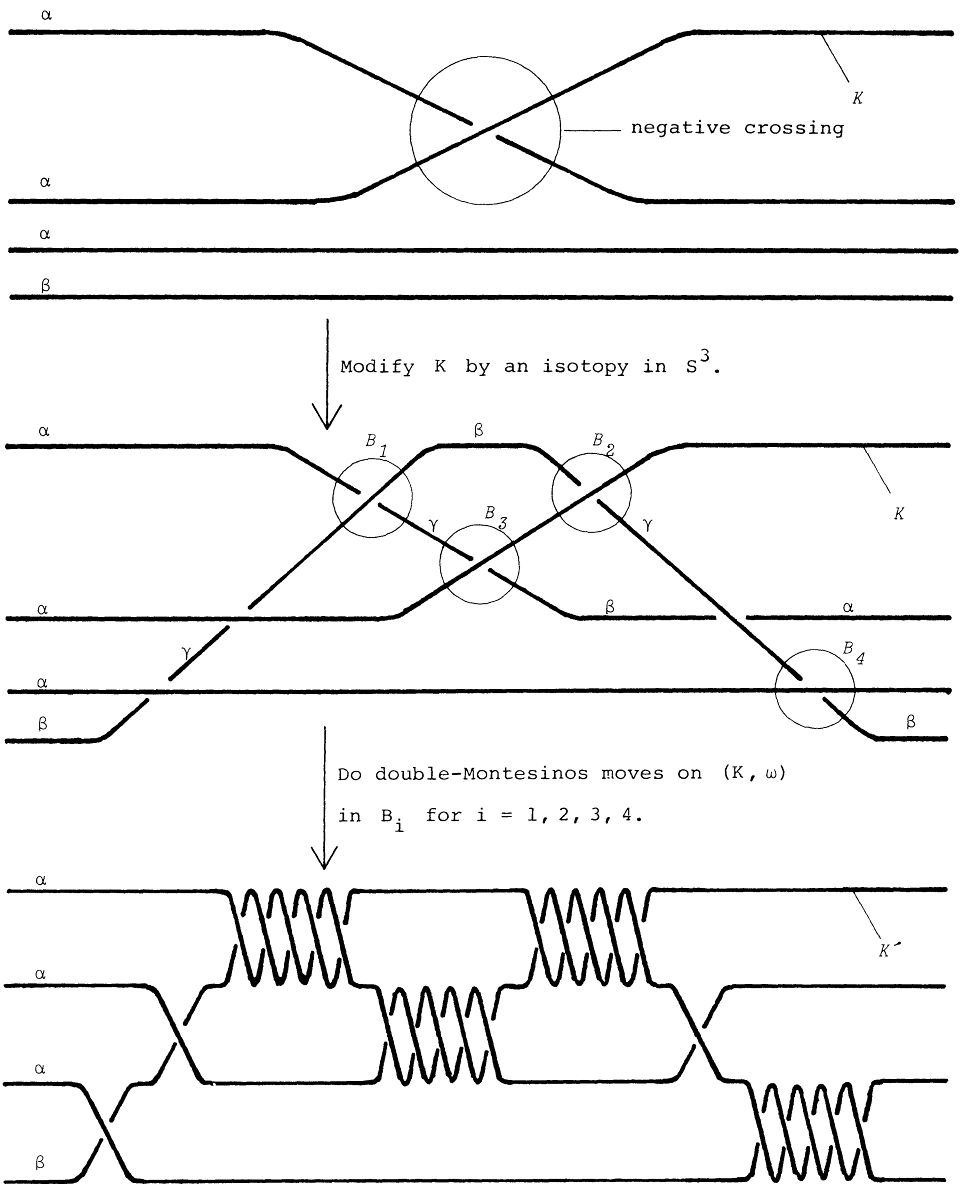

FIGURE 2

Let T be a torus in an atoroidal, irreducible 3-manifold M and D a compressing disk for T in M. Let f: $D \times I \to M$ be an embedding such that $f(D \times \{\frac{1}{2}\}) = D$ and $f(D \times I) \cap T = f(\partial D \times I)$, where $I = [0, 1]$. We say that $S = (T - \operatorname{int}(T \cap f(D \times I))) \cup f(D \times \{0\}) \cup f(D \times \{1\})$ is a 2-sphere obtained by doing *surgery on T along D*. Obviously $S \cap D = \varnothing$.

Since M is irreducible, S bounds a 3-ball B in M. If $B \cap D = \emptyset$, then T bounds a solid torus $B \cup f(D \times I)$ in M with a meridian disk D. If $B \supset D$, then T bounds a compact 3-manifold $N = \overline{(B - f(D \times I))}$ in M such that $(N, \partial D)$ is homeomorphic to $(E(K), m)$, where K is a knot in S^3 and $m \subset \partial E(K)$ is a meridian of a solid torus $N(K, S^3) = S^3 - \operatorname{int} E(K)$. Then we say that $(N, \partial D)$ is a *knot space-meridian pair*. Let l be a simple loop in ∂N which meets ∂D transversely at a single point (hence l is not contractible in ∂N) and is homologous to zero in N. Then we say that l is a *longitude* of $(N, \partial D)$.

Let A, B be two manifolds. Then we denote by $A \cong B$ that A is homeomorphic to B.

We prove the following three lemmas.

LEMMA 1. *Let M be a connected, closed 3-manifold which is irreducible and atoroidal. Let p: $M \to S^3$ be a 3-fold irregular branched covering branched over a knot K. If K is a composite knot, then M is a 2-fold branched covering of S^3 branched over a prime factor K_0 of K.*

REMARK. By Gordon and Litherland [3, Theorem 2], K_0 is simple. By Myers [12, Proposition 9.11], if K is fibered, then K_0 is also fibered.

Proof. Since K is composite, there exists a 2-sphere S embedded in S^3 which bounds two 3-balls B_1, B_2 in S^3 such that $B_1 \cap B_2 = S$ and $\alpha_i = B_i \cap K$ is a knotted arc in B_i for $i = 1, 2$. Since the representation associated to p is simple and S meets K transversely at two points, $p^{-1}(S)$ consists of two 2-spheres S_1, S_2 such that $p|S_1$: $S_1 \to S$ is a homeomorphism and $p|S_2$: $S_2 \to S$ is a 2-fold branched covering branched over $K \cap S$. Since M is irreducible, either $p^{-1}(B_1)$ or $p^{-1}(B_2)$ is disconnected. We may assume that $p^{-1}(B_1)$ consists of two components N_1 and N_2 such that $\partial N_i = S_i$ for $i = 1, 2$. Then $p|N_2$: $N_2 \to B_1$ is a 2-fold branched covering branched over α_1. If N_2 is a 3-ball, then α_1 is unknotted in B_1 by the Branched Covering Theorem [13], a contradiction. Thus $M - \operatorname{int} N_2$ is a 3-ball. We may extend $p|S_2$: $S_2 \to S$ to a 2-fold branched covering q: $\tilde{C} \to C$ branched over an unknotted arc α in C, where $\tilde{C}$, C are 3-balls. Then $p|N_2 \cup q$: $N_2 \cup_{S_2} \tilde{C} \to B_1 \cup_S C$ is a 2-fold branched covering branched over a knot $K_0 = \alpha_1 \cup \alpha$ in $B_1 \cup_S C \cong S^3$. Obviously we have $N_2 \cup_{S_2} \tilde{C} \cong M$. By the above remark, K_0 is simple. Hence, in particular, K_0 is a prime factor of K. This completes the proof. □

LEMMA 2. *Let T_1, T_2 be tori and p: $T_1 \to T_2$ a covering. Suppose that l is a simple loop in T_1 which is not contractible in T_1. Then l is isotopic to a simple loop l_1 in T_1 such that $p(l_1)$ is a simple loop in T_2 and $p \mid l_1$: $l_1 \to p(l_1)$ is a covering. (We say that l_1 is in good position with respect to p.)*

Proof. We suppose that every loop is oriented. Let α, β be generators of $\pi_1(T_2) \approx Z \times Z$. Then we suppose that a map $p \mid l$: $l \to T_2$ represents $n(p\alpha + q\beta)$ in $\pi_1(T_2)$, where $n, p, q \in Z$, $n \neq 0$ and $(p, q) = 1$. Let l_2 be a simple loop in T_2 which represents $p\alpha + q\beta$ in $\pi_1(T_2)$. Let π: $S^1 \to l_2$ be an n-fold cyclic covering and i: $l_2 \to T_2$ an inclusion. Since $p \mid l$ is homotopic to $i \circ \pi$, $i \circ \pi$ has a lift $\tilde{\pi}$ with respect to p. Then it is easy to show that $l_1 = \tilde{\pi}(S^1)$ satisfies the conclusions of Lemma 2. □

LEMMA 3. *Let M_0 be a compact, connected 3-manifold whose boundary consists of n tori $T_1, \ldots, T_n$ $(n \geq 1)$, and let M_k $(k = 1, \ldots, n)$ be a compact, connected 3-manifold such that ∂M_k is an incompressible torus in M_k. If $M = M_0 \cup_{T_1 = \partial M_1} M_1 \cdots \cup_{T_n = \partial M_n} M_n$ is atoroidal, then each T_k is compressible in M_0.*

Proof. If $n = 1$, the proof is trivial. We suppose $n > 1$. Then it suffices to prove that T_1 is compressible in M_0. We set $P = M_0 \cup_{T_1 = \partial M_1} M_1$ and $Q = M - \operatorname{int} M_1$. Then $M = P \cup_{T_2 = \partial M_2} M_2 \cdots \cup_{T_n = \partial M_n} M_n$. By induction on n, for $k > 1$, T_k is compressible in P.

We suppose that T_1 is incompressible in M_0. Since $T_1 = \partial M_1$ is incompressible in M_1, it also is in P. Since T_k $(k > 1)$ is compressible in P, $(j \circ i_k)_*$: $\pi_1(T_k) \to \pi_1(P)$ is not injective, where i_k: $T_k \subset M_0$ and j: $M_0 \subset P$. Since j_*: $\pi_1(M_0) \to \pi_1(P) \approx \pi_1(M_0) *_{\pi_1(T_1)} \pi_1(M_1)$ is injective, $(i_k)_*$ is not injective. Hence there exists a compressing disk D_k for T_k in M_0. By using an elementary innermost disk technique, we may assume $D_k \cap D_l = \varnothing$ for $2 \leq k < l \leq n$. Let S_k $(k = 2, \ldots, n)$ be a 2-sphere in M_0 obtained by doing surgery on T_k along D_k such that $S_k \cap S_l = \varnothing$ for $k \neq l$. Then S_k bounds a compact 3-manifold N_k in Q such that $N_k \supset M_k \cup D_k$. Since T_1 is compressible in Q,

$$j'_* \circ i'_*: \pi_1(T_1) \to \pi_1(Q - \operatorname{int}(N_2 \cup \cdots \cup N_k)) \to \pi_1(Q)$$
$$\approx \pi_1(Q - \operatorname{int}(N_2 \cup \cdots \cup N_k)) * \pi_1(N_2) * \cdots * \pi_1(N_k)$$

is not injective, where i': $T_1 \subset Q - \operatorname{int}(N_2 \cup \cdots \cup N_k)$ and j': $Q - \operatorname{int}(N_2 \cup \cdots \cup N_k) \subset Q$. Since j'_* is injective, i'_* is not injective. Hence T_1 is compressible in $Q - \operatorname{int}(N_2 \cup \cdots \cup N_k) \subset M_0$, a contradiction. Thus T_1 must be compressible in M_0. This completes the proof. □

2. Proof of Theorem. Let M be a closed, connected 3-manifold which is atoroidal and irreducible, and let $p\colon M \to S^3$ be a 3-fold irregular branched covering branched over a fibered knot K.

We suppose K is not simple, that is, int $E(K)$ contains an incompressible torus T which is not isotopic to $\partial E(K)$. Then $p^{-1}(T)$ consists of one, two, or three tori in M.

Let X be a compact orientable 2-manifold which is properly embedded in a compact 3-manifold Y. We denote by Y_X the compact 3-manifold obtained by splitting Y along X.

We use a weighted graph to study the configuration of $p^{-1}(T)$ in M.

To each component of $M_{p^{-1}(T)}$, we associate a vertex v with weight i and denote the component by $M(v)$. The weight i indicates that $p\,|\,M(v)$: $M(v) \to p(M(v))$ is an i-fold branched or unbranched covering. Let V be a solid torus in S^3 bounded by T. Obviously V contains K. We color a vertex v black if $p(M(v)) = V$, otherwise white.

To each component of $p^{-1}(T)$, we associate an edge e with weight i and direction, and denote the component by $T(e)$. The weight i indicates that $p\,|\,T(e)$: $T(e) \to T$ is an i-fold covering. We say that v is a vertex of e if $\partial M(v)$ contains $T(e)$. An edge e is directed, $v_1 \underset{e}{\to} v_2$, means $T(e)$ is compressible in the component of $M_{T(e)}$ which contains $M(v_2)$ (we note that M is atoroidal). An edge may have two directions. The two ends of an edge have opposite colors.

Thus we obtain the weighted graph Γ associated to $(M, p^{-1}(T))$.

The *valency* of a vertex v is the number of all edges of Γ with v as a common vertex.

LEMMA 4. *The graph Γ associated to $(M, p^{-1}(T))$ satisfies the following properties.*

(i) *Let v_0 be a white vertex of Γ with valency* 1 *and e_0 the unique edge with v_0 as a vertex. Then e_0 is directed only away from v_0.*

(ii) *Let v_1 be a black vertex of Γ with weight* 1 (*hence the valency of v_1 is* 1) *and e_1 the unique edge with v_1 as a vertex. Then e_1 is directed only toward v_1.*

(iii) *The total sum of the weights of all edges with v as a common vertex is equal to the weight of v.*

(iv) Γ *is a tree.*

(v) *The number of all black vertices of Γ is at most two. The number of white vertices is at most three.*

It follows that Γ is one of the five graphs Γ_i in Figure 3. (Lemma 4 does not determine the directing of the edge e in Γ_2 nor of e_1 in Γ_4.)

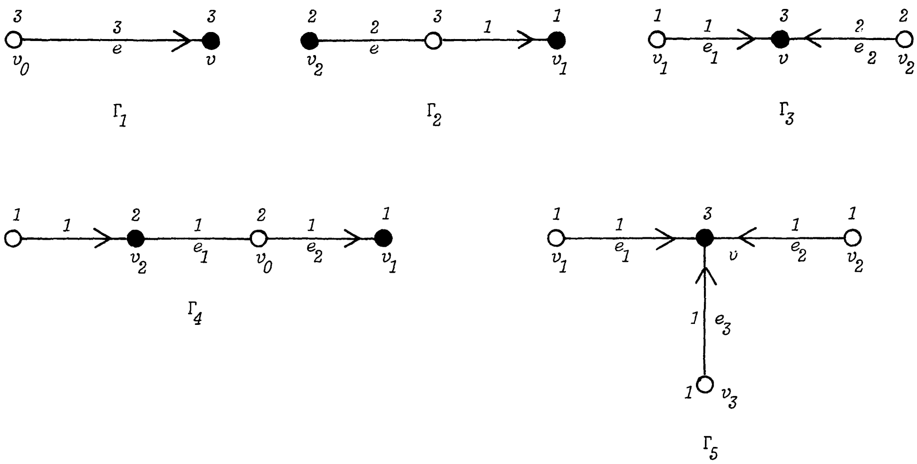

FIGURE 3

Proof of Lemma 4. (i) If $T(e)$ is compressible in $M(v_0)$, then T is compressible in $S^3 - \text{int } V$, a contradiction.

(ii) Since $p \mid M(v_1)$: $M(v_1) \to V$ is a homeomorphism, $M(v_1)$ is a solid torus. Hence $T(e_1) = \partial M(v_1)$ is compressible in $M(v_1)$.

(iii) If $p \mid M(v)$: $M(v) \to V$ (or $S^3 - \text{int } V$) is i-fold, then $p \mid \partial M(v)$: $\partial M(v) \to T$ is also i-fold. This gives (iii).

(iv) Let e be an edge of Γ. Since $T(e)$ bounds a compact 3-manifold N in M such that $\partial N = T(e)$ (see §1), $T(e)$ separates M into two components. Therefore Γ is a tree.

(v) If Γ has three black vertices v_1, v_2, v_3, then every $p \mid M(v_i)$: $M(v_i) \to V$ is 1-fold. Hence $p \mid M(v_i)$ is a homeomorphism. This contradicts that the branch set K of p is contained in V. □

Proof of Theorem. By Lemma 1 we may assume the branch set K is a prime, fibered knot. We prove the theorem by induction on $g(K)$. Let Γ be the graph associated to $(M, p^{-1}(T))$. By Lemma 2, we may assume that every non-contractible simple loop in $p^{-1}(T)$ is in good position with respect to $p \mid p^{-1}(T)$: $p^{-1}(T) \to T$.

Case 1. $\Gamma = \Gamma_1$.

Let D be a compressing disk for $T(e)$ in $M(v)$. We set $\partial D = \mu$. Then $m = p(\mu)$ is a meridian of V. It is easy to show that $p^{-1}(m)$ is either connected (i.e. $p^{-1}(m) = \mu$) or has three components μ_1 $(= \mu)$, μ_2, μ_3. If the latter case holds, we may extend $p|T(e)$: $T(e) \to T$ to a 3-fold unbranched covering q: $V_1 \to V$, where V_1 is a solid torus with meridians μ_1, μ_2, μ_3. Then $q \cup p|M(v_0)$: $V_1 \cup_{T(e)} M(v_0) \to S^3$ is a 3-fold unbranched covering. This contradicts that S^3 has no non-trivial covering. Hence we have $p^{-1}(m) = \mu$. Then we may extend $p|T(e)$: $T(e) \to T$ to a 3-fold cyclic branched covering r: $V_2 \to V$ branched over a core c of V, where V_2 is a solid torus with a meridian μ. Then $r \cup p|M(v_0)$: $V_2 \cup_{T(e)} M(v_0) \to S^3$ is a 3-fold cyclic branched covering branched over c. Since c in S^3 is the companion of K for T, c is fibered and $g(c) < g(K)$. If $(M(v_0), \mu)$ is a knot space-meridian pair, then $V_2 \cup_{T(e)} M(v_0) \cong S^3$. By the Branched Covering Theorem, c (hence V) is unknotted in S^3. Therefore $T = \partial V$ is compressible in a solid torus $S^3 - \operatorname{int} V$, a contradiction. Hence $M(v)$ is a solid torus with a meridian μ. Therefore we have $V_2 \cup_{T(e)} M(v_0) \cong M$. By [**3**, Theorem 2], c is simple. Thus $r \cup p|M(v_0)$ satisfies the conclusion of (i).

Case 2. $\Gamma = \Gamma_2$ and $\partial M(v_2)$ is compressible in $M(v_2)$.

Let D be a compressing disk for $T(e)$ in $M(v_2)$. We set $\mu = \partial D$. Then $m = p(\mu)$ is a meridian of V. If $p^{-1}(m) \cap T(e)$ consists of two components μ, μ', we may extend $p|T(e)$: $T(e) \to T$ to a 2-fold unbranched covering q: $V_1 \to V$, where V_1 is a solid torus with meridians μ, μ'. Then

$$q \cup (p|(M - \operatorname{int} M(v_2))): V_1 \cup_{T(e)} (M - \operatorname{int} M(v_2)) \to S^3$$

is a 3-fold unbranched covering, a contradiction. Therefore we have $p^{-1}(m) \cap T(e) = \mu$. Since $p|M(v_2)$: $M(v_2) \to V$ is a 2-fold (cyclic) branched covering, by the Equivariant Dehn's Lemma [**8**, Theorem 5], we may assume $g \cdot D = D$ for all $g \in G$, where G ($\cong Z_2$) is the group of the branched covering. By the argument of Gordon and Litherland [**3**], $p(D)$ is a meridian disk of V and $p(D) \cap K$ is a single point. By Schubert [**14**, §14, Satz 1], K is a composite knot. This contradicts our assumption. Thus Case 2 cannot occur.

Case 3. $\Gamma = \Gamma_2$ and $\partial M(v_2)$ is incompressible in $M(v_2)$.

We set $M_0 = M - \operatorname{int} M(v_2)$. Let D be a compressing disk for $T(e)$ in M_0. By a remark in §1, either M_0 is a solid torus with a meridian disk D, or $(M(v_2), \partial D)$ is a knot space-meridian pair. We set $\partial D = \mu$.

(3.1) We suppose M_0 is a solid torus. If $p|\mu$: $\mu \to p(\mu)$ is a 2-fold covering (resp. a homeomorphism), then we may extend $p|T(e)$: $T(e) \to T$ to q: $M_0 \to V_1$ which is a 2-fold branched covering branched over a core c of V_1 (resp. a 2-fold unbranched covering), where V_1 is a solid torus with a meridian $p(\mu)$. Then

$$p\,|\,M(v_2) \cup q\colon M = M(v_2) \cup_{T(e)} M_0 \to V \cup_T V_1$$

is a 2-fold branched covering branched over a link $K \cup c$ (resp. a knot K). We set $N = V \cup_T V_1$. Thus $p\,|\,M(v_2) \cup q$ satisfies the conclusions of (ii).

(3.2) We suppose that $(M(v_2), \mu)$ is a knot space-meridian pair. By the argument of (3.1), we may extend $p\,|\,T(e)\colon T(e) \to T$ to a 2-fold branched or unbranched covering $r\colon V_2 \to V_3$, where V_2, V_3 are solid tori with meridians μ, $p(\mu)$ respectively. Then

$$p\,|\,M(v_2) \cup r\colon M(v_2) \cup_{T(e)} V_2 \to V \cup_T V_3$$

is a 2-fold branched covering. Since $(M(v_2), \mu)$ is a knot space-meridian pair, $M(v_2) \cup_{T(e)} V_2$ is homeomorphic to S^3. Hence we have $\pi_1(V \cup_T V_3) = 1$, so $V \cup_T V_3$ is homeomorphic to S^3. By Fox [**2**, pp. 165–166], the branch set of $p\,|\,M(v_2) \cup r$ is connected. Therefore $r\colon V_2 \to V_3$ must be an unbranched covering, so $p\,|\,\mu\colon \mu \to p(\mu)$ is a homeomorphism. Let λ be a longitude of $(M(v_2), \mu)$. Since $l = p(\lambda)$ is homologous to zero in V, l is a meridian of V. Since $V \cup_T V_3 \cong S^3$, we may assume $l \cap p(\mu)$ consists of a single point. Since $p\,|\,\mu\colon \mu \to p(\mu)$ is a homeomorphism, $p^{-1}(l) \cap \mu$ consists of a single point. Hence $p^{-1}(l) \cap T(e)$ is connected, i.e. $p^{-1}(l) \cap T(e) = \lambda$. Therefore we may extend $p\,|\,T(e)\colon T(e) \to T$ to a 2-fold branched covering $s\colon V_4 \to V$ branched over a core c of V, where V_4 is a solid torus with a meridian λ. Then $s \cup p\,|\,M_0\colon V_4 \cup_{T(e)} M_0 \to S^3$ is a 3-fold irregular branched covering branched over c. Since c in S^3 is the companion of K for T, c is a fibered knot and $g(c) < g(K)$. We set $N = N(D, M_0)$. Since $\lambda \cap \mu$ consists of a single point, $B_1 = V_4 \cup_{T(e) \cap N} N$ is a 3-ball in $V_4 \cup_{T(e)} M_0$. Since $(M(v_2), \mu)$ is a knot space-meridian pair, $B_2 = M(v_2) \cup_{T(e) \cap N} N$ is a 3-ball in M. Since

$$V_4 \cup_{T(e)} M_0 - \operatorname{int} B_1 \cong \overline{(M_0 - N)} \cong M - \operatorname{int} B_2,$$

we have $V_4 \cup_{T(e)} M_0 \cong M$. Hence the result follows by induction.

Case 4. $\Gamma = \Gamma_3$.

By Lemma 3, $T(e_2)$ is compressible in $M(v)$. Let D_2 be a compressing disk for $T(e_2)$ in $M(v)$. We set $\mu_2 = \partial D_2$ and $m_2 = p(\mu_2)$. Since $p(D_2) \subset V$, m_2 is a meridian m of V. If $p^{-1}(m) \cap T(e_2)$ consists of two components μ_2, μ_2', then we may extend $p\,|\,T(e_2)\colon T(e_2) \to T$ to a 2-fold unbranched covering $q\colon V_1 \to V$, where V_1 is a solid torus with meridians μ_2, μ_2'. Then

$$q \cup p\,|\,M(v_2)\colon V_1 \cup_{T(e_2)} M(v_2) \to S^3$$

is a 2-fold unbranched covering, a contradiction. Hence we have $p^{-1}(m) \cap T(e_2) = \mu_2$. Then we may extend $p | T(e_2)$: $T(e_2) \to T$ to a 2-fold branched covering r: $V_2 \to V$ branched over a core c of V, where V_2 is a solid torus with a meridian μ_2. Then

$$r \cup p | M(v_2): V_2 \cup_{T(e_2)} M(v_2) \to S^3$$

is a 2-fold branched covering branched over c. If $(M(v_2), \mu_2)$ is a knot space-meridian pair, then $V_2 \cup_{T(e_1)} M(v_2) \cong S^3$. This gives a contradiction as in Case 1. Hence M_1 is a solid torus with a meridian μ_2. Therefore we have $V_2 \cup_{T(e_2)} M(v_2) \cong M$. Thus $r \cup p | M(v_2)$ satisfies the conclusion of (ii).

Case 5. $\Gamma = \Gamma_4$.

We may extend a homeomorphism $p | T(e_1)$: $T(e_1) \to T$ to a homeomorphism q: $V_1 \to V$, where V_1 is a solid torus bounded by $T(e_1)$. Then

$$q \cup p | \left(M(v_0) \cup_{T(e_2)} M(v_1)\right): V_1 \cup_{T(e_1)} \left(M(v_0) \cup_{T(e_2)} M(v_1)\right) \to S^3$$

is an unbranched 2-fold covering, a contradiction. Thus Case 5 cannot occur.

Case 6. $\Gamma = \Gamma_5$.,

Let D_i be a compressing disk for $T(e_i)$ in $M - \operatorname{int} M(v_i)$ for $i = 1, 2, 3$. By Lemma 3 we may assume $D_i \subset M(v)$ and $D_i \cap D_j = \varnothing$ for $i \neq j$. We set $\mu_i = \partial D_i$. Since $p(D_i) \subset V$, $m_i = p(\mu_i)$ is a meridian of V. We may assume $m_1 = m_2 = m_3$ $(= m)$. Since $p | M(v_i)$: $M(v_i) \to S^3 - \operatorname{int} V$ is a homeomorphism, $(M(v_i), \mu_i)$ is a knot space-meridian pair. Let λ_i be a longitude of $(M(v_i), \mu_i)$. We may assume $l = p(\lambda_1) = p(\lambda_2) = p(\lambda_3)$. Then l is a longitude of $(S^3 - \operatorname{int} V, m)$. We may extend a homeomorphism $p | T(e_i)$: $T(e_i) \to T$ to a homeomorphism q_i: $V_i \to \overline{V}$, where V_i (resp. $\overline{V}$) is a solid torus with a meridian λ_i (resp. l). Then

$$p | M(v) \cup \left(\bigcup_{i=1}^{3} q_i \right) : M(v) \cup_{T(e_1)} V_1 \cup_{T(e_2)} V_2 \cup_{T(e_3)} V_3 \to V \cup_T \overline{V}$$

is a 3-fold irregular branched covering over K in $V \cup_T \overline{V}$ $(\cong S^3)$. As in Case 3 we have

$$M(v) \cup_{T(e_1)} V_1 \cup_{T(e_2)} V_2 \cup_{T(e_3)} V_3 \cong M.$$

Obviously K in $V \cup_T \overline{V}$ is the preimage of K (in $V \cup_T (S^3 - \operatorname{int} V)$) for T. Hence the result follows by induction. This completes the proof. □

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