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CHARACTERIZING THE DIVIDED DIFFERENCE WEIGHTS FOR EXTENDED COMPLETE TCHEBYCHEFF SYSTEMS

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Newman and Rivlin have shown that there is a 1-1 correspondence between the nodes and weights of the nth order divided difference of nth degree polynomials. Their method applies only to polynomials. In this paper we develop a new approach and apply it to extend their results to the setting of Extended Complete Tchebycheff Systems.

0. Introduction. In [7] Newman and Rivlin (see also the reference there to S. Karlin's results) were able to characterize the weights which appear in the nth order divided difference formula with respect to the base functions $\{u_j(x) = x^j\}_{j=0}^n$ and to establish a 1-1 correspondence between these weights and the corresponding set of nodes, $0 = x_0 < x_1 < \cdots < x_n$, used in the formula. We propose in this paper to extend this result to the setting where the family $\{u_j(x)\}_{j=0}^n$ forms an Extended Complete Tchebycheff System (E.C.T.S.) on $[0, \infty)$. This means for each k, where $0 \leq k \leq n$, any non-trivial linear combination of the functions $\{u_0, \ldots, u_k\}$ has at most k zeros (including multiplicities) in $[0, \infty)$ where each $u_j \in C^n[0, \infty)$. We further assume that $u_0(x) \equiv 1$. For the remainder of this paper we shall postulate that these basic hypotheses concerning $\{u_j\}_{j=0}^n$ hold.

Among the E.C.T.S. for which these results are valid, we will highlight the families generated by the Cauchy Kernel and the Exponential Kernel.

1. Statement of problem. Let

$$(1) \quad S = \{\mathbf{x} = (x_1, \ldots, x_n) \subset R^n : 0 < x_1 < \cdots < x_n\}, \qquad x_0 \equiv 0.$$

A is defined to be the set of all $\mathbf{a} = (a_0, \ldots, a_n) \in R^{n+1}$ such that the following properties are valid

$$(2) \quad \begin{aligned} &\text{(i)} \quad (-1)^{n-i} a_i > 0 \qquad (i = 0, 1, \ldots, n); \\ &\text{(ii)} \quad \sum_{i=0}^n a_i = 0; \\ &\text{(iii)} \quad (-1)^{n-j} \sum_{i=j}^n a_i > 0, \quad j = 1, \ldots, n. \end{aligned}$$

The sets S and A are related through the classical concept of divided differences. For each $\mathbf{x} \in S$ and each real-valued function f defined on $[0, \infty)$, consider the nth order divided difference of f with respect to the points $(x_0, x_1, \ldots, x_n)$ defined as follows.

$$f[x_0, \ldots, x_n] = \frac{U\begin{bmatrix} u_0, \ldots, u_{n-1}, f \\ x_0, \ldots, x_n \end{bmatrix}}{U\begin{bmatrix} u_0, \ldots, u_n \\ x_0, \ldots, x_n \end{bmatrix}}, \tag{3}$$

where

$$U\begin{bmatrix} q_0, \ldots, q_n \\ x_0, \ldots, x_n \end{bmatrix} = \det\{q_i(x_j);\ i, j = 0, 1, \ldots, n\}.$$

We then set

$$a_i = (-1)^{n+i} \frac{U\begin{bmatrix} u_0 & , \ldots, & u_{n-1} \\ x_0, \ldots, x_{i-1}, & x_{i+1}, \ldots, x_n \end{bmatrix}}{U\begin{bmatrix} u_0, \ldots, u_n \\ x_0, \ldots, x_n \end{bmatrix}}, \qquad i = 0, 1, \ldots, n. \tag{4}$$

Clearly,

$$f[x_0, \ldots, x_n] = \sum_{i=0}^{n} a_i f(x_i).$$

The $\{a_i\}$ are called the weights of the divided difference formula. Cramer's Rule, together with (3), (4), shows that for a given $\mathbf{x} \in S$, $\mathbf{a} = (a_0, \ldots, a_n)$ satisfies (4) iff

$$\sum_{i=0}^{n} a_i u_j(x_i) = \delta_{nj}, \qquad j = 0, 1, \ldots, n, \tag{5}$$

where δ_{nj} is the Kronecker delta symbol.

Thus for each $\mathbf{x} \in S$, we can associate an $\mathbf{a}$ via the relationship (4). Let g be the map defined by (4), that is $g(\mathbf{x}) = \mathbf{a}$. The main purpose of this paper is to show that g is a 1-1 map of S onto A. As we indicated in the introduction, Newman and Rivlin proved this result for the special case of polynomials; that is, where $u_i = x^i$.

LEMMA 1. *g maps S into A.*

Proof. Since $(u_0,\ldots,u_n)$ form an Extended Complete Tchebycheff System (E.C.T.S.), it is clear from the definition of the weights a_i in (4) that $\mathbf{a} = g(\mathbf{x})$ satisfies (i) and (ii). (In this regard recall that $u_0 \equiv 1$.)

To prove (iii), for $0 \le j \le n-1$ pick $u^{(j)}$ in the linear subspace U spanned by $(u_0,\ldots,u_n)$ with the properties

(a) $u^{(j)}(x_i) = 1, i = 0, 1, \ldots, j,$

(b) $u^{(j)}(x_i) = 0, i = j+1, \ldots, n.$

Using (5) and the above it follows that

$$\sum_{i=0}^{j} a_i = \sum_{i=0}^{n} a_i u^{(j)}(x_i) = b_n,$$

where b_n is the coefficient of u_n in the expansion of $u^{(j)}$. From [5, p. 379] we infer that $\{(d/dx)u_j(x)\}_{j=1}^{n}$ forms an E.C.T.S. Thus by *Rolle's Theorem* $(d/dx)u^{(j)}(x)$ has a maximum set of $n-1$ simple zeros consisting of j zeros in (x_0, x_j) and $(n-j-1)$ zeros in (x_{j+1}, x_n). Further, since $u^{(j)}(x_j) = 1$ and $u^{(j)}(x_{j+1}) = 0$, $du^{(j)}/dx < 0$ in $[x_j, x_{j+1}]$ and thus $(-1)^{n-j}(du^{(j)}/dx)(x_n) > 0$. Using as data these $n-1$ zeros of $(d/dx)u^{(j)}(x)$ and x_n, we conclude by Cramer's Rule that $\operatorname{sgn}(d/dx)u^{(j)}(x_n) = \operatorname{sgn} b_n$; that is,

$$(-1)^{n-j} \sum_{i=0}^{j} a_i > 0.$$

By (2)(ii),

$$\sum_{i=0}^{j} a_i = \left(\sum_{i=0}^{n} a_i - \sum_{i=j+1}^{n} a_i \right) = - \sum_{i=j+1}^{n} a_i.$$

Finally, then

$$(-1)^{n-(j+1)} \sum_{i=j+1}^{n} a_i > 0. \qquad \square$$

LEMMA 2. *Let $\{\mathbf{x}^{(v)}\}_{v=1}^{\infty} \subset S$ be a sequence with the property that the corresponding sequence $\{\mathbf{a}^{(v)}\} \subset A$ (where $\mathbf{a}^{(v)} = g(\mathbf{x}^{(v)})$) has the feature that $\mathbf{a}^{(v)} \to \mathbf{a} \in A$. Then if $\mathbf{x}^{(v)} \to \mathbf{x}$, we can conclude that $\mathbf{x} \in S$.*

Proof. Assume the result is false. We treat two cases. Case (1): $x_i^{(v)} \to x_0 \equiv 0$ for all i. Thus using (5) for $j = n$ we find the limit function satisfies

$$\sum_{i=0}^{n} a_i u_n(0) = 1,$$

which contradicts (2)(ii). Case (2): For some i where $1 \leq i \leq n-1$, $x_0 < x_i = x_{i+1}$. Thus by exploiting the fact that **a** satisfies (2)(iii) and (5), we can find a set of numbers $\{b_j\}_{j=0}^{k}$, where $b_k \not\equiv 0$ with $0 \leq k \leq n-1$ so that for the $k+1$ *distinct* components of the limit vector **x**, say $\{x_{l_0}, \ldots, x_{l_k}\}$, we have

$$\sum_{i=0}^{k} b_i u_j(x_{l_i}) = 0 \qquad (j = 0, 1, \ldots, n-1).$$

This contradicts the fact that $\{u_j\}_{j=0}^{n-1}$ form an E.C.T.S. Thus the proof is complete. □

2. Main results. In this section we will develop the topological tools which we will use to prove our principal result; that is, g is a 1-1 map of S onto A. We will employ a differential equation approach which has been exploited by Fitzgerald and Schumaker [**4**]; Barrar, Loeb and Werner [**2**]; Barrar and Loeb [**1, 3**].

Our approach, in contrast to other attacks on these types of problems, has the important property that it does not require any type of a priori uniqueness. In this regard see Fitzgerald, Schumaker [**4**] or Newman, Rivlin [**7**] where such information is used.

Consider a fixed $\mathbf{z}^* \in A$. We want to demonstrate that there is exactly one $\mathbf{x}^* \in S$ which satisfies

$$\sum_{i=0}^{n} a_i^* u_j(x_i) = \delta_{nj} \qquad (j = 0, 1, \ldots, n).$$

Since $\sum_{i=0}^{n} a_i^* = 0$ and $u_0 \equiv 1$, this is equivalent to demonstrating it for the system

$$\sum_{i=1}^{n} a_i^*\big(u_j(x_i) - u_j(x_0)\big) = \delta_{nj}, \qquad j = 1, \ldots, n. \tag{6}$$

For each $\mathbf{x} \in S$, consider the system of n ordinary differential equations

$$\frac{d}{d\tau}\left[\sum_{i=1}^{n} \big((1-\tau)a_i + \tau a_i^*\big)\big(u_j(x_i(\tau)) - u_j(x_0)\big)\right] = 0, \qquad j = 1, \ldots, n, \tag{7}$$

where $\mathbf{a} = g(\mathbf{x})$ and the initial conditions are $\mathbf{x}(0) = \mathbf{x} = (x_1, \ldots, x_n)$. Here τ is the independent variable, $\mathbf{x}(\tau) = (x_1(\tau), \ldots, x_n(\tau))$, and $\mathbf{a} = (a_0, \ldots, a_n)$. Integrating (7) we find that

$$\sum_{j=1}^{n} \big((1-\tau)a_i + \tau a_i^*\big)\big(u_j(x_i(\tau)) - u_j(x_0)\big) \equiv c_j, \qquad j = 1, \ldots, n. \tag{8}$$

We evaluate the constants c_j by setting $\tau = 0$. One finds using (6) that

$$\delta_{nj} = \sum_{i=1}^{n} a_i\big(u_j(x_i) - u_j(x_0)\big) = c_j, \qquad j = 1,\ldots,n,$$

and indeed at $\tau = 1$,

$$\sum_{i=1}^{n} a_i^*\big(u_j(x_i(1)) - u_j(x_0)\big) = \delta_{nj} \qquad (j = 1,\ldots,n).$$

Thus, one notes that $\mathbf{a}^* = g(\mathbf{x}(1))$ and $\mathbf{x}(1)$ is a desired solution for $\mathbf{a}^*$. We see then that our main problem is to show that the system of differential equations has a solution in the interval [0, 1]. We proceed toward this goal.

For many important families of functions we will be able to verify the following assumption.

Assumption A. If $\{\mathbf{x}^{(v)}\}_{v=1}^{\infty} \subset S$ has the characteristic that $\mathbf{a}^{(v)} \equiv g(\mathbf{x}^{(v)}) \to \mathbf{a} \in A$ as $v \to \infty$, then $\{\mathbf{x}^{(v)}\}_{v=1}^{\infty}$ are bounded.

For the remainder of this section we shall postulate that *Assumption* A is valid for the E.C.T.S. $\{u_i\}_{i=0}^{n}$ on $[0, \infty]$ where $u_0 \equiv 1$.

Expanding (7) we obtain

$$(9) \qquad \begin{aligned} &\sum_{i=1}^{n} \left[\tau a_i^* + (1-\tau)a_i\right] u_j'(x_i(\tau)) \frac{dx_i}{d\tau}(\tau) \\ &\qquad = \sum_{i=1}^{n} (a_i - a_i^*)\left[u_j(x_i(\tau)) - u_j(x_0)\right] \qquad (i = 1,\ldots,n) \end{aligned}$$

with $u_j'(x) = \dfrac{d}{dx}u_j(x)$.

It is important to note that for $\tau \in [0, 1]$ and $\mathbf{x}(\tau) \in S$, the Jacobian matrix of the system (9),

$$(10) \qquad J(\tau) = \big\{(\tau a_i^* + (1-\tau)a_i)u_j'(x_i(\tau));\ i, j = 1,\ldots,n\big\},$$

is non-singular. This follows from the fact that $\{u_j'\}_{j=1}^{n}$ form a E.C.T.S. and that $(\tau\mathbf{a}^* + (1-\tau)\mathbf{a})$ satisfies (2)(i) when $\tau \in [0, 1]$.

Further, it is easy to check using Assumption A and Lemma 2 that $\{\mathbf{x}(\tau);\ \tau \in [0, 1]\}$ is bounded, and if $\{\tau_v\}_{v=1}^{\infty} \subset [0, 1]$ has the property that $\mathbf{x}(\tau_v) \to \mathbf{x}$, then $\mathbf{x} \in S$. These facts can be used to show that the system of differential equations has a solution over [0, 1]. The basic ingredients of such an existence proof are enunciated in [**1**, **2**].

For each $\mathbf{x} \in S$, let Φ be the map from $S \to B$ defined by $\Phi(\mathbf{x}) = \mathbf{x}(1)$ for $\mathbf{x} \in S$ where $B = \{\mathbf{x} \in S\colon g(\mathbf{x}) = \mathbf{a}^*\}$. If $\mathbf{x} \in B$, it is easy to verify that $\mathbf{x}(\tau) \equiv \mathbf{x}$ is a solution of (9) and, indeed, by the uniqueness of the solution of the system of differential equations, the only one. Thus Φ maps S onto B and since by the theory of differential equations Φ is continuous, Φ maps the connected set S onto the connected set B.

Let $\mathbf{x}^* \in B$. Then $\mathbf{x}^*$ is a solution of the non-linear system (6). Further, the Jacobian matrix of the system is

$$\left\{a_i^* u_j'(x_i^*);\ i, j = 1,\dots,n\right\}.$$

Since $\mathbf{a}^*$ satisfies (2)(i) and $\{u_j'(x)\}_{j=1}^n$ form a E.C.T.S., the matrix is non-singular. We can conclude by the *implicit function theorem* that $\mathbf{x}^*$ is an isolated point of B. Since $\mathbf{x}^*$ is an arbitrary point of the connected set B, it follows that B consists of exactly one point. Summarizing,

MAIN THEOREM. *For each* $\mathbf{a}^* \in A$, *there is exactly one* $\mathbf{x}^*$ *in* S *which satisfies*

$$\sum_{i=0}^{n} a_i^* u(x_i^*) = \delta_{jn} \qquad (i = 0, 1, \dots, n),$$

and the map g *defined by* (4) *is a* 1-1 *map which takes* S *onto* A.

3. Applications. In this section we present some examples of E.C.T.S. which satisfy Assumption A and thus satisfy the hypothesis of the Main Theorem.

Consider the exponential kernel $K(\lambda, x) = e^{\lambda x}$ and any set of n positive numbers $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n$ with $\lambda_0 = 0$. Then we set

$$u_i(x) = K(\lambda_i, x), \qquad i = 0, 1, \dots, n. \tag{11}$$

LEMMA 3. *The exponential family of functions defined in* (11) *has the property that if a sequence* $\{\mathbf{x}^{(v)}\}_{v=1}^{\infty} \subset S$ *yields a sequence* $\{\mathbf{a}^{(v)} = g(\mathbf{x}^{(v)})\}_{v=1}^{\infty}$ *with the characteristic that* $\mathbf{a}^{(v)} \to \mathbf{a} \in A$, *then the* $\{\mathbf{x}^{(v)}\}_{v=0}^{\infty}$ *are bounded.*

Proof. Let us assume that the components of $\mathbf{x}^{(v)}$ are not bounded. Then by going to a subsequence if necessary we can develop the following situation:

(12) (a) $\lim_{v \to \infty} x_n^{(v)} = \infty$;

(b) $\lim_{v \to \infty} \left(x_n^{(v)} - x_i^{(v)}\right) = c_i, \quad i = l,\dots,n$, where $l \geq 1$ and $c_i \geq c_{i+1}, \quad i = l,\dots,n-1$, with c_i finite;

(c) $\lim_{v \to \infty} \left(x_n^{(v)} - x_i^{(v)}\right) = \infty, \quad i = 1,\dots,l-1$.

Dividing each of the relationships

$$\sum_{i=0}^{n} a_i^{(v)} e^{\lambda_j x_i^{(v)}} = \delta_{nj}$$

by $e^{\lambda_j x_n^{(v)}}$ and letting $v \to \infty$, we find that the limits satisfy

$$\sum_{i=l}^{n} a_i e^{-\lambda_j c_i} = 0 \qquad (j = 1,\dots,n).$$

Let $c_{i_1} > c_{i_2} > \cdots > c_{i_k} = 0$ be the distinct values of $\{c_i\}_{i=l}^n$ where $k \le n - l + 1 \le n$. Then we can find numbers $b_1,\dots,b_k$ so that

$$f(\lambda) \equiv \sum_{i=l}^{n} a_i e^{-\lambda c_i} \equiv \sum_{m=1}^{k} b_m e^{-\lambda c_{i_m}},$$

where by property (2)(iii), $b_k \neq 0$. Thus since $f(\lambda_i) = 0$, $i = 1,\dots,n$ and $\{e^{-\lambda c_{i_m}}\}_{m=1}^k$ form an E.C.T.S., we have reached a contradiction. This completes the proof. □

We claim that Lemma 3 is also valid for the Cauchy kernel, $K(\lambda, x) = 1/(1 + \lambda x)$.

LEMMA 4. *Let* $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n$ *be given and set* $u_j(x) = 1/(1 + \lambda_j x)$ $(j = 0, 1,\dots,n)$. *Then Lemma* 3 *is valid for the* $\{u_j\}_{j=0}^n$.

Proof. Again assuming that $x_n^{(v)} \to \infty$, we can, by going to a subsequence if necessary, achieve the situation:

(a) $x_i^{(v)} \to \infty$, $i = l,\dots,n$, where $l \ge 1$;

(b) $x_i^{(v)} \to c_i$, $i = 0,\dots,l-1$, c_i finite with $c_i \le c_{i+1}$ and $c_0 = 0$.

For each relationship

$$\sum_{i=0}^{n} \frac{a_i^{(v)}}{1 + \lambda_j x_i^{(v)}} = 0,$$

letting $v \to \infty$, we find

$$\sum_{i=0}^{l-1} \frac{a_i}{1 + \lambda_j c_i} = 0 \qquad (j = 0,\dots,n-1).$$

Pick out the distinct elements $0 = c_{i_0} < \cdots < c_{i_{k-1}}$ of the set $\{c_i\}_{i=0}^{l-1}$ where $k \le l \le n$. Then there are k distinct numbers $b_0,\dots,b_{k-1}$ so that

$$f(\lambda) \equiv \sum_{i=0}^{l-1} \frac{a_i}{1 + c_i\lambda} = \sum_{m=0}^{k-1} \frac{b_m}{1 + c_{i_m}\lambda}$$

and where by properties (2)(i), (ii), (iii), $b_0 \neq 0$. Since $f(\lambda_j) = 0$ $(j = 0, 1, \ldots, n-1)$ we have contradicted the fact that the family $\{1/(1 + c_{i_m}\lambda)\}_{m=0}^{k-1}$ forms an E.C.T.S. □

Our results can be extended to treat multiple knots also.

As an example, we have the following result, which includes the results of [7].

LEMMA 5. *Let* $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_r$ *be given and consider the functions* $\{x^q e^{\lambda_p x};\ q = 0, 1, \ldots, m_p - 1;\ p = 0, 1, \ldots, r\}$. *If* $n + 1 = \sum_{p=0}^{r} m_p$ *and if we set* $u_j(x) = x^q e^{\lambda_p x}$ *with* $j = \sum_{t=-1}^{p-1} m_t + q$ *and* $m_{-1} = 0$, *then Lemma 3 is valid for the functions* $\{u_j\}_{j=0}^{n}$. *(The* λ_p *are called the knots and the* m_p *are designated as the multiplicities of the knots of the kernel* $K(x, \lambda) = e^{\lambda x}$. *It is well known that this set of functions is a E.C.T.S., see* [5, *p.* 9].)

Proof. Letting

$$f(\lambda, v) = \sum_{i=0}^{n} a_i^{(v)} e^{\lambda x_i^{(v)}},$$

we have

$$\frac{\partial^q f}{\partial \lambda^q}(\lambda, v) = \sum_{i=0}^{n} a_i^{(v)} \left(x_i^{(v)}\right)^q e^{\lambda x_i^{(v)}}.$$

The set of equations corresponding to (5) for $a_i = a_i^{(v)}$, $x_i = x_i^{(v)}$ can be written as

$$\text{(13)}\qquad \left.\frac{\partial^q}{\partial \lambda^q} f(\lambda, v)\right|_{\lambda=\lambda_p} = \delta_{p,r}\delta_{(q,m_r-1)} \qquad q = 0, 1, \ldots, m_p - 1; \quad p = 0, 1, \ldots, r.$$

Assuming $x_n^{(v)} \to \infty$, if $r \geq 1$, we divide $f(\lambda, v)$ by $e^{\lambda x_n^{(v)}}$, and apply Leibnitz's rule for differentiation of a product to find, using the notation of (12)(a), (b), (c), that in the limit as $v \to \infty$, (13), for $p \geq 1$, becomes

$$\text{(14)}\qquad \sum_{i=l}^{n} a_i c_i^q e^{\lambda_p c_i} = 0, \qquad q = 0, 1, \ldots, m_p - 1; p = 1, \ldots, r.$$

Combining equal c_i's as in Lemma 3, this becomes

$$\text{(15)}\qquad \sum_{s=1}^{w} b_s (c_{i_s})^q e^{\lambda_p c_{i_s}} = 0, \qquad q = 0, 1, \ldots, m_p - 1; p = 1, \ldots, r,$$

where $w \leq n + 1 - l$, $b_w \neq 0$ by (2)(iii), and $l \geq 1$. In (15) we are dealing with an E.C.T.S. of dimension $\leq n + 1 - l$ with typical term $x^q e^{\lambda_p x}$.

Further, the function in (15) has at least $n + 1 - m_0$ zeros. Thus $n + 1 - m_0 < n + 1 - l$, that is,

(16) $$m_0 > l \quad \text{if } r \geq 1.$$

For any r, we divide the equations in (13) for $\lambda = \lambda_0$ by $(x_n^{(v)})^q$ for each $q = 0, 1, \ldots, m_0 - 1$, and take the limit as $v \to \infty$. Using the notation of (12)(a), (b), (c) the result is a set of equations

$$\sum a_i (d_i)^q = 0, \quad d_i \leq d_{i+1}, \qquad q = 0, 1, \ldots, m_0 - 1.$$

Combining equal d_i's we obtain a set

(17) $$\sum_{s=1}^{g} b_{i_s} (d_{i_s})^q = 0, \qquad q = 0, 1, \ldots, m_0 - 1.$$

Note that $x_i^{(v)} - x_n^{(v)} \to c_i$ (finite) implies $x_i^{(v)}/x_n^{(v)} \to d_i = 1$. Thus $d_i = 1$ $(i = l, \ldots, n)$ with $g \leq l$ and $b_{i_g} \neq 0$. In (17) we are dealing with a non-zero function with m_0 zeros generated from a E.C.T.S. of dimension at most l. Therefore we must have

(18) $$m_0 < l.$$

If $r = 0$, (18) is a contradiction since $m_0 = n + 1$ and $l < n + 1$. If $r \geq 1$ both (16) and (18) must hold, which again is a contradiction. □

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