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NONLINEAR ERGODIC THEOREMS FOR AN AMENABLE SEMIGROUP OF NONEXPANSIVE MAPPINGS IN A BANACH SPACE

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Let C be a nonempty closed convex subset of a Banach space, S a semigroup of nonexpansive mappings t of C into itself, and $F(S)$ the set of common fixed points of mappings t. Then we deal with the existence of a nonexpansive retraction P of C onto $F(S)$ such that $Pt = tP = P$ for each $t \in S$ and Px is contained in the closure of the convex hull of $\{tx: t \in S\}$ for each $x \in C$. That is, we prove nonlinear ergodic theorems for a semigroup of nonexpansive mappings in a Banach space.

1. Introduction. Let C be a nonempty closed convex subset of a real Banach space E. Then a mapping $T: C \to C$ is called nonexpansive on C if

$$\|Tx - Ty\| \leq \|x - y\| \quad \text{for all } x, y \in C.$$

We denote by $F(T)$ the set of fixed points of T, that is,

$$F(T) = \{z \in C: Tz = z\}.$$

Let $S = \{S(t): t \geq 0\}$ be a family of nonexpansive mappings of C into itself such that $S(0) = I$, $S(t+s) = S(t)S(s)$ for all $t, s \in [0, \infty)$ and $S(t)x$ is continuous in $t \in [0, \infty)$ for each $x \in C$. Then S is said to be a nonexpansive semigroup on C.

The nonlinear ergodic theorem for nonexpansive mappings was originally studied in the framework of Hilbert spaces by Baillon [**1**], and later extended to Banach spaces by Bruck [**8**], Hirano [**15**], Reich [**21**] and others. A corresponding result for nonexpansive semigroups on C was given by Baillon [**2**], Baillon-Brézis [**3**] and Reich [**20**]. Nonlinear ergodic theorems for general commutative semigroups of nonexpansive mappings were given by Brézis-Browder [**4**] and Hirano-Takahashi [**16**]. Recentlly Takahashi [**26**] proved the following nonlinear ergodic theorem for a noncommutative semigroup of nonexpansive mappings: Let C be a nonempty closed convex subset of a real Hilbert space H, and let S be an amenable semigroup of nonexpansive mappings t of C into itself. Suppose

$$F(S) = \bigcap \{F(t): t \in S\} \neq \varnothing.$$

Then there exists a nonexpansive retraction P of C onto $F(S)$ such that $Pt = tP = P$ for all $t \in S$ and $Px \in \overline{\text{co}}\, Sx$ for all $x \in C$, where $Sx = \{tx: t \in S\}$ and $\overline{\text{co}}\, A$ is the closure of the convex hull of A. In this paper we shall prove analogous results for semigroups of nonexpansive mappings in Banach spaces. That is, we establish the existence of certain nonexpansive retractions onto the fixed point sets of amenable semigroups of nonexpansive mappings in Banach spaces. Theorem 2 is a generalization of Takahashi's nonlinear ergodic theorem.

2. Preliminaries. Let E be a real Banach space and E^* its dual, that is, the space of all continuous linear functionals f on E. The value of $f \in E^*$ at $x \in E$ will be denoted by $\langle x, f \rangle$. With each $x \in E$, we associate the set

$$J(x) = \left\{f \in E^*: \langle x, f \rangle = \|x\|^2 = \|f\|^2\right\}.$$

Using the Hahn-Banach theorem it is immediately clear that $J(x) \neq \varnothing$ for any $x \in E$. The multivalued operator $J: E \to E^*$ is called the duality mapping of E. Let $U = \{x \in E: \|x\| = 1\}$ be the unit sphere of E. Then the norm of E is said to be Gâteaux differentiable (and E is said to be smooth) if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each x and y in U. It is said to be Fréchet differentiable if, for each x in U, this limit is attained uniformly for y in U. Finally, it is said to be uniformly Fréchet differentiable (and E is said to be uniformly smooth) if the limit is attained uniformly for x, y in $U \times U$. It is well known that if E is smooth, then the duality mapping J is single valued. It is also known that if E has a Fréchet differentiable norm, then J is norm to norm continuous. Let K be a subset of E. Then we denote by $\delta(K)$ the diameter of K. A point $x \in K$ is a diametral point of K provided

$$\sup\{\|x - y\|: y \in K\} = \delta(K).$$

A closed convex subset C of a Banach space E is said to have normal structure if for each closed bounded convex subset K of C, which contains at least two points, there exists an element of K which is not a diametral point of K. It is well known that a closed convex subset of a uniformly convex Banach space has normal structure and a compact convex subset of a Banach space has normal structure.

Let S be an abstract semigroup and $m(S)$ the Banach space of all bounded real valued functions on S with the supremum norm. For each

$s \in S$ and $f \in m(S)$, we define elements f_s and f^s in $m(S)$ given by $f_s(t) = f(st)$ and $f^s(t) = f(ts)$ for all $t \in S$. An element $\mu \in m(S)^*$ (the dual space of $m(S)$) is called a mean on S if $\|\mu\| = \mu(1) = 1$. A mean μ is called left (right) invariant if $\mu(f_s) = \mu(f)$ ($\mu(f^s) = \mu(f)$) for all $f \in m(S)$ and $s \in S$. An invariant mean is a left and right invariant mean. A semigroup which has a left (right) invariant mean is called left (right) amenable. A semigroup which has an invariant mean is called amenable. Day **[10]** proved that a commutative semigroup is amenable. We also know that $\mu \in m(S)^*$ is a mean on S if and only if

$$\inf\{f(s): s \in S\} \leq \mu(f) \leq \sup\{f(s): s \in S\}$$

for every $f \in m(S)$. Let S be a right amenable semigroup. Then $Ss \cap St \neq \varnothing$ for all $s, t \in S$. See **[13]** and **[14]**. A right amenable semigroup is directed by an order relation $\geq$ defined by $t \geq s$ if and only if $t \in Ss$. Throughout this paper a right amenable semigroup is directed by the order relation defined above and a semigroup contains the identity. We know the following proposition **[25]**.

PROPOSITION 1. *Let S be a right amenable semigroup and μ a right invariant mean on S. Then we have*

$$\sup_s \inf_{s \leq t} f(t) \leq \mu(f) \leq \inf_s \sup_{s \leq t} f(t) \quad \textit{for all } f \in m(S).$$

Proof. Let f be an element of $m(S)$ and μ a right invariant mean on S. Then

$$\mu(f) = \mu(f^s) \leq \sup_t f^s(t) = \sup_t f(ts) = \sup_{s \leq t} f(t)$$

and, hence, $\mu(f) \leq \inf_s \sup_{s \leq t} f(t)$. Similarly, we have $\sup_s \inf_{s \leq t} f(t) \leq \mu(f)$.

3. Ergodic theorems in a reflexive Banach space. To establish the existence of "ergodic" nonexpansive retractions onto the fixed point sets of amenable semigroups of nonexpansive mappings in a reflexive Banach space, the following proposition obtained by Bruck [7] is very useful.

PROPOSITION 2. *Let X be a Hausdorff space, S a semigroup of mappings of X into X. If S is compact in the topology of pointwise convergence on X and for each $x \in X$, there exists a common fixed point of S in Sx, then there is in S a retraction of X onto $F(S)$ (the set of common fixed points of S).*

THEOREM 1. *Let C be a closed convex subset of a real reflexive Banach space E which has normal structure and let S be an amenable semigroup of nonexpansive mappings of C into itself. Suppose*

$$F(S) = \bigcap \{F(t): t \in S\} \neq \varnothing .$$

Then there exists a nonexpansive retraction P of C onto $F(S)$ such that $Pt = tP = P$ for every $t \in S$ and every S-invariant closed convex subset of C is P-invariant.

Proof. First we show, by making use of methods of [**6**] and [**7**], there exists a nonexpansive retraction r of C onto $F(S)$ such that every S-invariant closed convex subset of C is r-invariant. Put $G = \{s$: s is a nonexpansive mapping of C into itself, $F(s) \supset F(S)$ and every S-invariant closed convex subset of C is s-invariant$\}$. Then $S \subset G$. It is obvious that G is a semigroup of mappings of C into itself. We show that G is compact in the topology of pointwise weak convergence on C. Fix an element $v \in F(S)$. For each $x \in C$, let $W_x = \{y \in C: \|y - v\| \leq \|x - v\|\}$. Then since for any $s \in G$, $\|sx - v\| \leq \|x - v\|$, $Gx \subset W_x$ and W_x is weakly compact and convex. Since G is a subset of the product space $W = \prod_{x \in C} W_x$ and W is compact, to show that G is compact, it is sufficient to prove that G is closed in W. Let $\{s_\alpha\}$ be a net in G which converges to s in W. Then since for any $x, y \in C$ and $u \in F(S)$,

$$(1) \qquad \begin{aligned} \|sx - sy\| &= \Big\|\text{w-}\lim_{\alpha} (s_\alpha x - s_\alpha y)\Big\| \\ &\leq \liminf_{\alpha} \|s_\alpha x - s_\alpha y\| \leq \|x - y\|, \end{aligned}$$

and $su = \text{w-}\lim_\alpha s_\alpha u = u$, we have that s is nonexpansive and $F(s) \supset F(S)$. Since an S-invariant closed convex subset K of C is also weakly closed, we have $sK \subset K$. These imply $s \in G$ and, hence, G is closed in W. For any $x \in C$, consider Gx. Then, since for $s, t \in G$ and $0 \leq k \leq 1$, $ks + (1 - k)t \in G$ and G is a semigroup, Gx is an S-invariant bounded closed convex subset of C. So, by [**18**], there exists a common fixed point of S and, hence, a common fixed point of G. By Proposition 2 there exists a retraction $r \in G$ of C onto $F(G) = F(S)$.

Next we show there exists a nonexpansive mapping of C into itself such that $Qs = Q$ for all $s \in S$ and $Qx \in \overline{\text{co}}\, Sx$ for each $x \in C$. Let μ be an invariant mean on S and $x \in C$. Then since $F(S) \neq \varnothing$, $\{sx: s \in S\}$ is bounded and, hence, for each f in E^*, the real valued function $t \mapsto \langle tx, f\rangle$ is in $m(S)$. Denote by $\mu_t\langle tx, f\rangle$ the value of μ at this function. By linearity of μ, this is linear in f; moreover, since

$$|\mu_t\langle tx, f\rangle| \leq \|\mu\| \sup_t |\langle tx, f\rangle| \leq \left(\sup_t \|tx\|\right)\|f\|,$$

it is continuous in f. So by the Riesz theorem, there exists an $x_0 \in E^{**} = E$ such that $\mu_t\langle tx, f\rangle = \langle x_0, f\rangle$ for all $f \in E^*$. Setting $Qx = x_0$, we have that Q is nonexpansive. In fact, for any $j \in J(Qx - Qy)$,

$$\begin{aligned}\|Qx - Qy\|^2 &= \langle Qx - Qy, j\rangle = \mu_t\langle tx - ty, j\rangle \\ &\leq \left(\sup_t \|tx - ty\|\right)\|j\| \leq \|x - y\|\,\|Qx - Qy\|.\end{aligned}$$

From

$$\langle Qsx, f\rangle = \mu_t\langle tsx, f\rangle = \mu_t\langle tx, f\rangle = \langle Qx, f\rangle,$$

it follows that $Qs = Q$ for each $s \in S$. If $Qx \notin \overline{\text{co}}\, Sx$, then by the separation theorem, there exists a $f \in E^*$ such that

$$\langle Qx, f\rangle < \inf\{\langle z, f\rangle : z \in \overline{\text{co}} Sx\}.$$

So we have

$$\begin{aligned}\inf_t \langle tx, f\rangle &\leq \mu_t\langle tx, f\rangle = \langle Qx, f\rangle \\ &< \inf\{\langle z, f\rangle : z \in \overline{\text{co}}\, Sx\} \leq \inf_t \langle tx, f\rangle.\end{aligned}$$

This is a contradiction. Therefore, $Qx \in \overline{\text{co}}\, Sx$.

Now, let $P = rQ$. Then we have that P is a mapping of C onto $F(S)$. Since r and Q are nonexpansive, P is nonexpansive. From

$$P^2x = (rQ)(rQ)x = r(rQ)x = rQx = Px,$$

we have $P^2 = P$. Since Px is an element of $F(S)$, it follows that $tPx = Px$ for all $x \in C$ and $t \in S$. Since $Qt = Q$ for all $t \in S$, we have

$$Pt = (rQ)t = rQ = P.$$

Let K be an S-invariant closed convex subset of C and $x \in K$. Then since $Qx \in \overline{\text{co}}\, Sx \subset K$ and, hence, $Px = rQx \in K$, it follows that K is P-invariant. This completes the proof.

4. Ergodic theorems in a uniformly convex Banach space. Using Lemma 1 of [**15**], we can prove the following Lemma which is an extension of Lemma 2 of [**15**].

LEMMA 1. *Let C be a closed convex subset of a uniformly convex Banach space E and let S be an amenable semigroup of nonexpansive mappings of C*

into itself with a common fixed point. Let $x \in C, f \in F(S)$ *and* $0 < \alpha \le \beta < 1$. *Then for each* $\varepsilon > 0$ *there exists* $t_0 \in S$ *such that*

$$\|s(\lambda tx + (1-\lambda)f) - (\lambda stx + (1-\lambda)f)\| < \varepsilon$$

for all $s \in S$, $t \ge t_0$ *and* λ: $\alpha \le \lambda \le \beta$.

Proof. Since $f \in F(S)$ and S is right amenable, we obtain

$$\sup_s \inf_{s \le t} \|tx - f\| = \inf_s \sup_{s \le t} \|tx - f\|.$$

Put $r = \lim_t \|tx - f\|$, $c = \min\{2\lambda(1-\lambda)\colon \alpha \le \lambda \le \beta\}$ and $c' = \max\{2\lambda(1-\lambda)\colon \alpha \le \lambda \le \beta\}$. Let $\varepsilon > 0$. If $r = 0$, then there exists $t_0 \in S$ such that $\sup_{t_0 \le t} \|tx - f\| < \varepsilon/c'$. So we obtain that for all $s \in S$, $t \ge t_0$ and λ: $\alpha \le \lambda \le \beta$,

$$\begin{aligned}
&\|s(\lambda tx + (1-\lambda)f) - (\lambda stx + (1-\lambda)f)\| \\
&\quad \le \lambda \|s(\lambda tx + (1-\lambda)f) - stx\| + (1-\lambda)\|s(\lambda tx + (1-\lambda)f) - f\| \\
&\quad \le \lambda \|\lambda tx + (1-\lambda)f - tx\| + (1-\lambda)\|\lambda tx + (1-\lambda)f - f\| \\
&\quad = 2\lambda(1-\lambda)\|tx - f\| < 2\lambda(1-\lambda)\varepsilon/c' \le \varepsilon.
\end{aligned}$$

Let $r > 0$ and choose $d > 0$ so small that

$$(r+d)(1 - c\delta(\varepsilon/(r+d))) < r, \tag{2}$$

where δ is the modulus of convexity of the norm. Then there exists $t_0 \in S$ such that for all $t \ge t_0$, $\|tx - f\| < r + d$. Suppose

$$\|s(\lambda tx + (1-\lambda)f) - (\lambda stx + (1-\lambda)f)\| \ge \varepsilon$$

for some $t \ge t_0$, $s \in S$ and λ: $\alpha \le \lambda \le \beta$. Put $u = (1-\lambda)(sz - f)$ and $v = \lambda(stx - sz)$ where $z = \lambda tx + (1-\lambda)f$. Then $\|u\| \le (1-\lambda)\|z - f\| = \lambda(1-\lambda)\|tx - t\|$ and $\|v\| \le \lambda\|tx - z\| = \lambda(1-\lambda)\|tx - f\|$. By Lemma 1 of [**15**], we have

$$\begin{aligned}
\lambda(1-\lambda)\|stx - f\| &= \|\lambda u + (1-\lambda)v\| \\
&\le \lambda(1-\lambda)\|tx - f\|(1 - 2\lambda(1-\lambda)\delta(\varepsilon/(r+d))) \\
&\le \lambda(1-\lambda)\|tx - f\|(1 - c\delta(\varepsilon/(r+d))) \\
&\le \lambda(1-\lambda)(r+d)(1 - c\delta(\varepsilon/(r+d))).
\end{aligned}$$

On the other hand, there exists $s_0 \in S$ such that

$$(r+d)(1 - c\delta(\varepsilon/(r+d))) < \inf_{s_0 \le t} \|tx - f\|.$$

Since S is right amenable, we can choose u_0, $u_1 \in S$ such that $u_0 s_0 = u_1 st$. So we have

$$(r+d)(1-c\delta(\varepsilon/(r+d))) < \|u_0 s_0 x - f\| = \|u_1 stx - f\| \\ \leq \|stx - f\| \leq (r+d)(1-c\delta(\varepsilon/(r+d))),$$

which is a contradiction.

Let C be a closed convex subset of a Banach space E and D a closed subset of C. A retraction P: $C \to D$ is said to be sunny if for each $x \in C$, $Px = v$ implies $P(v + a(x - v)) = v$ whenever $v + a(x - v)$ belongs to C and $a \geq 0$; see **[5]** and **[19]**. The following Proposition is due to Reich **[19]**. For the proof, see Lemma 2.7 of **[19]**.

PROPOSITION 3. *Let C be a nonempty closed convex subset of a normed linear space E whose norm is Gâteaux differentiable and D a nonempty closed subset of C. If P is a sunny and nonexpansive retraction of C onto D, then*

$$\langle Px - x, J(y - Px)\rangle \geq 0$$

for all x in C and y in D, where J is the duality mapping on E.

THEOREM 2. *Let C be a closed convex subset of a uniformly convex and uniformly smooth Banach space E and let S be an amenable semigroup of nonexpansive mappings t of C into itself. Suppose $F(S) = \bigcap\{F(t): t \in S\}$ is nonempty. Then the following conditions are equivalent*:

(i) *For each $x \in C$, $\bigcap_s \overline{\text{co}}\{tx: t \geq s\} \cap F(S) \neq \varnothing$.*

(ii) *There exists a nonexpansive retraction P of C onto $F(S)$ such that $tP = Pt = P$ for all $t \in S$ and $Px \in \bigcap_s \overline{\text{co}}\{tx: t \geq s\}$ for each $x \in C$.*

Proof. It is obvious that (ii) implies (i). Suppose (i) is satisfied. Since C has normal structure, by Theorem 1, there exists a nonexpansive retraction of C onto $F(S)$. Then from Theorem 4.1 of **[23]**, there exists a sunny nonexpansive retraction r of C onto $F(S)$. Let Q be as in the proof of Theorem 1 and set $P = rQ$. Then P is a nonexpansive retraction of C onto $F(S)$ such that $Pt = tP = P$ for all $t \in S$. Let $x \in C$. Put $x_0 = Qx$ and $y = rx_0$. Then we show that $y \in \bigcap_s \overline{\text{co}}\{tx: t \geq s\}$. Suppose $y \notin \bigcap_s \overline{\text{co}}\{tx: t \geq s\}$. From the definition of y and Proposition 3, we have

$$\langle x_0 - y, J(y - v)\rangle \geq 0, \quad \text{for all } v \in F(S),$$

where J is the duality mapping of E. Therefore from Proposition 1, we have

(3) $$\inf_s \sup_{s \le t} \langle tx - y, J(y - v) \rangle \ge \mu_t \langle tx - y, J(y - v) \rangle = \langle x_0 - y, J(y - v) \rangle \ge 0$$

for each $v \in F(S)$. Let $z \in \bigcap_s \overline{\text{co}}\{tx: t \ge s\} \cap F(S)$. Fix a constant a such that $0 < a < 1$ and put $y_a = ay + (1 - a)z$. For each $t \in S$, let $y_t \in [y_a, tx] = \{\lambda tx + (1 - \lambda) y_a: 0 \le \lambda \le 1\}$ be such that $\|y_t - z\| = \min\{\|u - z\|: u \in [y_a, tx]\}$. Then $\|y_t - z\| \le \|y_a - z\| = a\|y - z\|$ and y_t satisfies the following inequality [**11**].

(4) $$\langle u - y_t, J(y_t - z) \rangle \ge 0 \quad \text{for all } u \in [y_a, tx].$$

Suppose y_t converges to y_a. Then, since J is norm-to-norm continuous, we have that, for given $\varepsilon > 0$, there exists $t_0 \in S$ such that

$$\langle tx - y_a, J(y_a - z) - J(y_t - z) \rangle \ge -\varepsilon$$

for all $t \ge t_0$. Therefore, we have for $t \ge t_0$,

$$\begin{aligned} \langle tx - y_a, J(y_a - z) \rangle &= \langle tx - y_a, J(y_a - z) - J(y_t - z) \rangle \\ &\quad + \langle tx - y_a, J(y_t - z) \rangle \\ &> -\varepsilon + 0 = -\varepsilon. \end{aligned}$$

Then it follows that for each $v \in \bigcap_s \overline{\text{co}}\{tx: t \ge s\}$,

(5) $$\langle v - y_a, J(y_a - z) \rangle \ge 0.$$

If we set $v = z$ in (5), then we have $y_a = z$, and hence $y = z$, which is a contradiction. So y_t does not converge to y_a. Then setting

$$y_t = a_t tx + (1 - a_t) y_a, \quad 0 \le a_t \le 1,$$

we obtain that a_t does not converge to 0. Hence, there exists a positive number c_0 so small that for each $t \in S$, there is a $t' \in S$ with $t' \ge t$ and $a_{t'} \ge c_0$. Let $T = \{t' \in S: a_{t'} \ge c_0\}$. Since $k = \lim_t \|tx - y_a\|$ exists and k is positive, we can choose $\varepsilon > 0$ so small that

(6) $$(R + \varepsilon)(1 - \delta(c_0 k / (R + \varepsilon))) < R,$$

where $R = a\|y - z\|$ and δ is the modulus of convexity of the norm. Then by Lemma 1 there exists $t_0 \in S$ such that

(7) $$\|s(c_0 tx + (1 - c_0) y_a) - (c_0 stx + (1 - c_0) y_a)\| < \varepsilon,$$

for all $s \in S$ and $t \ge t_0$. Let $t' \in T$ such that $t' \ge t_0$. Then since

$$\|a_{t'} t' x + (1 - a_{t'}) y_a - z\| = \|y_{t'} - z\| \le \|y_a - z\| = a\|y - z\| = R,$$

$\|y_a - z\| \leq R$ and $a_{t'} \geq c_0$, we obtain $\|c_0 t'x + (1 - c_0)y_a - z\| \leq R$. Using (7) we obtain

$$\|c_0 st'x + (1 - c_0)y_a - z\| \leq \|s(c_0 t'x + (1 - c_0)y_a) - z\| + \varepsilon$$
$$\leq \|c_0 t'x + (1 - c_0)y_a - z\| + \varepsilon \leq R + \varepsilon$$

for all $s \in S$. We also know

$$\|y_a - z\| = a\|y - z\| = R < R + \varepsilon$$

and

$$\|c_0 st'x + (1 - c_0)y_a - y_a\| = c_0\|st'x - y_a\| \geq c_0 k$$

for all $s \in S$. Then, by the definition of δ, we have for all $t \geq t'$,

(8) $$\left\|\frac{c_0}{2}tx + \left(1 - \frac{c_0}{2}\right)y_a - z\right\|$$
$$= \left\|\frac{1}{2}(c_0 tx + (1 - c_0)y_a - z) + \frac{1}{2}(y_a - z)\right\|$$
$$\leq (R + \varepsilon)(1 - \delta(c_0 k/(R + \varepsilon))) < R.$$

From (8) and $\|y_a - z\| = R$, $\|tx + \alpha(y_a - tx) - z\| \geq R$ for all $t \geq t'$ and $\alpha \geq 1$. Then we obtain

$$\langle tx + \alpha(y_a - tx) - y_a, J(y_a - z)\rangle \leq 0$$

for all $t \geq t'$ and $\alpha \geq 1$. From $y_a = ay + (1 - a)z$, we obtain

$$\langle tx - z, J(y - z)\rangle - a\langle y - z, J(y - z)\rangle \leq 0$$

and, hence,

$$\langle tx - y, J(y - z)\rangle \leq -(1 - a)\|y - z\|^2 \quad \text{for all } t \geq t'.$$

Then we have

(9) $$\inf_s \sup_{s \leq t} \langle tx - y, J(y - z)\rangle \leq -(1 - a)\|y - z\|^2.$$

This contradicts (3). Consequently, we obtain that y is contained in $\bigcap_s \overline{\text{co}}\{tx\colon t \geq s\}$.

COROLLARY 1 (*Takahashi* [**26**]). *Let C be a closed convex subset of a Hilbert space H and let S be an amenable semigroup of nonexpansive mappings t of C into itself. Suppose*

$$F(S) = \bigcap \{F(t)\colon t \in S\} \neq \varnothing.$$

Then there exists a nonexpansive retraction P of C onto $F(S)$ such that $Pt = tP = P$ for all $t \in S$ and $Px \in \bigcap_s \overline{\text{co}}\{tx: t \geq s\}$ for each $x \in C$.

Proof. Let Q be as in the proof of Theorem 1. Then we know $Qx \in \bigcap_s \overline{\text{co}}\{tx: t \geq s\} \cap F(S)$; see [**26**].

5. Ergodic theorems in a strictly convex Banach space. In this section we establish the existence of ergodic nonexpansive retractions onto the fixed point sets of commutative semigroups of nonexpansive mappings in a strictly convex Banach space.

LEMMA 2. *Let C be a closed convex subset of a strictly convex Banach space E and let S be a commutative semigroup of nonexpansive mappings of C into itself. Let $x \in C$ and suppose Sx is relatively compact. Then* (a) *$w(x) = \bigcap_s \overline{\{tx: t \geq s\}}$ is a minimal S-invariant nonempty closed set,* (b) *an element t in S is affine on $\overline{\text{co}}\, w(x)$ and* (c) *$\overline{\text{co}}\, w(x)$ contains a common fixed point of S.*

Proof. (a) It is easy to see that $w(x)$ is nonempty, closed S-invariant. To show the minimality of $w(x)$, it is sufficient to show that for each $y \in w(x)$, $w(x) \subset w(y)$. Let $y, z \in w(x)$. Then for given $\varepsilon > 0$ and $t \in S$, there exist $t' \in S$ and $s' \in S$ such that $\|y - t'x\| < \varepsilon/2$ and $\|z - s'tt'x\| < \varepsilon/2$. Then $s't \geq t$ and

$$\|s'ty - z\| \leq \|s'ty - s'tt'x\| + \|s'tt'x - z\| \leq \varepsilon.$$

Therefore $z \in w(y)$ and, hence, the minimality of $w(x)$ has been established. (b) Let $y, z \in w(x)$ and $t \in S$. For given $\varepsilon > 0$, there exists $t' \in S$ such that $t' \geq t$ and $\|y - t'y\| < \varepsilon$ since $y \in w(y)$. Then we show $\|z - t'z\| < \varepsilon$. In fact let $\{s_n\} \subset S$ be a sequence such that $z = \lim_n s_n y$. Then, since S is commutative, we have

$$\begin{aligned}\|z - t'z\| &\leq \|z - s_n y\| + \|s_n y - t's_n y\| + \|t's_n y - t's\| \\ &\leq 2\|z - s_n y\| + \|y - t'y\|.\end{aligned}$$

Therefore we have $\|z - t'z\| < \varepsilon$. Then we obtain

$$\begin{aligned}\|y - z\| &\geq \|ty - tz\| \geq \|t'y - t'z\| \\ &\geq \|y - z\| - \|y - t'y\| - \|z - t'z\| \\ &\geq \|y - z\| - 2\varepsilon.\end{aligned}$$

Since ε is arbitrary, $\|y - z\| = \|ty - tz\|$. Thus t is an isometry on $w(x)$.

Then by Proposition 2 of [**12**], t is affine on $\overline{co}\, w(x)$. (c) From (a) and (b), $\overline{co}\, w(x)$ is S-invariant. Therefore it contains a common fixed point of S [**24**].

THEOREM 3. *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E and let S be a commutative semigroup of nonexpansive mappings of C into itself. Suppose for each $x \in C$, Sx is relatively compact. Then there exists a nonexpansive retraction P of C onto $F(S)$ such that $Pt = tP = P$ for each $t \in S$ and $Px \in \bigcap_s \overline{co}\{tx\colon t \geq s\}$.*

Proof. Let $x \in C$, $x_0 \in F(S) \cap \overline{co}\, w(x)$ and $z \in F(S)$. Let $\varepsilon > 0$ and $s \in S$. Then, since $x_0 \in \overline{co}\, w(x)$, there exist elements t_i $(i = 1, 2, \cdots n)$ in $\{t\colon t \geq s\}$ and nonnegative numbers a_i $(i = 1, 2, \cdots n)$ with $\sum_{i=1}^n a_i = 1$ such that

$$\left\langle \sum_{i=1}^{n} a_i t_i x - x_0, J(x_0 - z) \right\rangle \geq -\varepsilon.$$

So there exists a t_i such that $\langle t_i x - x_0, J(x_0 - z)\rangle \geq -\varepsilon$. Then we have

$$\sup_{s \leq t} \langle tx - x_0, J(x_0 - z)\rangle \geq -\varepsilon.$$

Since ε is arbitrary, we obtain

$$\sup_{s \leq t} \langle tx - x_0, J(x_0 - z)\rangle \geq 0.$$

Then since Sx is relatively compact and, hence, $\overline{\{tx\colon t \geq s\}}$ is compact for each $s \in S$, $\overline{\{tx\colon t \geq s\}}$ contains a point y_s such that

$$\langle y_s - x_0, J(x_0 - z)\rangle \geq 0.$$

Therefore we obtain that there exists $y \in w(x) = \bigcap_s \overline{\{tx\colon t \geq s\}}$ such that

$$\langle y - x_0, J(x_0 - z)\rangle \geq 0. \tag{10}$$

(10) implies that for each a, $0 \leq a \leq 1$,

$$\langle ay + (1 - a)x_0 - x_0, J(x_0 - z)\rangle \geq 0.$$

Then by [**11**] we have

$$\|ay + (1 - a)x_0 - z\| \geq \|x_0 - z\| \quad \text{for all } a,\ 0 \leq a \leq 1. \tag{11}$$

While, since an element in S is affine and nonexpansive on $\overline{co}\, w(x)$, we have that for each a, $0 \leq a \leq 1$,

$$\lim_t \|aty + (1 - a)x_0 - z\| = \lim_t \|t(ay + (1 - a)x_0) - z\|. \tag{12}$$

Let $u \in w(x)$, $0 \leq a \leq 1$ and $\varepsilon > 0$. Then from (12) there exists $t_0 \in S$ such that

$$\big| \|aty + (1-a)x_0 - z\| - d \big| < \varepsilon/4$$

for all $t \geq t_0$, where $d = \lim_s \|asy + (1-a)x_0 - z\|$. Since $u, y \in w(x) = w(y)$, there exists $t' \in S$ and $t'' \in S$ such that $t', t'' \geq t_0$, $\|t'y - y\| < \varepsilon/4$ and $\|t''y - u\| < \varepsilon/4$. Then we have

$$\big| \|au + (1-a)x_0 - z\| - \|at''y + (1-a)x_0 - z\| \big| \leq a\|u - t''y\| < \varepsilon/4.$$

Therefore,

$$\begin{aligned} \|au + (1-a)x_0 - z\| &> \|at'' + (1-a)x_0 - z\| - \varepsilon/4 \\ &> \lim_t \|aty + (1-a)x_0 - z\| - \varepsilon/2 \\ &> \|at'y + (1-a)x_0 - z\| - 3\varepsilon/4 \\ &> \|ay + (1-a)x_0 - z\| - \varepsilon. \end{aligned}$$

Since ε is arbitrary, from (11) we obtain that

$$\|au + (1-a)x_0 - z\| \geq \|x_0 - z\| \quad \text{for all } a, 0 \leq a \leq 1.$$

This implies $\langle u - x_0, J(x_0 - z)\rangle \geq 0$ for all $u \in w(x)$. Therefore we have, for each $z \in F(S)$,

$$(13) \qquad \langle u - x_0, J(x_0 - z)\rangle \geq 0 \quad \text{for all } u \in \overline{\text{co}}\, w(x).$$

If $y \in \overline{\text{co}}\, w(x) \cap F(S)$, then by setting $u = z = y$ in (13), $y = x_0$. Therefore we have $\overline{\text{co}}\, w(x) \cap F(S) = \{x_0\}$. Now we set $Px = x_0$. Then P is well defined on C and P is a retraction of C onto $F(S)$ such that $tP = P$ for each $t \in S$ and $Px \in \bigcap_s \overline{\text{co}}\{tx : t \geq s\}$ for each $x \in C$. Since

$$\{Px\} = \overline{\text{co}}\, w(x) \cap F(S) \subset \overline{\text{co}}\, w(tx) \cap F(S) = \{Ptx\},$$

we also obtain $Px = Ptx$. We now show that P is nonexpansive. Let Q be as in the proof of Theorem 1. Then $Qx \in \overline{\text{co}}\, w(x)$ for each $x \in C$. In fact, if $Qx \notin \overline{\text{co}}\, w(x)$ for some $x \in C$, then there exists $f \in E^*$ such that

$$\langle Qx, f\rangle < \inf\{\langle z, f\rangle : z \in \overline{\text{co}}\, w(x)\}.$$

Since $\overline{\{tx : t \geq s\}}$ for each $s \in S$ is compact, we obtain $y_s \in \overline{\{tx : t \geq s\}}$ such that

$$\inf_{s \leq t} \langle tx, f\rangle = \langle y_s, f\rangle.$$

Then we can obtain $y \in w(x) = \bigcap_s \overline{\{tx : t \geq s\}}$ such that

$$\langle y, f \rangle \leq \sup_s \inf_{s \leq t} \langle tx, f \rangle.$$

So we obtain

$$\sup_s \inf_{s \leq t} \langle tx, f \rangle \leq \mu_t \langle tx, f \rangle = \langle Qx, f \rangle < \inf\{\langle z, f \rangle : z \in \overline{\mathrm{co}}\, w(x)\}$$
$$\leq \inf\{\langle z, f \rangle : z \in w(x)\} \leq \langle y, f \rangle \leq \sup_s \inf_{s \leq t} \langle tx, f \rangle.$$

This is a contradiction. Since $Qx \in \overline{\mathrm{co}}\, w(x)$, we obtain

$$\{PQx\} = F(S) \cap \overline{\mathrm{co}}\, w(Qx) = F(S) \cap \overline{\mathrm{co}}\, w(x) = \{Px\}$$

and, hence, $P = PQ$. Since Q is nonexpansive, it is sufficient to show that

$$\|PQx - PQy\| \leq \|Qx - Qy\| \quad \text{for } x, y \in C.$$

From (13) we have

$$\langle Qx - PQx, J(PQx - PQy) \rangle \geq 0$$

and

$$\langle Qy - PQy, J(PQy - PQx) \rangle \geq 0.$$

Then we obtain

$$\|PQx - PQy\|^2 \leq \langle J(PQx - PQy), Qx - Qy \rangle$$
$$\leq \|PQx - PQy\| \, \|Qx - Qy\|.$$

This completes the proof.

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